2024-10-29 LECTURE 14 MA3F1 SAUL SCHLEIMER

① RETRACTS

<u>DEF</u>: SUPPOSE $A \subset X$ IS A SUBSPACE. LET $i_A : A \to X$ BE THE INCLUSION MAP. SUPPOSE $r : X \to A$ HAS $r \circ i_A = Id_A$. THEN WE CALL r A <u>RETRACT</u> of X ONTO A. WE ALSO CALL A a <u>RETRACT</u> of X

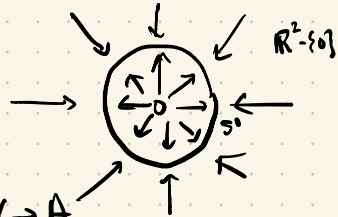
<u>EXAMPLES</u>

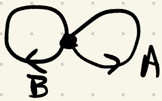
(i) $X \subset X$ IS A RETRACT VIA Id_X

(ii) $\{x_0\} \subset X$ " " " VIA CONSTANT MAP.

(iii) $S^{n-1} \subset \mathbb{R}^n - \{0\}$ IS A RETRACT VIA

$$r : \mathbb{R}^n - \{0\} \longrightarrow S^{n-1}$$
$$x \longmapsto x/|x|.$$



(iv) $X =$ 

CHOOSE $r : X \to A$ BY SENDING B TO ANY LOOP IN A.

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② <u>DEFORMATION RETRACT</u>

SUPPOSE $r : X \to A$ IS A RETRACT. SUPPOSE $i_A \circ r \underset{A}{\simeq} Id_X$: THAT IS THERE IS A HOMOTOPY $F : X \times I \longrightarrow X$ FROM Id_X to $i_A \circ r$ WITH $F(a,t) = a$ FOR ALL $a \in A$, $t \in I$.

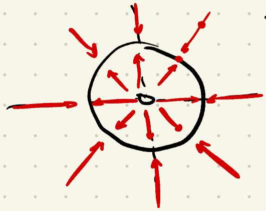
THEN WE CALL $r : X \to A$ a <u>DEFORMATION RETRACT</u>

<u>EXAMPLE</u>: $r : \mathbb{R}^n - \{0\} \longrightarrow S^{n-1}$. WE DEFINE

$$x \longmapsto x/|x|$$

$F : (\mathbb{R}^n - \{0\}) \times I \longrightarrow (\mathbb{R}^n - \{0\})$

$F(x,t) = (1-t)\cdot x + t \cdot x/|x|$



MOVE x ALONG LINE TO ORIGIN.

<u>PROP 1.17</u>: SUPPOSE $r : X \to A$ IS A RETRACT.

FIX $x_0 \in A \subset X$. THEN

(i) $r_* : \pi_1(X, x_0) \longrightarrow \pi_1(A, x_0)$ IS SURJECTIVE

(ii) $(i_A)_* : \pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$ IS INJECTIVE

(iii) IF r IS A DEF RETRACT THEN

$(i_A)_*$, r_* ARE BOTH ISOMORPHISMS.

<u>PROOF</u>: APPLY FUNCTORALITY AND 1.18. $\square$

<u>COROLLARY</u>: $\pi_1(S^1) \cong \pi_1(\mathbb{R}^2 - \{0\})$.

③ <u>NO RETRACT THM</u>

<u>THEOREM</u>: THERE IS NO RETRACT $r : \mathbb{D}^2 \longrightarrow S^1$

<u>PICTURE</u>:

$\mathbb{D}^2 = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$

<u>PROOF</u>: SUPPOSE $r : \mathbb{D}^2 \to S^1$ IS A RETRACT.

SO $r_* : \pi_1(\mathbb{D}^2, 1) \longrightarrow \pi_1(S^1, 1)$ IS SURJECTIVE. (1.17)

BUT $\pi_1(\mathbb{D}^2, 1) \cong \mathbb{1}$ [TRIVIAL GROUP VIA STRAIGHT-LINE HOMOTOPY]

AND $\pi_1(S^1, 1) \cong \mathbb{Z}$ (1.7). SO $\mathbb{1}$ SURJECTS $\mathbb{Z}$ $\divideontimes$ $\square$

③ <u>BROUWER FIXED POINT THEOREM</u>

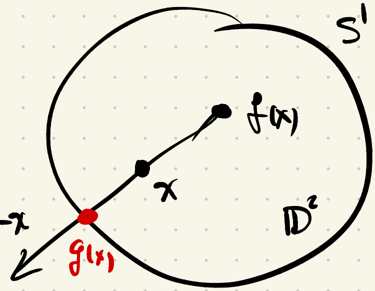
<u>THEOREM 1.19</u>: SUPPOSE $f : \mathbb{D}^2 \to \mathbb{D}^2$ IS CONTINUOUS.

THEN THERE IS SOME $x \in \mathbb{D}^2$ SO THAT $f(x) = x$.

<u>REMARK</u>: THIS IS A "PURE EXISTENCE" RESULT.

WE DO NOT GET AN ALGORITHM TO "FIND" x.

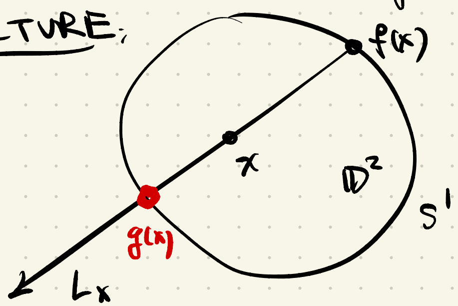
PROOF: SUPPOSE, FOR A CONTRADICTION, THAT $f(x) \neq x$, FOR ALL $x \in \mathbb{D}^2$. LET L_x BE THE RAY STARTING AT $f(x)$, IN THE DIRECTION OF x. **PICTURE**:



DEFINE: $g : \mathbb{D}^2 \to S^1$
BY: $g(x)$ IS THE UNIQUE POINT OF $S^1 \cap (L_x | [x, \infty))$

[SO, NOT ANY POINT OF $[f(x), x] \subset L_x$].

PICTURE:



NOTE THAT g IS CONTINUOUS [EXERCISE] AND IS A RETRACT.

$\divideontimes$ $\square$

THE EXERCISE IS NON-TRIVIAL!

SOLUTION: DEFINE $L_x(t) = (1-t)\cdot f(x) + t \cdot x$. $\boxed{\text{SKIP}}$

SO $|L_x(t)| = 1$ IFF $|L_x(t)|^2 = 1$

IFF $|(1-t) f(x) + t x|^2 = 1$

IFF $|f(x) + t(x - f(x))|^2 = 1$

$$\left(f(x) + t\,(x - f(x))\right)\left(\overline{f(x)} + t\,(\bar{x} - \overline{f(x)})\right) = 1$$

$$\left(|f(z)|^2 - 1\right) + t^2 |x - f(x)|^2 + 2t\,\mathrm{Re}\left(f(x)\cdot \overline{(x - f(x))}\right) = 0$$

SET $A = |x - f(x)|^2$, $B = 2\,\mathrm{Re}\left(f(x)\,\overline{(x - f(x))}\right)$, $C = |f(z)|^2 - 1$

NOTE $A > 0$ AND $0 \geq C$.

WE MUST SOLVE $At^2+Bt+C=0$ FOR $t\in\mathbb{R}$, $\boxed{t\geq 1}$.

SET: $t=\dfrac{-B+\sqrt{B^2-4AC}}{2A}$. WE MUST CHECK THIS HAS A SOLUTION t WITH $t\geq 1$.

① $B^2-4AC\geq 0$ BECAUSE $B^2\geq AC$ BECAUSE $B^2\geq 0\geq AC$.

② $\dfrac{-B+\sqrt{B^2-4AC}}{2A}\geq 1$ BECAUSE $\sqrt{B^2-4AC}\geq B+2A$

BECAUSE (EITHER $B+2A<0$ OR) $B^2-4AC\geq B^2+4AB+4A^2$

BECAUSE $-C\geq A+B$ BECAUSE $1-|f(x)|^2\geq |x-f(x)|^2+2\mathrm{Re}\left(f(x)\overline{(x-f(x))}\right)$

BECAUSE $1\geq |f(x)|^2+2\mathrm{Re}\left(f(x)\overline{(x-f(x))}\right)+|x-f(x)|^2$

BECAUSE $1\geq \left|f(x)+(x-f(x))\right|^2=|x|^2$.

IN PARTICULAR: $t=\dfrac{-B+\sqrt{B^2-4AC}}{2A}$ IS A CONTINUOUS FUNCTION OF $x\in\mathbb{D}^2$.

SKIP

$\square$