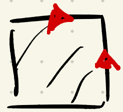
2024-10-07 LECTURE 4 MA3F1 SAUL SCHLEIMER

① OUR FRIEND THE SQUARE

(i) PROVE THAT $I^2 \cong B^2 \cong$ 

(ii) PROVE THAT $M^2 \cong$ 

② HOMOTOPY EQUIVALENT

DEF: SUPPOSE X, Y ARE SPACES. SUPPOSE $f : X \to Y$ AND $g : Y \to X$ ARE MAPS WITH

$$f \circ g \simeq \mathrm{Id}_Y$$

$$g \circ f \simeq \mathrm{Id}_X.$$

THEN WE WRITE $X \simeq Y$ AND SAY X AND Y ARE HOMOTOPY EQUIVALENT [AND WE CALL f, g HOMOTOPY EQUIVALENCES AND ALSO HOMOTOPY INVERSES.]

NOTATIONS: $X \cong Y$ HOMEOMORPHIC

$f : X \xrightarrow{\cong} Y$ HOMEOMORPHISM

$X \simeq Y$ HOMOTOPY EQUIVALENT

$f : X \xrightarrow{\simeq} Y$ HOMOTOPY EQUIVALENCE

KEY EXERCISE

$x \longmapsto x/|x|$

$$\begin{tikzcd} \mathbb{R}^n - \{0\} \arrow[r, bend left] \arrow[r, phantom, "\simeq"] & S^{n-1} \arrow[l, bend left] \end{tikzcd}$$

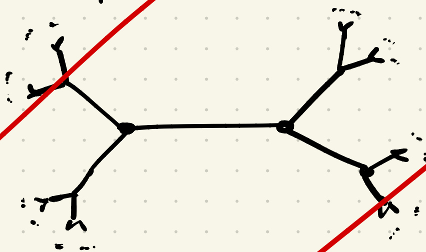
$x \longleftarrow\!\shortmid\ x$

NOTE: $X \cong Y$ IMPLIES $X \simeq Y$.] skipped

③ CONTRACTIBILITY:

SAY A SPACE X IS CONTRACTIBLE IF $X \simeq \{pt\}$.

EXAMPLE: B^n, $\mathbb{R}^n$ ARE CONTRACTIBLE.

EXAMPLE: 

T_3: THE REGULAR THREE-VALENT TREE IS CONTRACTIBLE.

THEOREM: S^n IS NOT CONTRACTIBLE

$n=0$: EXERCISE

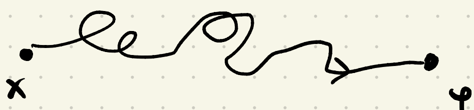
$n=1$: THIS MODULE (MA3F1)

$n \geq 2$: DEGREES (MA3H6)

Skipped

④ PATHS:

DEF: SUPPOSE X IS A SPACE. FIX $x, y \in X$. A PATH IN X FROM x TO y IS A MAP $f: I \longrightarrow X$ WITH $f(0)=x$, $f(1)=y$.



DEF: SUPPOSE f, g ARE PATHS IN X WITH $f(1)=g(0)$. THEN WE DEFINE A PATH $f*g : I \longrightarrow X$ BY $f*g(t) = \begin{cases} f(2t) & t \leq \frac{1}{2} \\ g(2t-1) & t \geq \frac{1}{2} \end{cases}$

WE CALL $f*g$ THE CONCATENATION OF f WITH g.

NOTE: ALL PATHS (IN PATH CONN X) ARE HOMOTOPIC TO CONSTANT PATHS.

GIVEN $f: I \longrightarrow X$ CONSIDER $F: I \times I \longrightarrow X$ DEF BY $F(s,t) = f(s \cdot (1-t))$

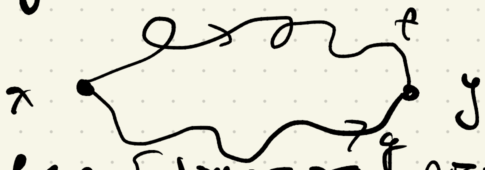
So $f_0 = f$, $f_1 \equiv f(0)$ CONST.

[REELING IN FISHING LINE]

TO OBTAIN A THEORY WE NEED TO IMPOSE CONDITIONS AT THE ENDPOINTS.

DEF: SUPPOSE X IS A SPACE, $x, y \in X$ ARE POINTS. SUPPOSE $f, g: I \longrightarrow X$ ARE PATHS FROM x TO y.

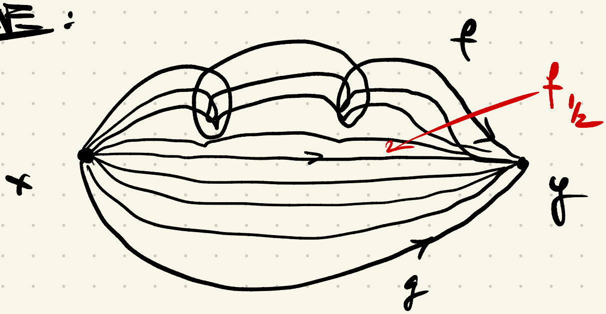
PICTURE:



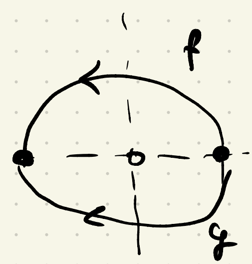
WE SAY $f \simeq_{\partial} g$ [HOMOTOPIC REL ENDPOINTS]

IF THERE IS A HOMOTOPY $F: I \times I \longrightarrow X$ WITH $f = f_0$, $g = f_1$ AND $f_t(0) = x$, $f_t(1) = y$ FOR ALL $t \in I$.

PICTURE:



NON-EXAMPLE $f, g: I \longrightarrow \mathbb{R}^2 - \{0\}$



$f(t) = (\cos(\pi t), \sin(\pi t))$

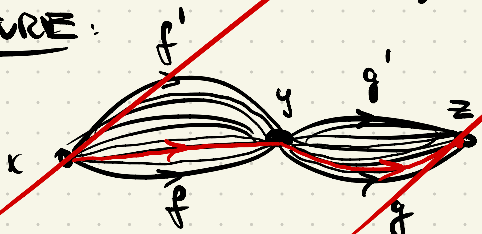
$g(t) = (\cos(\pi t), -\sin(\pi t))$

NOTE $f \simeq g$ BUT NOT $f \underset{\partial}{\simeq} g$. [TRY TO PROVE THIS!]

EXERCISE: $f \underset{\partial}{\simeq} g$ IS AN EQU. REL. [PROP 1.2]

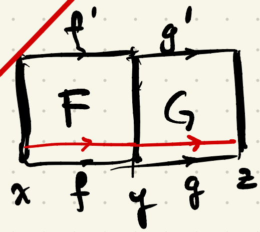
EXERCISE: SUPPOSE $\left.\begin{array}{l} f(1) = g(0). \\ f \underset{\partial}{\simeq} f', g \underset{\partial}{\simeq} g'. \end{array}\right\}$ THEN $f * g \underset{\partial}{\simeq} f' * g'$

PICTURE:



PROOF: USE GLUEING LEMMA.

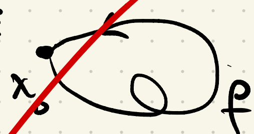
DIAGRAM:



(5) LOOPS

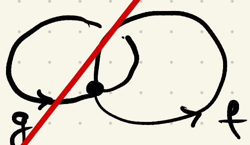
DEF: SUPPOSE X IS A SPACE. SUPPOSE $x_0 \in X$ IS A POINT: CALL x_0 THE BASEPOINT.
CALL THE PAIR (X, x_0) A POINTED SPACE.

DEF: A PATH $f : I \longrightarrow X$ WITH $f(0) = f(1) = x_0$ IS A LOOP BASED AT x_0. PICTURE



NOTE: IF f, g ARE LOOPS BASED AT x_0 THEN SO IS $f * g$!

PICTURE:

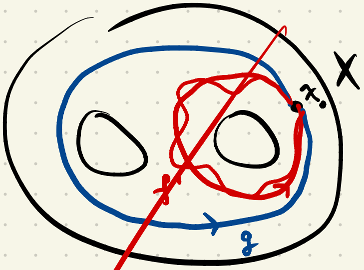


SAY IN $\mathbb{R}^3$

$f * g$: DO f THEN g

FIX (X, x_0). IF $f : I \longrightarrow X$ IS A LOOP BASED at x_0
DEFINE $[f] = \left\{ g : I \longrightarrow X \;\middle|\; \begin{array}{l} g \text{ LOOP AT } x_0 \\ f \underset{\partial}{\simeq} g \end{array} \right\}$

THAT IS: THE $\underset{\partial}{\simeq}$ EQUIV CLASS OF f.



MORALLY: IF $f \simeq_\partial f'$ THEN CAN "WIGGLE" TO TURN f INTO f'. BUT CANNOT "JUMP OVER HOLES"

DEFINE: THE SET

$$\pi_1(X, x_0) = \left\{ [f] \;\middle|\; \begin{array}{c} f : I \to X \\ \text{LOOP BASED} \\ \text{AT } x_0 \end{array} \right\}$$

DEFINE: THE PRODUCT $[f]\cdot[g] = [f * g]$

[EXERCISE: THIS IS WELL-DEFINED.].

PROP 1.3: $(\pi_1(X, x_0), \cdot)$ IS A GROUP.

[THE HARDEST PART IS ASSOCIATIVITY!]