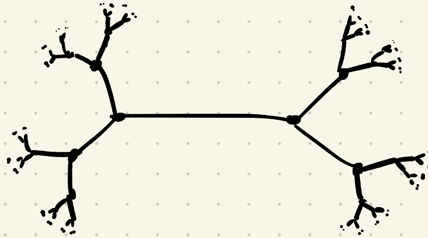
2024-11-19 MA3F1 LECTURE 23 SAUL SCHLEIMER

① EXAMPLE of CW COMPLEX

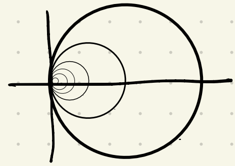
T_3 : THE INFINITE REGULAR THREE-VALENT TREE.



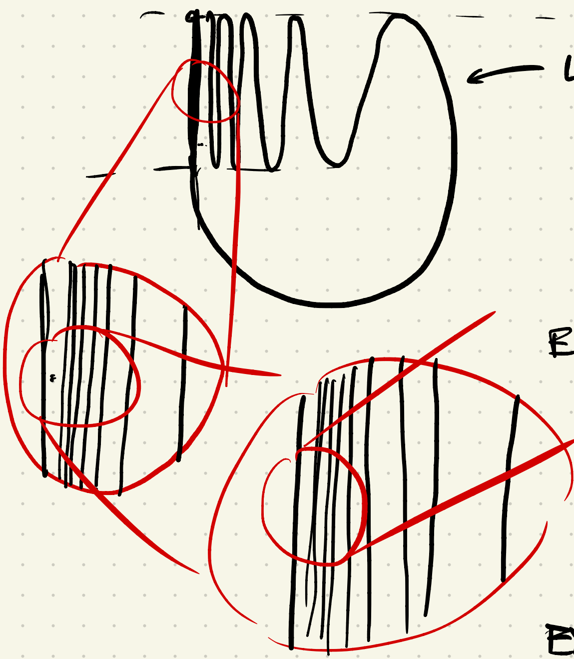
② NON-EXAMPLES:

(1) THE CANTOR SET

(2) THE EARRING SPACE



(3) THE TOPOLOGIST'S CIRCLE

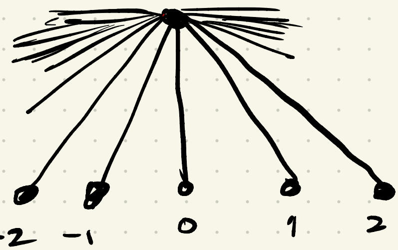


← LIKE $\sin(1/x)$

ETC!

(4)

$$C = \left\{ (1-t)(n,0) + t \cdot (0,1) \;\middle|\; n \in \mathbb{Z},\ t \in [0,1] \right\}$$



EXERCISE: THE CONE C IS **NOT** HOMEOMORPHIC TO $\bigvee_{n=-\infty}^{\infty} [0,1]$

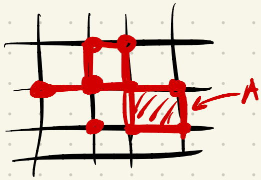
③ SUBCOMPLEXES

SUPPOSE X IS A CW-COMPLEX. $A \subset X$ IS A **SUBCOMPLEX**

IF (i) A IS CLOSED

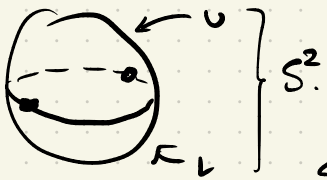
AND (ii) A IS A UNION of CELLS of X.

EXAMPLE:



EXAMPLE: $X^{(n)} \subset X$ IS A SUBCOMPLEX.

EXAMPLE: S^n HAS A CELL STRUCTURE X WITH A PAIR of CELLS IN EACH DIMENSION $0, 1, 2, \ldots, n-1, n$. ALSO $X^{(k)} \cong S^k$. SO FIND SEQUENCE $S^0 \subset S^1 \subset S^1 \subset S^2 \subset \cdots \subset S^{n-1} \subset S^n$.



WE CAN BUILD $\mathbb{RP}^n \cong S^n/_{x \sim -x}$

[QUOTIENT BY ANTIPODAL MAP]

SO $\mathbb{RP}^n$ HAS CELL STRUCTURE WITH A SINGLE CELL IN EACH DIMENSION.

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④ **MANY USEFUL FACTS** SUPPOSE X IS A CW COMPLEX.

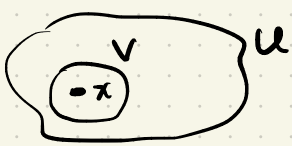
PROP A.1 SUPPOSE $K \subset X$ IS COMPACT. THEN THERE IS A FINITE SUBCOMPLEX $A \subset X$ SO THAT $K \subset A$.

PROP A.2 X IS NORMAL [THUS HAUSDORFF]

PROP A.4 X IS LOCALLY CONTRACTIBLE

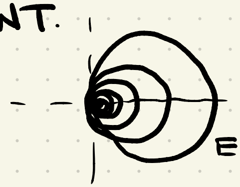
THAT IS: FOR ALL $x \in X$, FOR ALL OPEN $U \subset X$ WITH $x \in U$ THERE IS SOME OPEN $V \subset X$ SO THAT $x \in V \subset U$, AND V IS CONTRACTIBLE ($V \underset{h.e.}{\simeq} \{pt\}$).

PICTURE



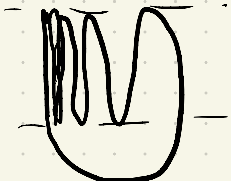
EXAMPLE: $\mathbb{R}^2$ IS LOCALLY CONTRACTIBLE. (B/C B^2 IS CONTRACTIBLE).

NONEXAMPLES THE BARRING SPACE IS NOT LOCALLY CONTRACTIBLE, AT ITS BASEPOINT.



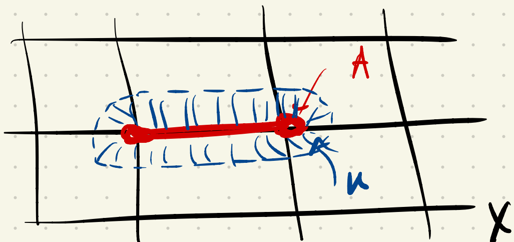
NOTE: LOCALLY CONTRACTIBLE IMPLIES LOCALLY CONNECTED.

SIMILARLY: THE TOPOLOGIST'S CIRCLE IS NOT LOCALLY CONNECTED.



PROP A.5 SUPPOSE $A \subset X$ IS A SUBCOMPLEX. THEN THERE IS $U \subset X$ OPEN SO THAT $A \subset U$ AND U DEF.RETRACTS TO A.

PICTURE



(5) EXERCISES FROM HATCHER [PAGE 529]

(1) SUPPOSE X IS CW. SUPPOSE $p: \hat{X} \to X$ IS A COVERING MAP. THEN THERE IS A (UNIQUE) LIFT of THE CW-STR ON X TO $\hat{X}$.

(3) SUPPOSE X IS CW. THE X IS PATH CONN IF AND ONLY $X^{(1)}$ IS PATH CONN.

HERE IS OUR FINAL USEFUL FACT

THEOREM [PAGE 97 - KIND of] SUPPOSE X IS CW. SUPPOSE $x_0 \in X^{(0)}$. THEN THE INCLUSION $\iota: X^{(2)} \to X$ INDUCES $i_*: \pi_1(X^{(2)}, x_0) \xrightarrow[\text{ISOM.}]{\cong} \pi_1(X, x_0)$

THAT IS: π_1 IS DETERMINED BY THE TWO-SKELETON.

EXERCISES

(1) SHOW S^n HAS A CW-COMPLEX STR WITH EXACTLY A SINGLE 0-CELL AND A SINGLE n-CELL

(2) SHOW S^n HAS A CW-COMPLEX STRUCTURE WITH A PAIR of k-CELLS IN DIMENSIONS $k=0,1,2,\cdots,n$.

(3) DEFINE $\mathbb{RP}^n = S^n/x\sim -x$: THE QUOTIENT of S^n UNDER THE ANTIPODAL MAP. GIVE $\mathbb{RP}^n$ A CW-COMPLEX STR.