2024-10-17 MA3F1 LECTURE 9 SAUL SCHLEIMER

AS ALWAYS GOAL IS

THEOREM 1.7: $\pi_1(X, x_0) \cong \mathbb{Z}$ [ISOMORPHIC GROUPS]

---

(1) HOMOTOPY LIFTING:

WE NEED AN "INVERSE" TO "HOMOTOPIES DESCEND"

DEF: SUPPOSE $p: Z \to X$ IS A MAP. WE SAY p HAS THE HOMOTOPY LIFTING PROPERTY if

① GIVEN A HOMOTOPY $F: Y \times I \to X$ AND

② GIVEN A MAP $\hat{g}: Y \times \{0\} \to Z$ WITH $p \circ \hat{g} = f_0$ [A LIFT of f_0]

THERE IS A UNIQUE $\tilde{F}: Y \times I \to Z$ WITH $\hat{f}_0 = \tilde{g}$ AND $p \circ \tilde{F} = F$.

DIAGRAM:

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "\tilde{g}"] \arrow[d, hook] & Z \arrow[d, "p"] \\
Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\tilde{F}"] & X
\end{tikzcd}$$

THINK of F AS A BIT LIKE A DIFF. EQU AND $\hat{g}$ LIKE A BOUNDARY CONDITION. AND $\hat{F}$ LIKE A SOLUTION!

SPECIAL CASE: $Y = \{pt\}$: THUS F IS A PATH. THEN WE OBTAIN PATH LIFTING BY FINDING $\tilde{F}$.

PICTURE:

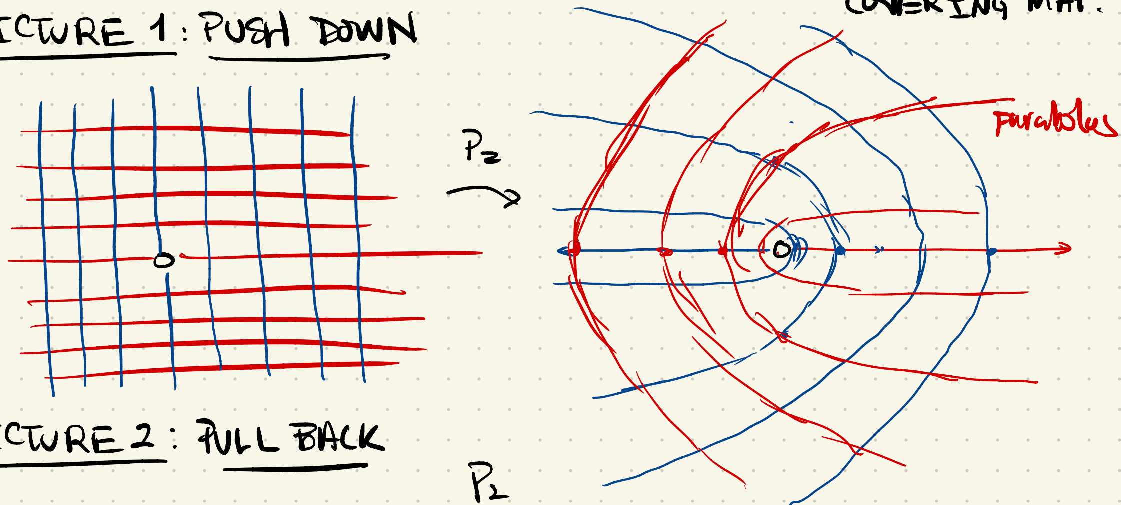
THE LIFT of THE PATH IS DETERMINED BY THE INITIAL POINT (AND THE PATH)

② AND COVERING SPACES.

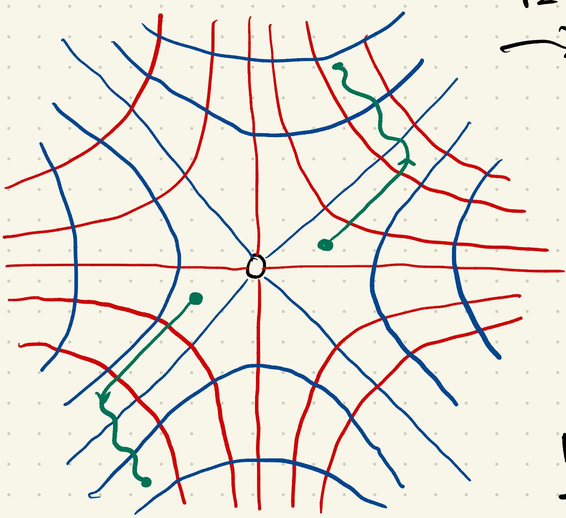
PROP 1.30: COVERING MAPS ($p: \tilde{X} \to X$) HAVE THE HOMOTOPY LIFTING PROPERTY.

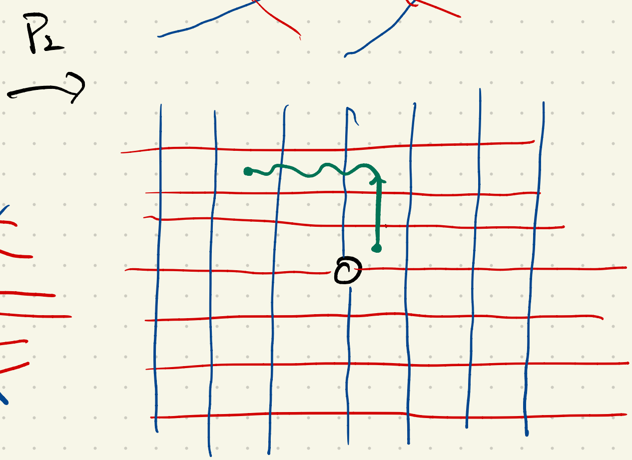
EXAMPLE: DEFINE
$$p: \mathbb{C} - \{0\} \longrightarrow \mathbb{C} - \{0\}$$
$$z \longmapsto z^2$$
} **EXERCISE:** THIS IS A COVERING MAP.

PICTURE 1: PUSH DOWN



PICTURE 2: PULL BACK





EXERCISE: USE COMPUTER TO DRAW PICTURES FOR $p_3(z) = z^3$.

VERY SPECIAL CASE: IF $\alpha: I \to X$ IS CONSTANT PATH THEN SO IS ANY LIFT $\hat{\alpha}: I \to \tilde{X}$. [BETTER PROOF: APPLY UNIQUENESS! □

PROOF: $p \circ \tilde{\alpha}(t) = \alpha(t) = \alpha(0)$. SO $\tilde{\alpha}(t) \in p^{-1}(\alpha(0))$. BUT POINTS of $p^{-1}(\alpha(0))$ ARE SEPARATED BY DISJOINT OPEN SETS AND I IS CONNECTED. $\square$.

(3) THEOREM 1.7: $\Phi : \mathbb{Z} \longrightarrow \pi_1(S^1, 1)$ IS AN ISOMORPHISM

PROOF: Φ IS HOMOMORPHISM BY LAST TIME.

SURJECTIVE: SUPPOSE $[\alpha] \in \pi_1(S^1, 1)$. SO $\alpha : I \longrightarrow S^1$ WITH $\alpha(0) = \alpha(1) = 1$. APPLY (1.30) TO GET $\tilde{\alpha} : I \longrightarrow \mathbb{R}$ WITH $p \circ \tilde{\alpha} = \alpha$ AND $\tilde{\alpha}(0) = 0$.

DIAGRAM:

$$
\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & \mathbb{R} \arrow[d, "p"] \\
I \arrow[r, "\alpha"'] \arrow[ur, dashed, "\tilde{\alpha}"] & S^1
\end{tikzcd}
$$

NOTE: $(p \circ \tilde{\alpha})(1) = p(\tilde{\alpha}(1)) = \alpha(1) = 1$.

SO $\tilde{\alpha}(1) = k$ FOR SOME $k \in \mathbb{Z}$.

APPLY STRAIGHT LINE HOMOTOPY IN $\mathbb{R}$.

$$F(s,t) = (1-t) \cdot \tilde{\alpha}(s) + t \cdot \tilde{\omega}_k(s)$$

TO FIND $\tilde{\alpha} \simeq_{\partial} \tilde{\omega}_k$. SO [HOMOTOPIES DESCEND]

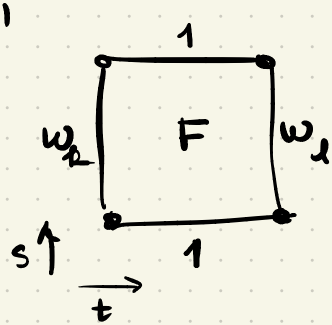
WE HAVE $p \circ \tilde{\alpha} \simeq_{\partial} p \circ \tilde{\omega}_k$

SO $\alpha \simeq_{\partial} \omega_k$

SO $[\alpha] = [\omega_k] = \Phi(k) \in \pi_1(S^1, 1)$ ✓

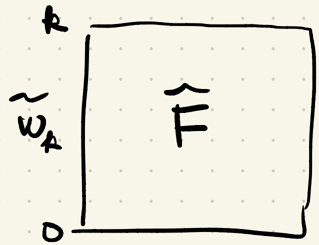
INJECTIVE: SUPPOSE $\Phi(k) = \Phi(l)$. [ALTERNATE PROOF: SUPPOSE $\Phi(k) = \Phi(0)$.

SO $[w_k] = [w_l]$. SO THERE IS A HOMOTOPY $w_k \simeq_{\partial} w_l$, SAY BY $F: I^2 \longrightarrow S^1$

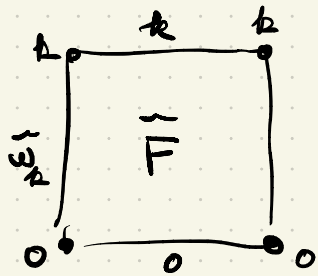


WE LIFT USING THE INITIAL CONDITION $\begin{cases} I \times \{0\} \longrightarrow \mathbb{R} \\ (s,0) \longrightarrow \tilde{w}_k(s) \end{cases}$

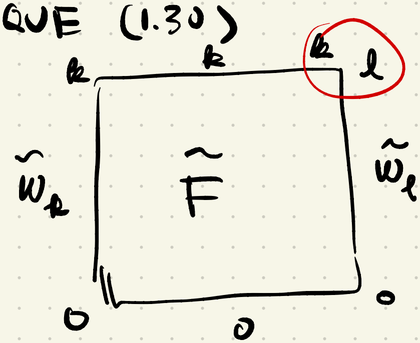
APPLY (1.30) TO GET



CONSTANT PATHS LIFT TO CONSTANT PATHS



PATH LIFTING IS UNIQUE (1.30)



THUS $k = l$.

□.

NEXT WEEK: PROOF of 1.30.

IF $l=0$ THEN INSTEAD USE LEMMA ABOUT LIFTING CONSTANT PATHS.