2024-10-24 LECTURE 12 MA3F1 SAUL SCHLEIMER

CURRENT GOAL: $\mathbb{B}^2$ HAS THE FIXED POINT PROPERTY. 1.9

WE FIRST DEVELOP A BIT OF THEORY.

① INDUCED HOMOMORPHISMS.

**DEF** SUPPOSE $(X, x_0)$ AND $(Y, y_0)$ ARE POINTED SPACES. SUPPOSE $f: X \to Y$ IS A MAP WITH $f(x_0) = y_0$. DEFINE $f_*: \pi_1(X, x_0) \longrightarrow (Y, y_0)$

BY $[\alpha] \longmapsto [f \circ \alpha]$.

**LEMMA**: ① $f_*$ IS WELL-DEFINED

② $f_*$ IS A HOMOMORPHISM.

**PROOF**: ① SUPPOSE $\alpha \in \mathrm{LOOPS}(X, x_0)$. THEN $f \circ \alpha \in \mathrm{LOOPS}(Y, y_0)$. SUPPOSE $\alpha \overset{\partial}{\simeq} \alpha'$. THEN, AS HOMOTOPIES PUSH FORWARD, $f \circ \alpha \overset{\partial}{\simeq} f \circ \alpha'$, GIVING ①.

② WE COMPUTE AS FOLLOWS.

$$\begin{aligned}
f_*([\alpha][\beta]) &= f_*([\alpha * \beta]) && \text{DEF MULTI.} \\
&= [f \circ (\alpha * \beta)] && \text{DEF } f_* \\
&= [(f \circ \alpha) * (f \circ \beta)] && \text{CONCAT PUSHES FORWARD.} \\
&= [f \circ \alpha][f \circ \beta] && \text{DEF MULTI.} \\
&= f_*([\alpha]) f_*([\beta]) && \text{DEF } f_*.
\end{aligned}$$

②

**EXAMPLE**: $(\mathrm{Id}_X)_* = \mathrm{Id}_{\pi_1(X, x_0)}$

**EXAMPLE**: $p_2: S^1 \longrightarrow S^1$, $z \longmapsto z^2$ } THE SQUARING MAP.

NOTE $p_2(1) = 1$ SO $(p_2)_*: \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, 1)$.

NOTE $(p_2 \circ \omega_k)(t) = p_2(\exp(2\pi i k t))$
$$= \exp(2\pi i\, 2k t)$$
$$= \omega_{2k}(t).$$

DIAGRAM of GROUPS

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\times 2"] \arrow[d, "\Phi"', "\cong"] & \mathbb{Z} \arrow[d, "\Phi"', "\cong"] \\
\pi_1(S^1,1) \arrow[r, "(p_2)_*"] & \pi_1(S^1,1)
\end{tikzcd}
\qquad
\begin{tikzcd}
k \arrow[r, mapsto] \arrow[d, mapsto] & 2k \arrow[d, mapsto] \\
{[\omega_k]} \arrow[r, mapsto] & {[\omega_{2k}]}
\end{tikzcd}$$

THAT IS $(p_2)_*$ IS THE DOUBLING MAP.

EXERCISE: WHAT IS $(p_\ell)_*$ (FOR $\ell \in \mathbb{Z}$)?

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## ② FUNCTORIALITY

LEMMA: SUPPOSE $(X,x_0), (Y,y_0), (Z,z_0)$ ARE POINTED SPACES. SUPPOSE $f: X \to Y$, $g: Y \to Z$ ARE MAPS WITH $f(x_0) = y_0$, $g(y_0) = z_0$. THEN $\boxed{(g\circ f)_* = g_* \circ f_*.}$

DIAGRAM:

$$\begin{tikzcd}
X \arrow[r, "f"'] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

PROOF: SUPPOSE $[\alpha] \in \pi_1(X,x_0)$. WE COMPUTE

$$(g\circ f)_*[\alpha] = [g \circ f \circ \alpha]$$
$$= g_*[f\circ\alpha] = g_*(f_*[\alpha]) = g_* \circ f_* \qquad \square$$

## ③ HOMEOMORPHISM INVARIANCE

**COROLLARY**: SUPPOSE $f:(X,x_0) \to (Y,y_0)$ IS A HOMEO.

THEN $f_*$ IS A GROUP ISOMORPHISM.

**PROOF**: LET $g$ BE THE GIVEN INVERSE.

SO $g \circ f = Id_X$ AND $f \circ g = Id_Y$

SO $g_* \circ f_* = Id_{\pi_1(X,x_0)}$ AND $f_* \circ g_* = Id_{\pi_1(Y,y_0)}$.

SO $g_*, f_*$ ARE INVERSES, THUS ISOMORPHISMS. $\square$

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## ④ HOMOTOPY INVARIANCE.

IN FACT, $\pi_1$ IS "LESS SENSITIVE".

**PROPOSITION**: SUPPOSE $(X,x_0)$ IS POINTED. SUPPOSE

$f, f' : X \to Y$ ARE HOMOTOPIC VIA $F: X \times I \to Y$.

SET $y_0 = f(x_0)$, $y_0' = f'(x_0)$ AND $h: I \to Y$, $t \mapsto F(x_0, t)$

[SO $h$ IS A PATH FROM $y_0$ to $y_0'$].

THEN $\boxed{\beta_h \circ f_* = f'_*}$

**DIAGRAM**:

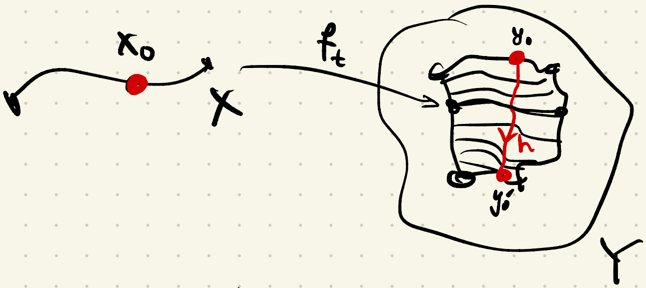
$$\begin{tikzcd}
 & \pi_1(Y,y_0) \arrow[d, "\beta_h"] \\
\pi_1(X,x_0) \arrow[ur, "f_*"] \arrow[dr, "f'_*"'] \arrow[r, phantom, "\circ"] & \pi_1(Y,y_0') \\
 &
\end{tikzcd}$$

IF $h$ IS CONSTANT PATH THEN

$f_* = f'_*$

Done too quickly.

PICTURE:




**PROOF**: FIX $[\alpha] \in \pi_1(X, x_0)$.

so: $\beta_h \circ f_* [\alpha] = \beta_h [f \circ \alpha] = [\bar{h} * (f \circ \alpha) * h]$

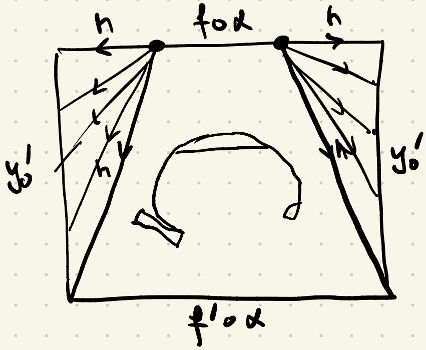
SO IT SUFFICES TO SHOW

$\bar{h} * (f \circ \alpha) * h \overset{\partial}{\simeq} f' \circ \alpha$.

DEFINE $G: I \times I \to Y$ BY

$G(s,t) = F(\alpha(s), t)$.

WE USE THE HOMOTOPY:




$\square$