2024-10-22 MA3F1 LECTURE 11 SAUL SCHLEIMER

① DIAGRAM:

$$\begin{tikzcd} Y\times\{0\} \arrow[r, "g"] \arrow[d, hook] & \tilde{X} \arrow[d, "p"] \\ Y\times I \arrow[r, "F"] \arrow[ur, dashed, "\tilde{F}"] & X \end{tikzcd}$$

WE BUILD $\tilde{F}_N : N\times I \longrightarrow \tilde{X}$ BY RECURSION.

BASE CASE: $\hat{F}_N^0 : N\times\{0\} \longrightarrow \tilde{X}$ IS DEFINED BY

$\hat{F}_N^0(y,0) = g(y)$.

RECURSIVE STEP: SUPPOSE $\hat{F}_N^k$ GIVEN.

NOTE THAT $F(N\times[t_k, t_{k+1}]) \subset U_\alpha$ FOR SOME α

SO THERE IS SOME β WITH $\hat{F}_N^k(y_0,t_k) \in V_\alpha^\beta$.

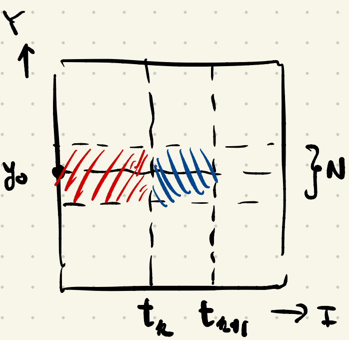
SHRINK N BY REPLACING $N\times\{t_k\}$ BY

$$(N\times\{t_k\}) \cap (\hat{F}_N^k)^{-1}(V_\alpha^\beta).$$

NOW DEFINE

$$\hat{F}_N^{k+1}(y,t) = \begin{cases} \hat{F}_N^k(y,t), & \text{if } t\le t_k \\ g_\alpha^\beta \circ F(y,t), & \text{if } t\ge t_k \end{cases}$$

PICTURE



AND DEFINE $\tilde{F}_N = \hat{F}_N^n$.

THIS PROVES

(A) EXISTENCE of $\hat{F}_N$.

SUPPOSE $\tilde{F}_N'$ IS DEFINED

ON $N\times I$ SOMEHOW

(PERHAPS VIA A DIFF. PARTITION $\{t_k\}$).

WE MUST SHOW $\hat{F}_N = \tilde{F}_N'$. SO FIX $z \in N$.

NOTE $\hat{F}_N(z,0) = g(z) = \hat{F}_N'(z,0)$.

NOW PICK $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m = 1$

SO THAT $F(\{z\} \times [t_k, t_{k+1}]) \subset U_\alpha$ $[\forall k\ \exists \alpha]$

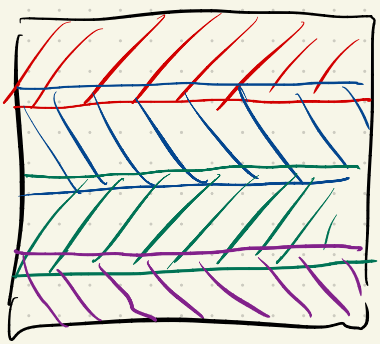
WE <u>INDUCT</u>: $\hat{F}_N|_{z \times [0,t_k]} = \hat{F}_N'|_{\{z\} \times [0,t_k]}$

SUPPOSE
$$\left.\begin{array}{l} \hat{F}_N(\{z\} \times [t_k, t_{k+1}]) \subset V_\alpha^\beta \\ \hat{F}_N'(\{z\} \times [t_k, t_{k+1}]) \subset V_\alpha^{\beta'} \end{array}\right\}$$
BUT $V_\alpha^\beta \cap V_\alpha^{\beta'}$ CONTAINS $\hat{F}_N(z,t_k) = \tilde{F}_N'(z,t_k)$

SO $\beta = \beta'$. THIS PROVES $\hat{F}_N = \hat{F}_N'$

SO WE HAVE (B) UNIQUENESS.

THE PROOF OF (C): $\hat{F}_M|_{(M \cap N) \times I} = \hat{F}_N|_{(M \cap N) \times I}$
FOLLOWS FROM (B)!



FINALLY:
$$\hat{F} = \bigcup_N \tilde{F}_N$$
AND WE ARE DONE.

1.30 1.7 ✓

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② <u>APPLICATION OF 1.7</u>:   DONE, BUT QUICKLY

<u>THEOREM</u> [BROUWER FIXED POINT THM]

SUPPOSE $f: B^n \to B^n$ IS CONTINUOUS.

THEN THERE IS SOME $x \in B^n$ WITH $f(x) = x$.

$n=1$: FOLLOWS FROM INTERM. VALUE THE

$n=2$: FOLLOWS FROM THEOREM 1.7

$n\geq 3$: NEXT MODULE [HOMOLOGY of S^n].

REMARKS:

(i) THE PROOF IS NON-CONSTRUCTIVE!

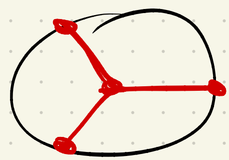
(ii) COMPARE TO THE CONTRACTION MAPPING PRINCIPLE.

(iii) CONTINUITY ESSENTIAL.

EXAMPLES: S^n $(n\geq 0)$ HAS CTS MAPS w/o FIXED POINTS.

$\mathbb{R}^n$, $\mathbb{T}^n$ $(n\geq 1)$ HAS CTS MAPS w/o FIXED POINTS.

EXERCISE: DEFINE Y TO BE THE TRIPOD



$$Y=\left\{z\in\mathbb{C} \;\middle|\; |z|\leq 1,\ \arg(z)=\frac{2\pi k}{3},\ k\in\mathbb{Z}\right\}$$

SHOW Y HAS THE FIXED POINT PROPERTY.

EXERCISE: TRY TO PROVE D^2 HAS THE FIXED POINT PROPERTY.