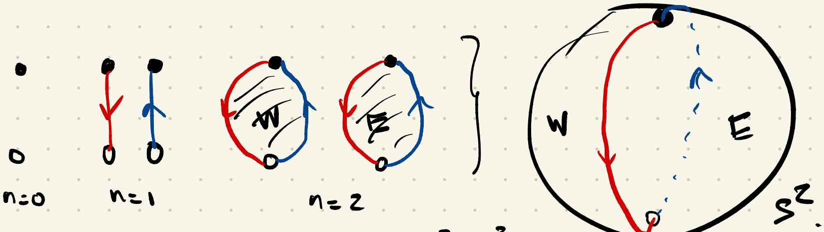
① EXAMPLES

THE REAL PROJECTIVE SPACE: RECALL $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$.

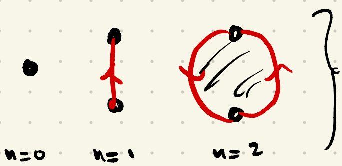
DEFINE $\mathbb{RP}^n = S^n / x \sim -x$ THE n-DIM'L PROJECTIVE SPACE.

EXERCISE: $\mathbb{RP}^n \cong \mathbb{R}^{n+1} - \{0\} \big/ x \sim \lambda x$ FOR $\lambda \in \mathbb{R} - \{0\}$.

THE LATTER IS THE "SPACE of LINES IN $\mathbb{R}^{n+1}$".
SIMILARLY, S^n IS THE "SPACE of DIRECTIONS IN $\mathbb{R}^{n+1}$".
HERE IS A CELL STRUCTURE FOR S^2:

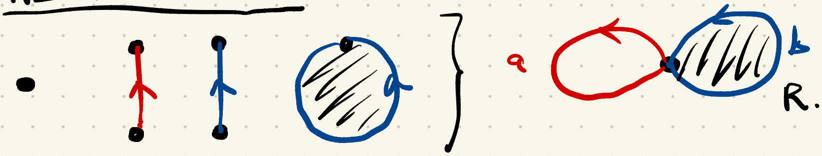


WE TAKE THE QUOTIENT $\mathbb{RP}^2 = S^2 / x \sim -x$.
THIS HAS CELL STRUCTURE



DOES NOT EMBED IN $\mathbb{R}^3$!

THE FAKE ROSE:



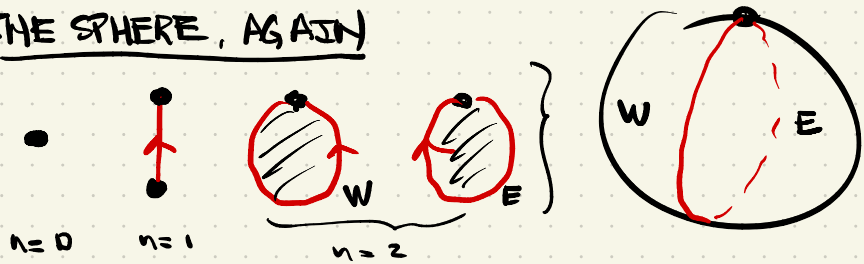
NOTE THAT $R \underset{h.e.}{\simeq} S^1$. SO $\pi_1(R, x_0) \cong \mathbb{Z}$.

ALSO SvK IMPLIES THAT $\pi_1(R, x_0) \cong \pi_1(R_2)/N \cong F_2/N$

WHERE F_2 IS GEN BY a, b AND $N = \langle\langle b \rangle\rangle$.

SO $\mathbb{Z} \cong \pi_1(R) \cong \langle a, b \mid b \rangle$.

THE SPHERE, AGAIN



SO $1 \cong \pi_1(S^2, x_0) \cong \langle a \mid a, a \rangle$

NOTE:
$N = \langle\langle a, a \rangle\rangle = \langle\langle a \rangle\rangle$.

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② COMPUTING π_1 of CW COMPLEXES

PROP (1.26) [FOR CW COMPLEXES]

SUPPOSE X IS CW COMPLEX. FIX $x_0 \in X^{(0)}$.

FOR ⓐ + ⓑ SUPPOSE $C \subset X$ IS A SUBCOMPLEX WITH $x_0 \in C^{(0)}$.

LET $\iota: C \longrightarrow X_0$ BE THE INCLUSION MAP. SUPPOSE $X - C = e^n_\alpha$ IS A SINGLE n-CELL.

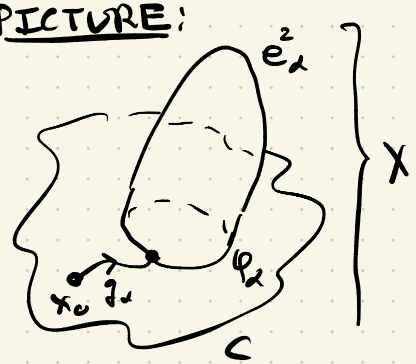
SUPPOSE $\varphi_\alpha : S^{n-1}_\alpha \longrightarrow C$ IS THE ATTACHING MAP.

SUPPOSE $s_\alpha \in S^{n-1}_\alpha$ AND $g_\alpha : I \longrightarrow C$ IS A PATH FROM x_0 TO s_α

ⓐ SUPPOSE $n = 2$. IDENTIFY $S^1_\alpha = [0,1]/_{0 \sim 1}$ AND FORM $\gamma_\alpha = g_\alpha * \varphi_\alpha * \bar{g}_\alpha \in \text{LOOPS}(C, x_0)$.

DEFINE $N = \langle\langle \gamma_\alpha \rangle\rangle$.

PICTURE:



THEN $\iota_*: \pi_1(C, x_0) \to \pi_1(X, x_0)$ IS SURJECTIVE WITH $\mathrm{KER}(\iota_*) = N$.

(b) SUPPOSE $n \geq 3$. THEN ι_* IS AN ISOMORPHISM.

(c) THE INCLUSION $X^{(2)} \longrightarrow X$ INDUCES ISOMORPHISM $\pi_1(X^{(2)}, x_0) \longrightarrow \pi_1(X, x_0)$

PROOF: (c) FOLLOWS FROM (b) + INDUCTION IF X IS FINITE [ATTACH THE CELLS of $X - X^{(2)}$ ONE-BY-ONE]. IF X IS NOT FINITE WE REDUCE TO THE FINITE CASE USING (A1) AS FOLLOWS. LET $\iota: X^{(2)} \longrightarrow X$. WTS ι_* IS ISOMORPHISM.

SURJECTIVE: FIX $[\gamma] \in \pi_1(X, x_0)$. SO $\gamma \in \mathrm{LOOPS}(X, x_0)$

SINCE I IS COMPACT SO IS $\gamma(I)$. (A1) IMPLIES THERE IS A FINITE SUBCOMPLEX $C \subset X$ SO THAT

$\gamma(I) \subset C$. NOTE $x_0 \in \gamma(I)$ SO $x_0 \in C^{(2)}$.

WE HAVE INCLUSIONS:

$$\begin{tikzcd} C^{(2)} \arrow[r, "s"] \arrow[d, "p"'] & C \arrow[d, "q"] \\ X^{(2)} \arrow[r, "t"] & X \end{tikzcd}$$

THESE GIVE HOMOMORPHISMS.

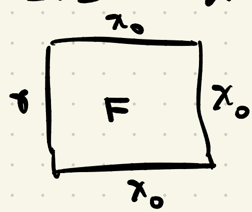
$$\begin{tikzcd} \pi_1(C^{(2)}, x_0) \arrow[r, "s_*"] \arrow[d, "p_*"'] & \pi_1(C, x_0) \ni [\gamma] \arrow[d, "q_*"] \\ \pi_1(X^{(2)}, x_0) \arrow[r, "t_*"] & \pi_1(X, x_0) \ni [\gamma] \end{tikzcd}$$

(b) IMPLIES s_* IS ISOMORPHISM.

LET $[\gamma'] = (s_*)^{-1}([\gamma])$. NOTE $\gamma' \in \text{LOOPS}(C^{(2)}, x_0)$ AND $\gamma' \simeq_\partial \gamma$. NOTE $(t_* \circ p_*)[\gamma'] = [\gamma]$.

SINCE γ WAS ARB IN LOOPS(X, x_0) t_* IS SURJECTIVE. ✓

<u>INJECTIVE</u>: SUPPOSE $\gamma \in \text{LOOPS}(X^{(2)}, x_0)$. SUPPOSE $[\gamma] = [e] \in \pi_1(X, x_0)$. SO THERE IS $F : I \times I \longrightarrow X$ A BASED NULL-HOMOTOPY GIVING $\gamma \simeq_\partial e$.



SO $F(I \times I)$ IS COMPACT, SO CONTAINED IN SOME FIN. SUBCOMPLEX. THUS $[\gamma] = [e]$ IN $\pi_1(C, x_0)$. SO $[\gamma] = [e]$ IN $\pi_1(C^{(2)}, x_0)$ BY (b).

[THAT IS: THE ISOMORPH. s_* IMPLIES THERE IS SOME $F' : I \times I \to C^{(2)}$ SO THAT $\gamma \overset{?}{\simeq} e$ IN $C^{(2)}$. SINCE $C^{(2)} \subset X^{(2)}$ WE ARE DONE WITH (c) ✓

(a), (b) NEXT TIME!