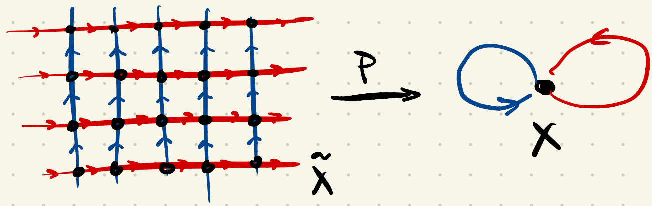
LECTURE 7 MA3F1 SAUL SCHLEIMER 2024-10-14

<u>RECALL</u>: A MAP $p: \tilde{X} \to X$ IS A <u>COVERING MAP</u> IF THERE IS AN OPEN COVER $\mathcal{U} = \{U_\alpha\}$ of X SO THAT FOR ALL α THERE IS A PARTITION $p^{-1}(U_\alpha) = \bigsqcup_\beta V_\alpha^\beta$ SO THAT, FOR ALL β THE MAP $p|V_\alpha^\beta : V_\alpha^\beta \to U_\alpha$ IS A HOMEOMORPHISM.

<u>MORALLY</u>: $\tilde{X}$ LOCALLY IS A COPY of X.

<u>EXAMPLE</u>:



<u>RECALL</u>: COVERS $p: Y \to X$, $q: Z \to X$ ARE <u>ISOMORPHIC</u> IF THERE IS A HOMEO. $h: Y \to Z$ SO THAT

$$p = q \circ h$$

$$\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "p"'] & \circ & Z \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

① <u>DECK GROUPS</u>: SUPPOSE $p: \tilde{X} \to X$ IS A COVER.

<u>DEFINE</u>: $\mathrm{DECK}(p) = \left\{ h: \tilde{X} \to \tilde{X} \;\middle|\; \begin{array}{l} h \text{ IS A HOMEO} \\ \text{AND } p = p \circ h \end{array} \right\}$

THAT IS: THE SET of "AUTOMORPHISMS" (SELF ISOMORPHISMS)

<u>EXERCISE</u>: $(\mathrm{DECK}(p), \circ)$ IS A GROUP.

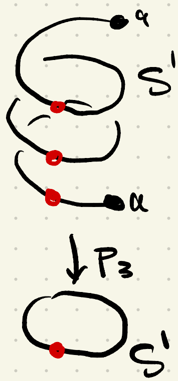
<u>PF</u>: $p \circ \mathrm{Id}_{\tilde{X}} = p$, IF $p = p \circ h$ THEN $p \circ h^{-1} = p$. $\square$.

TERMINOLOGY: CALL ELTS of $\mathrm{DECK}(p)$ DECK TRANSFORMATIONS.

EXAMPLES SUPPOSE $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

(1) LET $p_k : S^1 \longrightarrow S^1$, $z \longmapsto z^k$. THIS IS A COVERING

PICTURE:



EXERCISE

$\mathrm{DECK}(p_k) \cong \mathbb{Z}/k\mathbb{Z}$

[AND NATURALLY ISOM TO $U_k = \{\exp(\frac{2\pi i l}{k}) \mid l \in \mathbb{Z}\}$ THE k^{th} ROOTS of UNITY.]

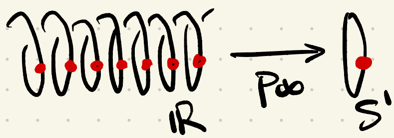
[THINK ABOUT $k=2$]

(2) DEFINE $p_\infty : \mathbb{R} \longrightarrow S^1$, $t \longmapsto \exp(2\pi i t)$

EXERCISE:

$\mathrm{DECK}(p_\infty) \cong \mathbb{Z}$

PICTURE:



(3) THE EXAMPLE AT THE BEGINNING of LECTURE HAS $\mathrm{DECK}(p) \cong \mathbb{Z}^2$.

(4) LIFTING:

SUPPOSE $p: \tilde{X} \longrightarrow X$ IS A COVERING MAP.

SUPPOSE $f: Y \longrightarrow X$ IS ANY MAP. WE CALL A MAP $\tilde{f}: Y \longrightarrow \tilde{X}$ IS A LIFT of f IF $f = p \circ \tilde{f}$.

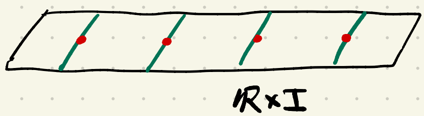
DIAGRAM:

$$\begin{tikzcd} & \hat{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

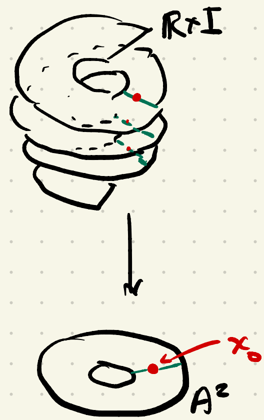
AN EXAMPLE: THE STRIP $\mathbb{R}\times I$ COVERS THE ANNULUS $S^1\times I$ VIA THE MAP $p:\mathbb{R}\times I \longrightarrow S^1\times I$

$$(s,t) \longmapsto (\exp(2\pi i s), t)$$

THINK OF A SPIRAL STAIRCASE (OR A CARPARK, OR A SLINKY ...) PICTURE



$\cong$



AN EXAMPLE OF LIFTING: LET $\omega_2 : I \longrightarrow (A^2, x_0)$ BE GIVEN BY $\omega_2(s) = (\exp(4\pi i s), \frac{1}{2})$