2024-11-14 LECTURE 21 MA3F1 SAUL SCHLEIMER

(1) **THEOREM** [SEIFERT–VAN KAMPEN] (1.20)

SUPPOSE $(X, x_0)$ IS A POINTED SPACE. SUPPOSE $\{(A_\alpha, x_0)\}_\alpha$ IS AN OPEN COVER OF $X$, WITH $A_\alpha$ PATH-CONNECTED FOR ALL $\alpha$. DEFINE $i_\alpha : A_\alpha \hookrightarrow X$ AND $\Phi = \underset{\alpha}{*} (i_\alpha)_*$

$$\boxed{\text{(1) SUPPOSE } B_{\alpha\beta} \text{ PATH CONN. FOR ALL } \alpha, \beta. \text{ THEN } \Phi \text{ IS SURJECTIVE}}$$

DEFINE $B_{\alpha\beta} = A_\alpha \cap A_\beta$, $i_{\alpha\beta} : B_{\alpha\beta} \to A_\alpha$. AND $B_{\alpha\beta\gamma} = A_\alpha \cap A_\beta \cap A_\gamma$.

DEFINE $$\mathcal{U} = \left\{ (i_{\alpha\beta})_*(\omega)\,(i_{\beta\alpha})_*(\omega^{-1}) \;\middle|\; \begin{array}{l} \text{ALL } \alpha, \beta, \\ \text{ALL } \omega \in \pi_1(B_{\alpha\beta}, x_0) \end{array} \right\}$$

AND $N = \langle\langle \mathcal{U} \rangle\rangle$ THE <u>NORMAL CLOSURE</u> OF $\mathcal{U}$.

[THAT IS: THE INTERSECTION OF ALL NORMAL SUBGROUPS OF $\underset{\alpha}{*} \pi_1(A_\alpha, x_0)$ CONTAINING $\mathcal{U}$]

$$\boxed{\text{(2) SUPPOSE } B_{\alpha\beta\gamma} \text{ IS PATH CONNECTED FOR ALL } \alpha, \beta, \gamma. \text{ THEN } \ker(\Phi) = N.}$$

<u>THAT IS:</u>

$$\pi_1(X, x_0) \underset{\Phi}{\cong} \frac{\underset{\alpha}{*} \pi_1(A_\alpha, x_0)}{N}.$$

<u>POINT:</u>

<u>SPACES</u>

$$\begin{tikzcd}
 & X & \\
A_\alpha \arrow[ur, "i_\alpha"] & & A_\beta \arrow[ul, "i_\beta"'] \\
 & B_{\alpha\beta} \arrow[ul, "i_{\alpha\beta}"] \arrow[ur, "i_{\beta\alpha}"'] &
\end{tikzcd}$$

<u>GROUPS</u>

$$\begin{tikzcd}
 & \pi_1(X) & \\
 & \pi_1(A_\alpha) * \pi_1(A_\beta) \arrow[u, "\Phi"] & \\
\pi_1(A_\alpha) \arrow[uur, bend left, "(i_\alpha)_*"] \arrow[ur] & & \pi_1(A_\beta) \arrow[uul, bend right, "(i_\beta)_*"'] \arrow[ul] \\
 & \pi_1(B_{\alpha\beta}) \arrow[ul, "(i_{\alpha\beta})_*"] \arrow[ur] &
\end{tikzcd}$$

SO IF $w \in \pi_1(B_{\alpha\beta})$ IT APPEARS TWICE IN $\pi_1(A_\alpha) * \pi_1(A_\beta)$

ONCE AS $(i_{\alpha\beta})_*(w)$ AND ONCE AS $(i_{\beta\alpha})_*(w)$

SO $\underline{\Phi}\left(\left((i_{\alpha\beta})_*(w)\right)\cdot\left((i_{\beta\alpha})_*(w^{-1})\right)\right) = [e]$. $\Big\{$ SO ELEMENTS of $\pi_1(B_{\alpha\beta})$ LEAD TO NONUNIQ. of FACTORISATION

THUS $N < \mathrm{KER}(\Phi)$. THE OTHER INCLUSION IS WHERE THE WORK IS.

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② REDUCTIONS, EXPANSIONS, EXCHANGES.

SUPPOSE $[f] \in \pi_1(X, x_0)$ FACTORS AS $[f_1][f_2]\cdots[f_n]$

SUPPOSE $[f_i][f_{i+1}] \in \pi_1(A_\alpha, x_0)$.

THEN $\left\{\begin{array}{c} [f_1]\cdots[f_i][f_{i+1}]\cdots[f_n] \\ \text{EXPAND} \uparrow\ \downarrow \text{REDUCE} \\ [f_1]\cdots[f_i * f_{i+1}]\cdots[f_n] \end{array}\right.$

SUPPOSE $w \in \pi_1(B_{\alpha\beta}, x_0)$ AND $[f_i] = (i_{\alpha\beta})_*(w)$, $[g_i] = (i_{\beta\alpha})_*(w)$

THEN $\left\{\begin{array}{c} [f_1]\cdots[f_i]\cdots[f_n] \\ \downarrow\uparrow \text{ EXCHANGE} \\ [f_1]\cdots[g_i]\cdots[f_n] \end{array}\right.$

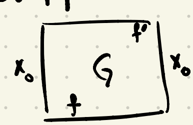
CLAIM: SUPPOSE $B_{\alpha\beta\gamma}$ IS PATH CONN FOR ALL $\alpha, \beta, \gamma$.
THEN ANY TWO FACTORISATIONS of ANY $[f] \in \pi_1(X, x_0)$ ARE CONNECTED BY SOME SEQ of EXPANSIONS, REDUCTIONS, AND EXCHANGES. [EXERCISE: CLAIM PROVES ②]

PROOF: FIX $f \in \mathrm{LOOPS}(X, x_0)$. SUPPOSE

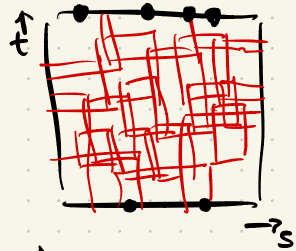
$$f \underset{\partial}{\simeq} f_1 * f_2 * \cdots * f_k$$
$$f \underset{\partial}{\simeq} f_1' * f_2' * \cdots * f_\ell'$$

IS A PAIR of FACTORISATIONS of $f$. FIX HOMOTOPY

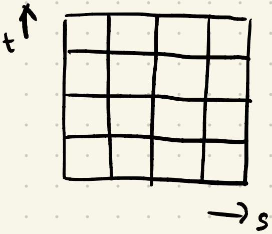
$G: I \times I \longrightarrow X$ WITH $g_0 = f_1 * \cdots * f_k$, $g_1 = f_1' * \cdots * f_\ell'$,



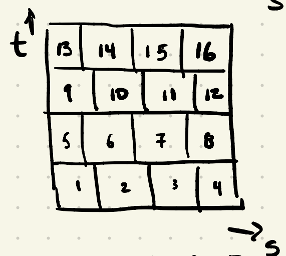
FOR ALL $(s,t)\in I\times I$ THERE IS SOME RECTANGLE $R$ ABOUT $(s,t)$ SO THAT $G(R)\subset A_\alpha$ FOR SOME $\alpha$. FIX FINITE COVER of $I\times I$ BY SUCH RECTANGLES. PICTURE

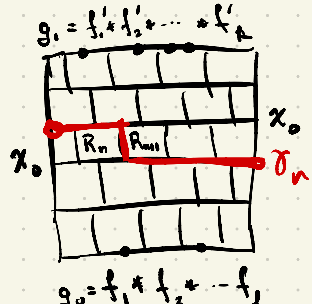



EXTEND ALL EDGES TO GET A TILING of $I\times I$ BY RECTANGLES



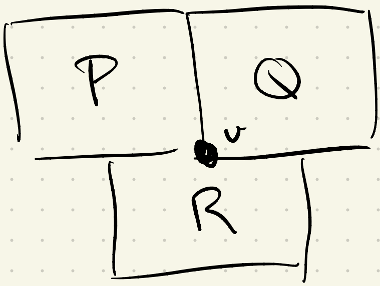

FINALLY SHIFT EVERY OTHER ROW JUST A BIT TO GET RID of ALL VERTICES of VALENCE FOUR. NUMBER THE RESULTING RECTANGLES FROM LEFT TO RIGHT ACROSS ROWS, FROM BOTTOM TO TOP. LET $\gamma_n$ BE THE PATH FROM LEFT TO RIGHT SEPERATING $\bigcup_{i\le n} R_i$ FROM $\bigcup_{i\ge n+1} R_i$




SO: $G|\gamma_n$ LIES IN $\mathrm{LOOPS}(X,x_0)$.

SUPPOSE $v$ IS A VERTEX of THE TILING.

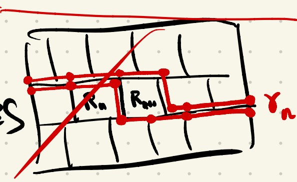



SO THERE ARE $A_\alpha, A_\beta, A_\gamma$ WITH $G(P)\subset A_\alpha$, $G(Q)\subset A_\beta$, $G(R)\subset A_\gamma$.

SO $G(v)\in B_{\alpha\beta\gamma}$, WHICH IS PATH CONN.

PICK $g_v : I \to B_{\alpha\beta\gamma}$ FROM $x_0$ TO $G(v)$.

WE FACTORISE $[G|\gamma_n]$ BY BREAKING INTO EDGES BELOW OR LEFT AND INSERTING $\bar{g}_v * g_v$ FOR ALL VERTICES




~~CLAIM: REDUCTIONS, EXPANSIONS, AND EXCHANGES SUFFICE TO CONNECT THE FACTORISATION~~

of $[G|\gamma_n]$ TO THAT of $[G|\gamma_{n+1}]$.

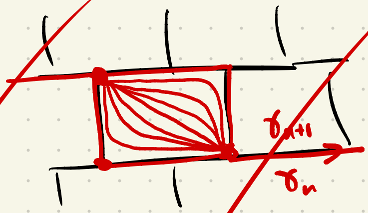
STEP 1: EXPAND TO ENSURE LOWER VERTICES LIE IN $\gamma_n$.

STEP 2: EXCHANGE TO ENSURE EDGES of $\gamma_n - \gamma_{n+1}$ LIE IN $\pi_1(A_\alpha, x_0)$ $[G(R_{n+1}) \subset A_\alpha]$

STEP 3: REDUCE ALONG BOUNDARY of $R_{n+1}$

STEP 4: HOMOTOPE ACROSS $R_{n+1}$ [LEAVES FACTORISATION UNCHANGED]

PICTURE




NOW REVERSE STEPS 3, 2, 1 TO OBTAIN FACTOR. of $\gamma_{n+1}$

FINALLY: $\left\{\begin{matrix}[g_0] \\ [g_1]\end{matrix}\right\}$ CONNECTS TO $\left\{\begin{matrix}[G|\gamma_0] \\ [G|\gamma_N]\end{matrix}\right\}$ RESPECTIVELY, USING THE MOVES AS ABOVE.

PICTURE