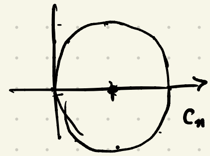
2024-11-12 LECTURE 20 MA 3F1 SAUL SCHLEIMER.

① EARRING SPACE.

WE WOULD LIKE $\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y)$.

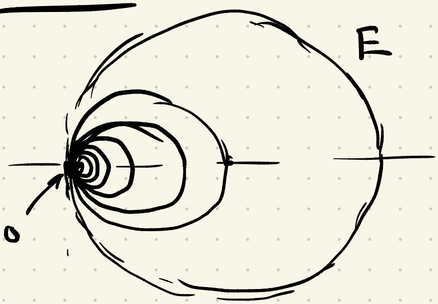
THIS IS NOT QUITE TRUE...

EXAMPLE: $C_n = \{ x \in \mathbb{R}^2 \mid \|x - \tfrac{1}{n}\| = \tfrac{1}{n} \}$



DEFINE $E = \bigcup_n C_n \subset \mathbb{R}^2$.

PICTURE:



WE USE $\vec{0}$ AS THE BASE POINT.

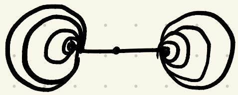
EXERCISES

(i) $E \vee E \cong E$.

(ii) $\pi_1(E)$ IS UNCOUNTABLE.

(iii) $\pi_1(E \vee E) \not\cong \pi_1(E) * \pi_1(E)$.

THE LAST IS VERY DIFFICULT!

HINT: CONSIDER $X =$  WITH TWO POINTS WHERE IT IS NOT "SEMI-LOCALLY SIMP. CONN." SO APPLY THM 1.2 of [EDA 2002]

② SEIFERT — VAN KAMPEN

THEOREM 1.20 SUPPOSE (X, x_0) IS PATH CONNECTED. SUPPOSE $\{A_\alpha\}_\alpha$ IS AN OPEN COVER SO THAT FOR ALL α

(i) $x_0 \in A_\alpha$

(ii) A_α IS PATH CONNECTED.

DEFINE $\iota_\alpha : A_\alpha \hookrightarrow X$.

DEFINE $\Phi = \underset{\alpha}{*} (\iota_\alpha)_* : \underset{\alpha}{*} \pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0)$

$\boxed{\text{(1) SUPPOSE } A_\alpha \cap A_\beta \text{ IS PATH-CONN FOR ALL } \alpha, \beta. \text{ THEN } \Phi \text{ IS SURJECTIVE.}}$

DEFINE $B_{\alpha\beta} = A_\alpha \cap A_\beta$, AND $\iota_{\alpha\beta} : B_{\alpha\beta} \longrightarrow A_\alpha$

DEFINE $B_{\alpha\beta\gamma} = A_\alpha \cap A_\beta \cap A_\gamma$

DEFINE

$$U = \left\{ (\iota_{\alpha\beta})_*(\omega) \cdot (\iota_{\beta\alpha})_*(\omega)^{-1} \;\middle|\; \begin{array}{c} \text{FOR ALL } \alpha, \beta, \omega \text{ WITH} \\ \omega \in \pi_1(B_{\alpha\beta}, x_0) \end{array} \right\}$$

DEFINE $N = \langle\langle U \rangle\rangle$ THE NORMAL CLOSURE of U

$\boxed{\text{(2) SUPPOSE } B_{\alpha\beta\gamma} \text{ IS PATH CONN FOR ALL } \alpha, \beta, \gamma. \text{ THEN } \ker(\Phi) = N}$

THAT IS: $\boxed{\pi_1(X, x_0) \cong \underset{\alpha}{*} \pi_1(A_\alpha, x_0) \big/ N}$

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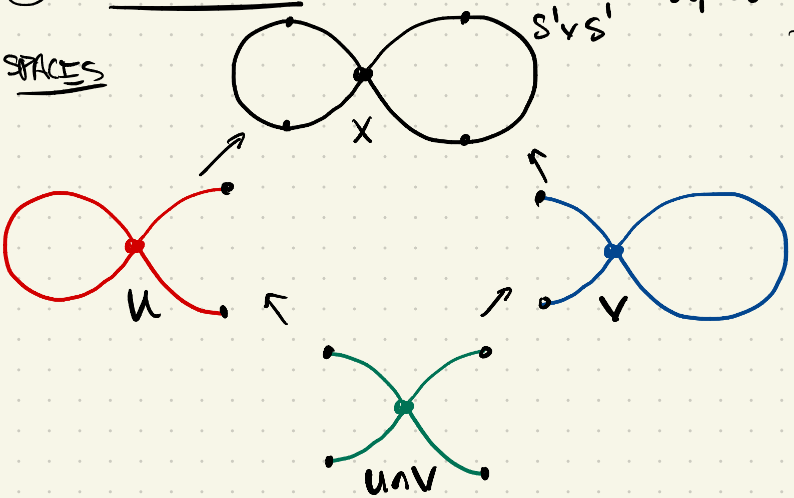
(3) APPLICATION $\qquad \pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$

SPACES



FUND. GROUPS

$$\begin{tikzcd}
 & \mathbb{Z} * \mathbb{Z} & \\
\mathbb{Z} \arrow[ur] & & \mathbb{Z} \arrow[ul] \\
 & 1 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

④ **PROOF of ①:** FIX $[f] \in \pi_1(X, x_0)$

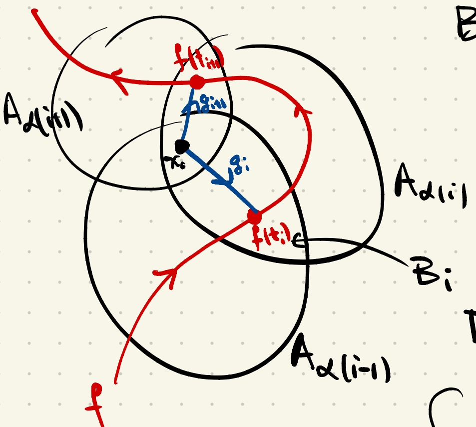
A FACTORISATION of $[f]$ IS A LIST $[[f_1], [f_2], \dots [f_n]]$ SO THAT FOR ALL i THERE IS A $\alpha(i)$ WITH $[f_i] \in \pi_1(A_{\alpha(i)}, x_0)$ AND $f \overset{\wedge}{\underset{\partial}{\simeq}} f_1 * f_2 * \dots * f_n$ [CONCAT. of PATHS].

THAT IS: $[f]$ FACTORS IFF $[f] \in \text{IMAGE}(\Phi)$.

CLAIM: EVERY $[f] \in \pi_1(X, x_0)$ FACTORS.

PROOF: $\{ f^{-1}(A_\alpha) \}_\alpha$ IS AN OPEN COVER of I.

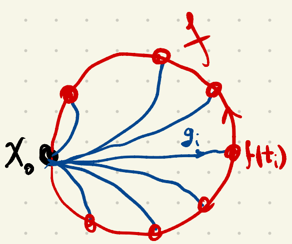
LEBBSGUE: THERE IS A PARTITION $0 = t_0 < t_1 < \dots < t_n = 1$ of $I = [0,1]$ SO THAT FOR ALL i THERE IS $\alpha(i)$ WITH $f([t_i, t_{i+1}]) \subset A_{\alpha(i)}$. SET $B_i := A_{\alpha(i-1)} \cap A_{\alpha(i)}$



BY HYPOTHESES, $x_0 \in B_i$, $f(t_i) \in B_i$, AND B_i PATH-CONN

PICK $g_i : I \longrightarrow B_i$

$g_i(0) = x_0$, $g_i(1) = f(t_i)$



DEFINE:

$$\begin{cases} f_0 = f|[t_0, t_1] * \overline{g_1} \\ f_i = g_i * f|[t_i, t_{i+1}] * \overline{g_{i+1}} \\ f_n = g_{n-1} * f|[t_{n-1}, t_n] \end{cases}$$

SO $f \overset{\partial}{\simeq} f_0 * f_1 * f_2 * \dots * f_n$ [g_i ALL CANCELL!] $\square$