2024-12-02 SAUL SCHLEIMER LECTURE 28 MA3F1

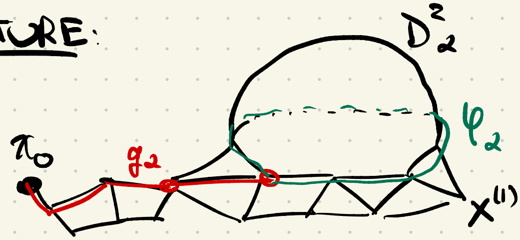
① FROM COMPLEXES TO PRESENTATIONS

WE GIVE AN "ALGORITHM" WHICH, GIVEN A CW COMPLEX X, PRODUCES A PRESENTATION of $\pi_1(X)$.

SUPPOSE THAT X IS THE GIVEN CONNECTED CW COMPLEX. FIX $x_0 \in X^{(0)}$ AND $T \subset X^{(1)}$ A SPANNING TREE. LET $S \subset X^{(1)} - T$ BE THE NON-TREE EDGES of $X^{(1)} - T$.

<u>PROP</u>: $\pi_1(X^{(1)}, x_0) \cong F_S$.

SUPPOSE D^2_α IS A TWO-CELL IN X. SO $\varphi_\alpha : S^1_\alpha \to X^{(1)}$ IS THE ATTACHING MAP. FIX $s_\alpha \in S^1_\alpha$ AND $g : I \to X^{(1)}$ FROM x_0 TO $\varphi_\alpha(s_\alpha)$. DEFINE $\gamma_\alpha \in \text{LOOPS}(X^{(1)}, x_0)$ BY $\gamma_\alpha = g_\alpha * \varphi_\alpha * \bar{g}_\alpha$ <u>PICTURE</u>:



LET r_α BE THE RESULTING WORD IN F_S [AS $\pi_1(X^{(1)}, x_0) \cong F_S$]

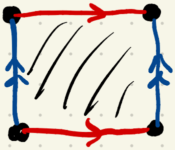
SET $R = \{ r_\alpha \mid D^2_\alpha \text{ IS A TWO-CELL IN } X^{(2)} \}$.

<u>THEN</u>, BY PROP 1.26(b) $\pi_1(X^{(2)}, x_0) \cong F_S / \langle\langle R \rangle\rangle$

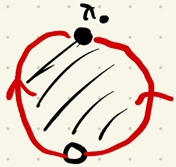
THUS, BY PROP 1.26(c) $\boxed{\pi_1(X, x_0) \cong \langle S \mid R \rangle}$

② <u>HEXAGON SPACE</u>: A NICE FAMILY of SPACES COMES FROM THE $2n$-GON, GLUING OPPOSITE SIDES

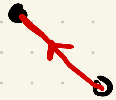
WE HAVE SEEN THE 4-GON ALREADY.

 $\cong T^2$.

HERE IS THE TWO-GON EXAMPLE



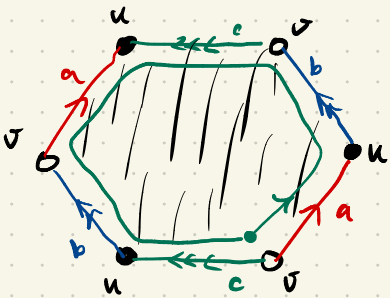
$\cong$

NOTE $X^{(1)} =$  IS A TREE T

$S = X^{(1)} - T = \emptyset$

AND SO $\pi_1(X, x_0) \cong 1$.

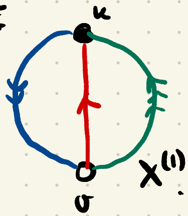
HERE IS THE HEXAGON EXAMPLE:



TAKE $T = c$ AND $S = \{a, b\}$.
THE ATTACHING MAP GIVES
THE LOOP $a * b * \cancel{c} * a^{-1} * b^{-1} * \cancel{c}$ IN $X^{(1)}$. BUT $c \subset T$ SO IS
A TREE EDGE, SO $r = a * b * a^{-1} * b^{-1}$

AND $\pi_1(X, v) \cong \langle a, b \mid a b a^{-1} b^{-1} \rangle \cong \mathbb{Z}^2$

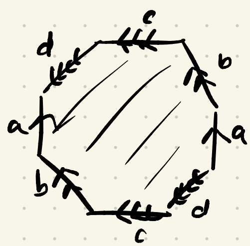
PICTURE



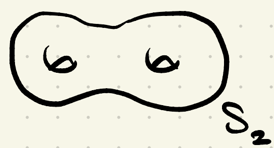
SINCE THERE IS MORE THAN ONE VERTEX THE SPANNING TREE IS NON-TRIVIAL.

EXERCISE: SHOW THE $4g$-GON SURFACE IS HOMEOMORPHIC

TO $S_g = \#_g T^2$.



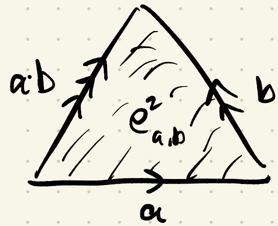
$\cong$



AND SO (OR OTHERWISE) GIVE A PRESENTATION of
of $\pi_1(S_g)$.

③ FROM GROUPS TO SPACES

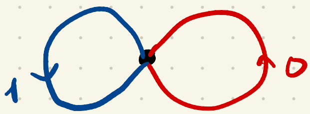
SUPPOSE G IS A GROUP. WE DEFINE A TWO-COMPLEX $X = X_G$ AS FOLLOWS: $X^{(0)} = x_0$ IS A POINT. THERE IS A ONE-CELL e^1_a FOR EACH $a \in G$. SO $X^{(1)} \cong \bigvee_{a \in G} S^1$. THE TWO-CELLS $e^2_{a,b}$ ARE TRIANGLES



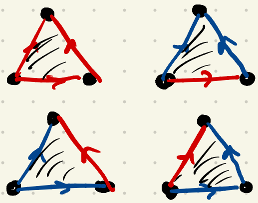
FOR ALL $a, b \in G$.

SO $X_G = \{x_0\} \sqcup \bigsqcup_{a \in G} D^1_a \sqcup \bigsqcup_{a,b \in G} D^2_{a,b} \Big/ \text{ATTACHING MAPS}.$

EXAMPLE: $G = \mathbb{Z}/2\mathbb{Z}$.



ATTACH



NOTE

$$\pi_1(X_G) \cong F_G \Big/ a * b * (a \cdot b)^{-1} \cong G.$$

[SKIP THE MORE GENERAL CASE OF PRESENTATION COMPLEXES.]

④ UNIVERSAL COVERS:

DEF: SUPPOSE $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ IS A COVERING MAP. IT IS A UNIVERSAL COVERING IF $\pi_1(\tilde{X}, \tilde{x}_0) \cong \mathbb{1}$.

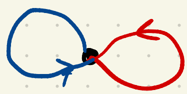
THEOREM (1.36-38) SUPPOSE X IS A CW COMPLEX, $x_0 \in X^{(0)}$, AND X IS CONNECTED. THEN A UNIVERSAL COVER $(\tilde{X}, \tilde{x}_0)$ EXISTS AND IS UNIQUE UP TO UNIQUE ISOMORPHISM.

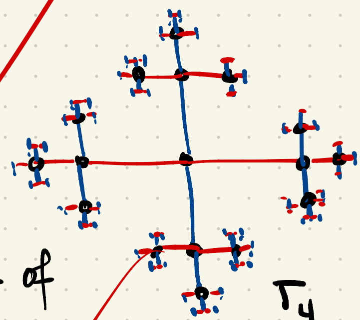
EXAMPLES: ① IF $\pi_1(X, x_0) \cong 1$ THEN $\tilde{X} = X$ AND $Id_X : \tilde{X} \to X$ IS A UNIVERSAL COVER. [SO, $\mathbb{R}^n$, S^n $(n \geq 2)$, $\mathbb{B}^n$, ...]

② $\left.\begin{array}{l} p: \mathbb{R}^1 \to S^1 \\ t \mapsto \exp(2\pi i t) \end{array}\right\}$ IS A UNIV. COVER

③ $p: \mathbb{R}^n \to \mathbb{T}^n$ IS AS WELL. SPECIAL CASE of:

④ $\widetilde{X \times Y} \cong \tilde{X} \times \tilde{Y}$ IF $\tilde{X}, \tilde{Y}$ EXIST.

⑤  R_2 HAS UNIVER. COVER.

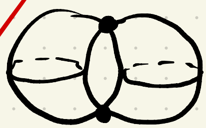


T_4

T_4 THE REGULAR FOUR-VALENT TREE.

EXERCISE: FIND THE UNIVERSAL COVER of

① $\mathbb{RP}^1 \times \mathbb{RP}^1$ AND

②  } TWO POINT UNION of S^2 AND S^2.

NON-EXAMPLE THE EARRING SPACE  HAS NO UNIVERSAL COVER.

BLUEPRINT FOR $(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)$. FIX (X, x_0)

① DEFINE A SET $\tilde{X}_0$ AND A POINT $\tilde{x}_0$.

② DEFINE A FUNCTION $p: \tilde{X} \to X$. CHECK $p(\tilde{x}_0) = x_0$.

③ DEFINE A TOPOLOGY ON $\tilde{X}$.

④ PROVE p IS A COVERING MAP.

⑤ PROVE $\tilde{X}$ IS PATH-CONNECTED.

⑥ PROVE $\pi_1(\tilde{X}) = 1$. ✓