LECTURE 18 MA3F1 SAULSCHLEIMER 2024-11-07

① INDEX vs DEGREE

PROP 1.32 SUPPOSE $p:(\hat{X},\hat{x}_0) \to (X,x_0)$ IS A PATH CONNECTED COVER. THEN

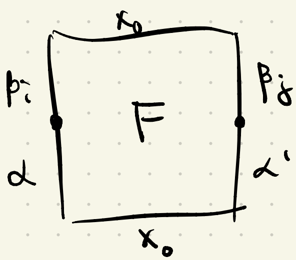
$$\deg(p) = [\pi_1(X,x_0) : p_*(\pi_1(\hat{X},\hat{x}_0))]$$

PROOF: LET $\{\hat{x}_i\}_i = p^{-1}(x_0)$. PICK [AXIOM of CHOICE!] PATHS $\hat{\beta}_i : I \to \hat{X}$ FROM $\hat{x}_0$ TO $\hat{x}_i$. DEFINE $\beta_i = p \circ \hat{\beta}_i$. NOTE $\beta_i \in \mathrm{LOOPS}(X,x_0)$.

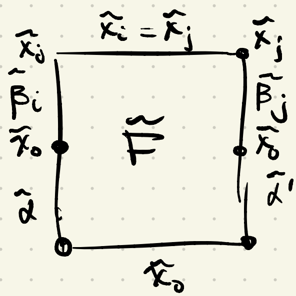
SET $G = \pi_1(X,x_0)$, $H = p_*(\pi_1(\hat{X},\hat{x}_0))$. NOTE $[\beta_0] \in H$.

CLAIM ①: $H\cdot[\beta_i] = H\cdot[\beta_j]$ IFF $i=j$.

PROOF: SUPPOSE $H[\beta_i] = H[\beta_j]$. SO THERE ARE $\tilde{\alpha}, \tilde{\alpha}' \in \mathrm{LOOPS}(\hat{X},\hat{x}_0)$ SO THAT, TAKING $\alpha = p\circ\tilde{\alpha}$, $\alpha' = p\circ\tilde{\alpha}'$, WE HAVE $\alpha * \beta_i \underset{\partial}{\simeq} \alpha' * \beta_j$ VIA F.



WE APPLY 1.30 TO OBTAIN



SO $\hat{x}_i = \hat{x}_j$ BY UNIQUENESS of LIFTING. SO $i=j$ ①

SKIP

CLAIM ② FOR ANY $[\alpha] \in G$, THERE IS i SO THAT $[\alpha] \in H[\beta_i]$.

PROOF: LIFT α TO $\tilde{\alpha}: I \to \hat{X}$ WITH $\tilde{\alpha}(0) = \hat{x}_0$. SO $\tilde{\alpha}(1) \in p^{-1}(x_0)$. SO THERE IS SOME i WITH $\tilde{\alpha}(1) = \hat{x}_i$. LET $\tilde{\gamma}$ BE THE REVERSE of $\hat{\beta}_i$.

SKIP

SO $\hat{\alpha} * \tilde{\gamma} \in \mathrm{LOOPS}(\tilde{X}, \tilde{x}_0)$. SO $[p \circ (\hat{\alpha} * \tilde{\gamma})] \in H$

THAT IS $[\alpha][\beta_i]^{-1} \in H$ AND SO $[\alpha] \in H[\beta_i]$ ②

SO: $\left.\begin{array}{ccc} p^{-1}(x_0) & \longrightarrow & H\backslash G \\ \tilde{x}_i & \longmapsto & H[\beta_i] \end{array}\right\}$ IS A BIJECTION. $\square$

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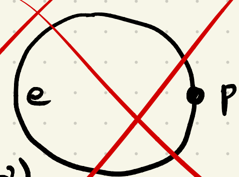
② OUR FRIEND THE TORUS

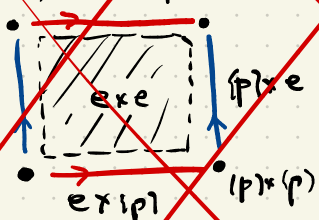
RECALL $\mathbb{T}^2 = S^1 \times S^1$. WE WRITE S^1 AS A UNION OF A POINT p AND AN EDGE e

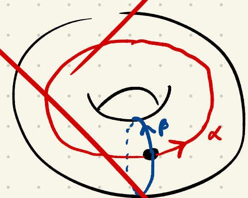


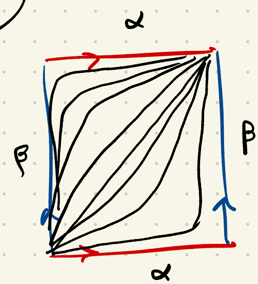
SO $\mathbb{T}^2 = S^1 \times S^1 = (e \cup \{p\}) \times (e \cup \{p\})$

$= (e \times e) \cup (e \times \{p\}) \cup (\{p\} \times e) \cup (\{p\} \times \{p\})$

SO GET







SO WE HAVE A HOMOTOPY: $\alpha * \beta \simeq_\partial \beta * \alpha$

SO $[\alpha], [\beta] \in \pi_1(\mathbb{T}^2)$ AND $[\alpha][\beta] = [\beta][\alpha]$

IN FACT: $\pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2$ GENERATED BY $[\alpha], [\beta]$.

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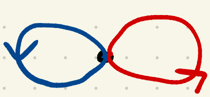
③ POINTED UNION

: SUPPOSE $\{(X_\alpha, x_\alpha)\}_\alpha$ IS A COLLECTION of POINTED SPACES. WE DEFINE THE ONE-POINT UNION

$$\bigvee_\alpha (X_\alpha, x_\alpha) = \left(\frac{\bigsqcup X_\alpha}{x_\alpha \sim x_\beta}, [x_\alpha] \right).$$

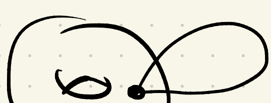
WHEN THERE ARE JUST TWO SPACES WRITE $X \vee Y$.

PICTURE:



$S^1 \vee S^1$.

PICTURE



$\mathbb{T}^2 \vee S^1$

GOAL: UNDERSTAND $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} = F_2$

THE FREE GROUP of RANK TWO

④ FREE PRODUCTS: SUPPOSE $\{G_\alpha\}_\alpha$ IS A COLLECTION of GROUPS. WE MUST DEFINE $\ast_\alpha G_\alpha$ THE FREE PRODUCT of THE G_α [JUST WRITE $G*H$ IF ONLY TWO GPS]

DEF: A WORD w OVER $\{G_\alpha\}_\alpha$ IS A FINITE LIST $w=[g_1, g_2, \cdots, g_n]$ WHERE FOR ALL i THERE IS $\alpha(i)$ SO THAT $g_i \in G_{\alpha(i)}$

WE CALL $n=|w|$ THE LENGTH of w.

THE TRIVIAL WORD [DENOTED $\varepsilon = [\ ]$] IS THE UNIQUE WORD of LENGTH ZERO. [ALSO CALLED EMPTY WORD]

SUPPOSE $u=[g_1, \cdots, g_m]$ $v=[h_1, \cdots, h_n]$ ARE WORDS. DEFINE $u*v = [g_1, \cdots g_m, h_1, \cdots, h_n]$ THE CONCATENATION of THE WORDS.

EXERCISE: IF u, v, w ARE WORDS THEN

(i) $(u*v)*w = u*(v*w)$

(ii) $u*\varepsilon = \varepsilon * u = u$

(iii) $|u*v| = |u|+|v|$

HOWEVER: $u*v \neq v*u$ (GENERALLY) AND WE DON'T HAVE INVERSES...

④ REDUCTIONS

[DEFINE, FOR $u=[g_1, \cdots g_n]$, $\bar{u} = [g_n^{-1}, \cdots, g_1^{-1}]$ THIS "WANTS TO BE u^{-1}"]

DEF: SUPPOSE $u=[g_1, \cdots, g_n]$ IS A WORD OVER $\{G_\alpha\}_\alpha$ SUPPOSE $g_i \in G_{\alpha(i)}$ FOR $i=1, \cdots, n$. WE SAY u IS REDUCED IF

(i) $g_i \neq 1_{\alpha(i)}$ [IDENTITY IN $G_{\alpha(i)}$]

(ii) $\alpha(i) \neq \alpha(i+1)$

REDUCTION: IF $u = [g_1, \ldots, g_n]$ IS NOT REDUCED THEN EITHER

(i) SOME $g_i = 1_{\alpha(i)}$: THEN DEFINE

$$u' = [g_1 \cdots g_{i-1}, g_{i+1} \cdots g_n] \quad \text{OR}$$

(ii) SOME $\alpha(i) = \alpha(i+1)$: THE DEFINE

$$u' = [g_1, g_2, \ldots, g_i \cdot g_{i+1}, \cdots g_n]$$

NOTE $|u'| = |u| - 1$.

EXAMPLE: $G \cong \mathbb{Z} = \langle a \rangle$, $H \cong \mathbb{Z} = \langle b \rangle$

USE $\bar{a} = a^{-1}$, $\bar{b} = b^{-1}$. WORD $u = [a, \bar{a}, b, \bar{b}, 1_a, a]$

$$\begin{aligned} [a, \bar{a}, b, \bar{b}, 1_a, a] &\longrightarrow [a, \bar{a}, b, \bar{b}, a] \\ &\longrightarrow [1_a, b, \bar{b}, a] \\ &\longrightarrow [1_a, 1_b, a] \\ &\longrightarrow [1_b, a] \\ &\longrightarrow [a] \quad \text{REDUCED.} \end{aligned}$$

[GAVE PART of LATTICE of REDUCTIONS.

NOTE: $\varepsilon = [\,]$ IS REDUCED, AS IS $[g_1]$ IF $g_1 \neq 1_{\alpha(1)}$

~~LEMMA: THE RESULT of REDUCTION IS UNIQUE [INDEPENDENT of CHOICES].~~

~~PROOF: NONTRIVIAL INDUCTION. [DISJOINT REDUCTIONS / OVERLAPPING "] $\square$~~

~~WE WILL GIVE A GROUP STRUCTURE TO~~

~~$\ast_i G_\alpha = \{$ REDUCED WORDS OVER $\{G_\alpha\}\}$. NEXT TIME!~~