2024-10-28 MA3F1 LECTURE 13 SAUL SCHLEIMER.

LAST TIME: IF $f:(X,x_0) \to (Y,y_0)$ IS A HOMEO
THEN $f_*: \pi_1(X,x_0) \to \pi_1(Y,y_0)$ IS A ISOM.

① HOMOTOPY INVARIANCE

EXERCISE: S^1 IS NOT HOMEO TO $\mathbb{C}-\{0\}$

BUT $\pi_1(S^1) \cong \mathbb{Z} \cong \pi_1(\mathbb{C}-\{0\})$.

PROPOSITION: SUPPOSE $f, f': X \to Y$ ARE HOMOTOPIC VIA $F: X\times I \to Y$. FIX $x_0 \in X$. LET $y_0 = f(x_0)$, $y_0' = f'(x_0)$.

LET
$$\left.\begin{aligned} h: I &\longrightarrow Y \\ t &\longmapsto F(x_0,t) \end{aligned}\right\}$$
BE THE TRACK of x_0 in Y

THEN:

$$\begin{tikzcd} & \pi_1(Y,y_0) \arrow[dd, "\beta_h"] \\ \pi_1(X,x_0) \arrow[ur, "f_*"] \arrow[dr, "f'_*"'] \arrow[r, phantom, "\circ"] & \\ & \pi_1(Y,y_0') \end{tikzcd}$$

$$\boxed{\beta_h \circ f_* = f'_*}$$

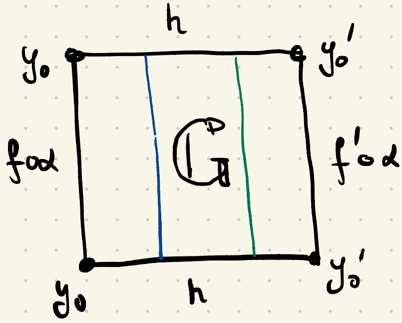
PROOF: LET $[\alpha] \in \pi_1(X,x_0)$ [SO $\alpha \in \mathrm{LOOPS}(X,x_0)$]

NOTE THAT
$$\begin{aligned} (\beta_h \circ f_*)[\alpha] &= \beta_h(f_*[\alpha]) \\ &= \beta_h[f\circ\alpha] \\ &= [\bar{h} * (f\circ\alpha) * h] \end{aligned}$$
AND $f'_*[\alpha] = [f'\circ\alpha]$

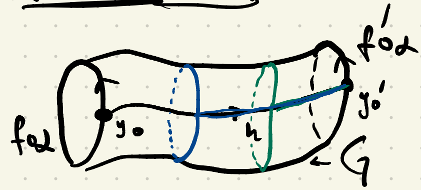
SO WE MUST PROVE
$$\bar{h} * (f\circ\alpha) * h \overset{\partial}{\simeq} f'\circ\alpha$$

DEFINE
$$\left.\begin{aligned} G: I^2 &\longrightarrow Y \\ (s,t) &\longmapsto F(\alpha(s),t) \end{aligned}\right\}$$
SO $g_0 = f\circ\alpha$, $g_1 = f'\circ\alpha$.

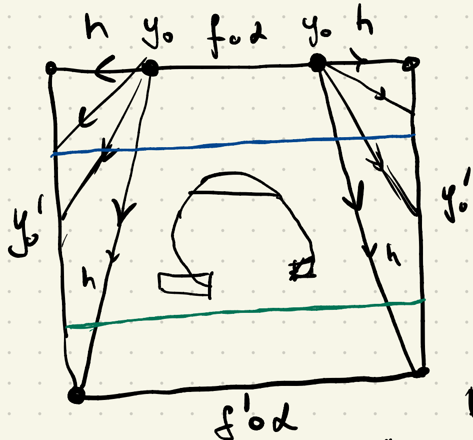
THAT IS



PICTURE IN Y



WE REPARAMETRISE TO GET



SO

$$\bar{h} * (f\circ\alpha) * h \overset{\partial}{\simeq} f'\circ\alpha$$

AS DESIRED. $\square$

PROP 1.18: SUPPOSE $f : X \longrightarrow Y$ IS A HOMOTOPY EQUIV. FIX $x_0 \in X$. SET $y_0 = f(x_0)$.

THEN $f_* : \pi_1(X, x_0) \xrightarrow{\cong} \pi_1(Y, y_0)$

IS AN ISOMORPHISM.

EXAMPLES: $\mathbb{R}^n \simeq \{pt\}$, $\mathbb{R}^n - \{0\} \simeq S^{n-1}$.

PROOF of PROP: LET $g : Y \longrightarrow X$ BE THE GIVEN HOMOTOPY INVERSE. DIAGRAM

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, phantom, "\simeq"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

SO: $Id_X \simeq g\circ f$

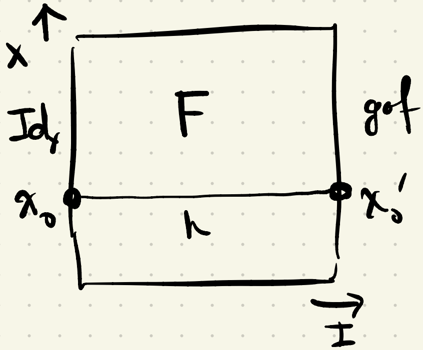
$Id_Y \simeq f\circ g$.

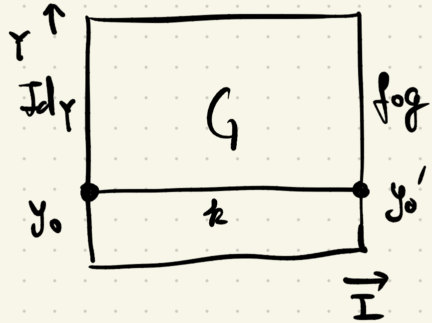
LET $\left.\begin{array}{l} F : X\times I \longrightarrow X \\ G : Y\times I \longrightarrow Y \end{array}\right\}$ BE THE GIVEN HOMOTOPIES.

SET $x_0' = g(y_0) = g(f(x_0))$. SET $y_0' = f(x_0') = f(g(y_0))$

DEFINE
$$h : I \longrightarrow X \qquad\qquad k : I \longrightarrow Y$$
$$t \longmapsto F(x_0, t) \qquad\qquad t \longmapsto G(y_0, t)$$

PICTURES





BY PREVIOUS PROPOSITION

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[d, "\beta_h"'] \arrow[dr, phantom, "\circ"] & \pi_1(Y, y_0) \arrow[d, "\beta_k"] \arrow[dl, "g_*"'] \\
\pi_1(X, x_0') \arrow[r, "f'_*"'] & \pi_1(Y, y_0')
\end{tikzcd}$$

THAT IS:

$g_* \circ f_* = \beta_h$

$f'_* \circ g_* = \beta_k$

ALSO: PROP 1.5 SAYS β_h, β_k ARE ISOMORPHISMS.
SO g_* IS INJ AND SURJ. SO g_* IS AN ISOM.
SINCE $f_* = (g_*)^{-1} \circ \beta_h$ DEDUCE f_* IS AN ISOM. □