2024-10-10 LECTURE 6 MA3F1 SAUL SCHLEIMER

(1) OUR FRIEND THE CIRCLE.

EXERCISES

① $S^1 \times \mathbb{R} \cong \mathbb{R}^2 - \{(0,0)\}$

② $S^1 \simeq S^1 \times \mathbb{R} \simeq S^1 \times I$.

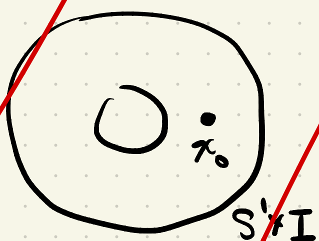
WE PICK $x_0 = (1,0) \in S^1$. WE DEFINE $\omega_n : I \longrightarrow S^1$ BY $\omega_n(t) = (\cos(2\pi n t), \sin(2\pi n t))$.

SO: ① $\omega_0 = e$ IS THE CONSTANT LOOP.

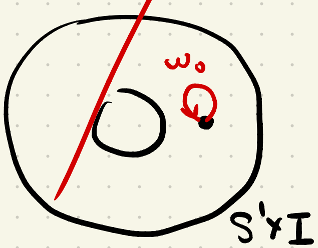
ALSO ② $\omega_{-n} = \overline{\omega_n}$ ARE "REVERSES" OF EACH OTHER.

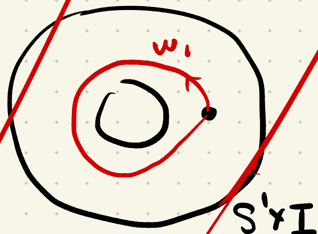
③ $\omega_m * \omega_n \simeq_\partial \omega_{m+n}$ ④ $\omega_n * \omega_n = \omega_{2n}$ [SPECIAL CASE!]

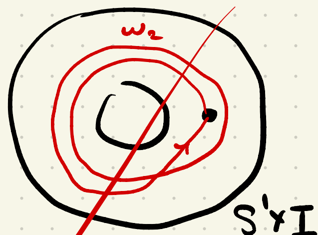
PICTURES



THICKEN S^1 TO GET $S^1 \times I$, SO WE CAN SEE.







SO ω_n "WINDS" n TIMES AROUND S^1.

THEOREM 1.7: $\mathbb{Z} \xrightarrow{\cong} \pi_1(S^1, x_0)$

$n \longmapsto [\omega_n]$

② BASE POINTS:

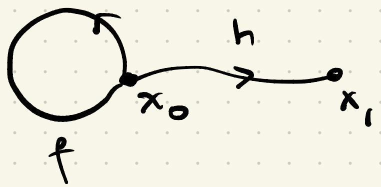
Suppose $h: I \longrightarrow X$ IS A PATH FROM x_0 to x_1

WE DEFINE

$$\beta_h : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$
$$[f] \longmapsto [\bar{h} * f * h]$$

Exercise: Check this is well-defined.

PICTURE.



PROP 1.5: β_h IS AN ISOMORPHISM of GROUPS.

PROOF: β_h IS A HOMOMORPHISM

$$\begin{aligned}
\beta_h([f][g]) &= \beta_h([f * g]) \\
&= [\bar{h} * f * g * h] \\
&= [\bar{h} * f * h * \bar{h} * g * h] \\
&= [\bar{h} * f * h][\bar{h} * g * h] \\
&= \beta_h([f])\, \beta_h([g])
\end{aligned}$$

β_h IS AN ISOMORPHISM BECAUSE

$$\beta_h \circ \beta_{\bar{h}}([f]) = \beta_h([h * f * \bar{h}]) = [\bar{h} * h * f * \bar{h} * h]$$
$$= [f].$$

SO $\beta_h \circ \beta_{\bar{h}} = \mathrm{Id}_{\pi_1(X, x_1)}$ AND SIMILARLY

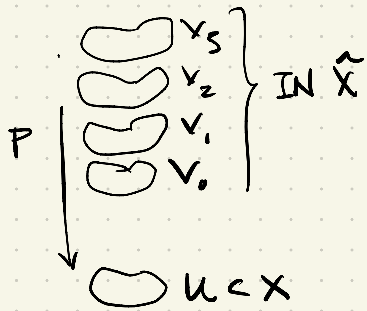
$$\beta_{\bar{h}} \circ \beta_h = \mathrm{Id}_{\pi_1(X, x_0)}$$

(3) COVERING SPACES:

DEF: SUPPOSE X IS A TOP. SPACE. SUPPOSE Ω IS THE GIVEN TOPOLOGY. A SUBSET $\mathcal{U} \subset \Omega$ IS AN OPEN COVERING OF X IF $\bigcup_{U \in \mathcal{U}} U = X$.

DEF: SUPPOSE $p: \tilde{X} \to X$ IS CONTINUOUS. WE SAY p IS A COVERING MAP IF THERE IS AN OPEN COVER $\mathcal{U} = \{U\}$ SO THAT, FOR ALL $U \in \mathcal{U}$, $p^{-1}(U) = \sqcup V_\beta$ IS A DISJOINT UNION of SETS V_β SO THAT $p|V_\beta : V_\beta \to U$ IS A HOMEOMORPHISM.

PICTURE: STACK of PANCAKES



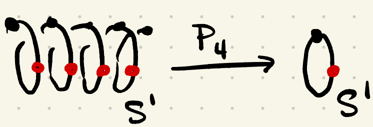
EXAMPLES:

(1) $Id_X : X \to X$ IS A COVERING MAP.

(2) REGARD $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$.

DEFINE
$$\begin{aligned} P_d : S^1 &\longrightarrow S^1 \\ z &\longmapsto z^d. \end{aligned}$$
FOR EXAMPLE IF $d=2$ THIS IS THE SQUARING MAP.

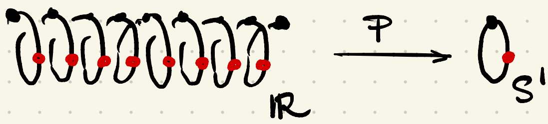
PICTURE:



(3) $p : \mathbb{R} \longrightarrow S^1$

$t \longmapsto \exp(2\pi i t)$

PICTURE



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(4) ISOMORPHISM of COVERING SPACES.

SUPPOSE $p: Y \to X$, $q: Z \to X$ ARE COVERING MAPS. WE SAY THE COVERS ARE ISOMORPHIC IF THERE IS A HOMEOMORPHISM $h: Y \to Z$ SO THAT $\boxed{p = q \circ h}$

$$\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circ"] & Z \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

EXAMPLE: $P_d : S^1 \to S^1$

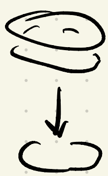
$z \longmapsto z^d$

IS ISOMORPHIC TO P_c iff $c = \pm d$.

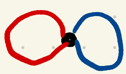
[USE $h = P_{\pm 1}$]

DEF: CALL $p : \tilde{X} \to X$ AN n-FOLD COVERING MAP IF IT IS A COVERING AND, FOR ALL $x \in X$, $|p^{-1}(x)| = n$.

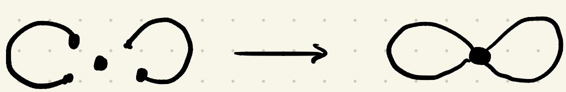
EXAMPLE:



$$\left.\begin{array}{c} S^1 \sqcup S^1 \\ \downarrow \\ S^1 \end{array}\right\}$$ IS A 2-FOLD COVERING (DOUBLE COVER)

DEFINE X =  THE ROSE WITH TWO PETALS.

THIS IS BUILT FROM ONE VERTEX AND TWO INTERVALS BY GLUING THE ENDPOINTS TO THE VERTEX



HERE ARE THE DOUBLE COVERS OF X (UP TO ISOM)

FIND SOME TRIPLE COVERS (UP TO ISOM).

(5) DECK GROUPS SUPPOSE $p : \hat{X} \to X$ IS A COVERING. A HOMEOMORPHISM $\tau : \hat{X} \to \hat{X}$ IS A DECK TRANSFORMATION OF p IF $p \circ \tau = p$.

DIAGRAM

$$\begin{tikzcd} \hat{X} \arrow[rr, "\tau"] \arrow[dr, "p"'] & & \hat{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

EXAMPLE: $\tau : \mathbb{R} \to \mathbb{R}$ DEF BY $\tau(t) = t+1$ IS A DECK TRANSFORM FOR $p : \mathbb{R} \to S^1$
$t \longmapsto \exp(2\pi i t)$

DEF: SUPPOSE $p : \hat{X} \to X$ IS A COVERING

$$\mathrm{DECK}(p) = \{ \tau : \hat{X} \to \hat{X} \mid \tau \text{ IS A DECK TRANSFORM.} \}$$

EXERCISE: $\mathrm{DECK}(p : \mathbb{R} \to S^1) \cong \mathbb{Z}$

EXERCISE: $\mathrm{DECK}(p)$ IS A GROUP.

PROOF: $Id_{\hat{X}} \in \mathrm{DECK}(p)$. [ID]

SUPPOSE $\tau, \sigma, \gamma \in \mathrm{DECK}(p)$. THEN

$p \circ \tau = p$ SO $p = p \circ \tau^{-1}$. [INVERSE]

$p \circ (\sigma \circ \tau) = (p \circ \sigma) \circ \tau = p \circ \tau = p$ [CLOSURE]

$(\sigma \circ \tau) \circ \gamma = \sigma \circ (\tau \circ \gamma)$ [ASSOC]. //