MA3F1 SAUL SCHLEIMER 2024-09-30

LECTURE 1: RE- AND OVERVIEW.

① TOPOLOGICAL SPACES

DEFINITION: SUPPOSE X IS A SET. A TOPOLOGY ON X IS A COLLECTION OF SUBSETS $\Omega \subset \mathcal{P}(X)$ SO THAT

(i) $\emptyset, X \in \Omega$

(ii) If $U, V \in \Omega$ THEN $U \cap V \in \Omega$

(iii) If $\{U_i\} \subset \Omega$ THEN $\bigcup_i U_i \in \Omega$

CALL $U \in \Omega$ AN OPEN SET of (X, Ω).

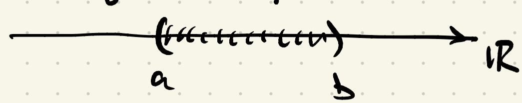
DEF: A SUBSET $B \subset \Omega$ IS A BASIS for (X, Ω) if

(*) for all $U \in \Omega$, for all $x \in U$, THERE IS SOME $V \in B$ SO THAT $x \in V \subset U$.

THAT IS: EVERY $U \in \Omega$ IS A UNION of SETS FROM B

EXAMPLE: $\{0\}$ THE SPACE WITH ONE POINT. IT HAS ONLY TWO OPEN SETS. [ALSO CALLED $\mathbb{R}^0$].

EXAMPLE: $\mathbb{R}$ WITH BASIS $B = \{ (a,b) \mid a < b \}$ THE "OPEN" INTERVALS.



WE USUALLY SURPRESS THE NOTATION Ω.

② NEW SPACES FOR OLD: PRODUCTS.

DEF: SUPPOSE X, Y ARE SPACES. THEN WE GIVE $X \times Y$ THE PRODUCT TOPOLOGY WITH BASIS

$$B = \{ U \times V \mid U, V \text{ OPEN IN } X, Y \}$$

NOTE: $X \times Y$ IS USED **BOTH** FOR THE <u>SPACE</u> AND THE UNDERLYING SET. WHAT A SYMBOL "MEANS" DEPENDS ON ITS <u>CONTEXT</u>.

<u>EXAMPLE</u>: $\mathbb{R}^0 = \{pt\}$
$$\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$$

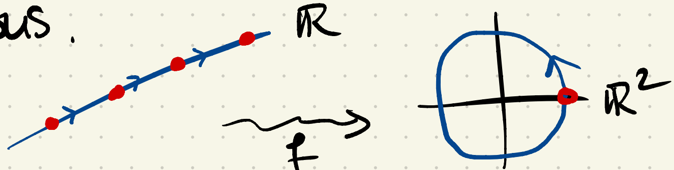
③ <u>SAMENESS</u>:

<u>DEF</u>: SUPPOSE X, Y ARE SPACES. A FUNCTION $f : X \longrightarrow Y$ IS <u>CONTINUOUS</u> IF $f^{-1}(V)$ IS OPEN FOR ALL $V \subset Y$ OPEN.

$[\, f^{-1}(V) = \{ x \in X \mid f(x) \in V \} \,]$.

<u>EXAMPLES</u>. (i) $Id_X : X \longrightarrow X$ IS CONTINUOUS.

(ii) $f : \mathbb{R}^1 \longrightarrow \mathbb{R}^2$, $f(t) = (\cos(t), \sin(t))$ IS CONTINUOUS.



(iii) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(t) = \begin{cases} 0 & \text{if } t = 0 \\ t/|t| & \text{if } t \neq 0 \end{cases}$ <u>NOT</u> CONTINUOUS.

<u>TERMINOLOGY</u>: CALL CONT. FCNS <u>MAPS</u>

<u>DEF</u>: A MAP $f : X \longrightarrow Y$ IS A <u>HOMEOMORPHISM</u> IF THERE IS A MAP $g : Y \longrightarrow X$ SO THAT

(i) $f \circ g = Id_Y$

(ii) $g \circ f = Id_X$

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

EXAMPLES: (i) Id_X IS A HOMEOMORPHISM.

(ii) $f: \mathbb{R} \longrightarrow \mathbb{R},\ f(x) = x+1$
(iii) $f: \mathbb{R} \longrightarrow \mathbb{R} : f(x) = x^3$ } ALSO

(iv) $g: \mathbb{R} \longrightarrow \mathbb{R},\ g(x) = x^2$ IS NOT.

FOUNDATIONAL PROBLEMS of TOPOLOGY:

(i) WHICH PROPERTIES of SPACES ARE INVARIANTS of HOMEOMORPHISM?

(ii) GIVEN SPACES X, Y: ARE THEY HOMEOMORPHIC?

EXAMPLE (i) $\mathbb{R}^0$ & $\mathbb{R}^1$ ARE NOT HOMEOMORPHIC.
(ii) $\mathbb{R}^1$ & $\mathbb{R}^2$ " " "

INVARIANCE of DOMAIN [BROUWER, 1910]

$\mathbb{R}^m \cong \mathbb{R}^n$ iff $m = n$.

TO PROVE THIS TAKES A YEAR!

(4) NEW SPACES FOR OLD

SUPPOSE (X, Ω) IS A SPACE. SUPPOSE $A \subset X$.

DEFINE $\Omega_A = \{ U \cap A \mid U \in \Omega \}$ AND CALL (A, Ω_A) A SUBSPACE of X [EQUIPPED WITH THE SUBSPACE TOPOLOGY].

EXAMPLES: (i) $[0,1] \subset \mathbb{R}$ CLOSED INTERVAL
$[0,1) \subset \mathbb{R}$ HALF-OPEN "
$(0,1) \subset \mathbb{R}$ OPEN "

WHICH of THESE IS HOMEO TO $\mathbb{R}$?

EXAMPLES

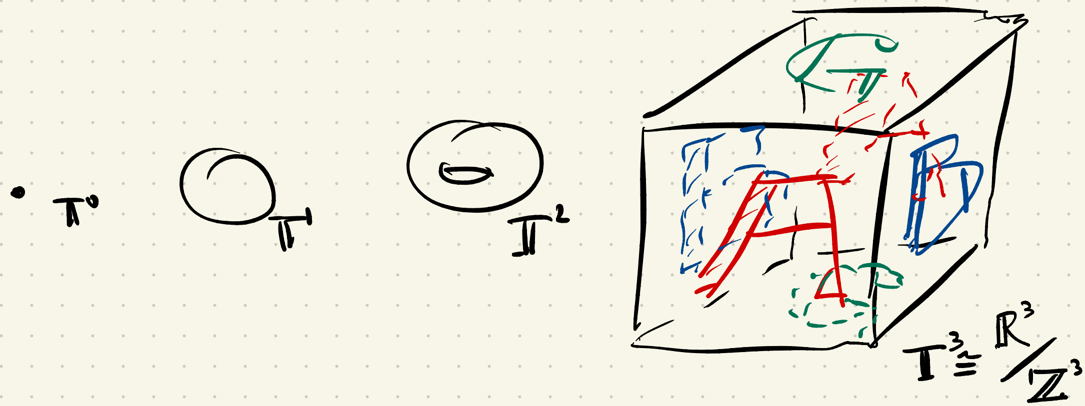
SPHERES $\quad S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$

BALLS $\quad B^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$

TORI $\quad \mathbb{T}^0 = \{pt\}$

$$\mathbb{T}^{n+1} = \mathbb{T}^n \times S^1.$$

PICTURES



GOT HERE.

EXERCISE: $\mathbb{T}^2 \not\cong S^2$.

EXERCISE: IN FACT $S^2 \not\cong X \times X$ FOR **ANY** SPACE X.