Please let me know if any of the problems are unclear, have typos, or have mistakes. Please turn in your solution to Exercise 5.3 on Friday (2022-03-18) by noon, on Moodle. Below, if coefficients are not given, they are assumed to be $\mathbb{Z}$.

Exercise 5.1. We give S^1 the usual CW–complex structure with exacly one zero-cell and one one-cell. Let $T^3 = S^1 \times S^1 \times S^1$ be the three-torus.

1. Describe the resulting product CW–complex structure on T^3. Count the cells in each dimension, record the various attaching maps, and describe the k–skeleta. Give a sketch.
2. Use the above to give explicit generators for the cohomology groups $H^k(T^3)$.
3. In terms of these generators, describe the cup product structure on $H^*(T^3)$.
4. Let X be the subcomplex of the two-skeleton obtained by deleting a single (open) two-cell. Give explicit generators for the cohomology groups and describe the cup product structure on $H^*(X)$.

Exercise 5.2. Show that X and Y are not homeomorphic.

1. $X = S^2 \times S^4$ and $Y = \mathbb{CP}^3$.
2. $X = S^2 \times S^2$ and $Y = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. (Here # is the oriented *connect sum* and the bar over the second projective plane indicates that its orientation is reversed.)

Exercise 5.3. Prove that a direct proof that a surface S is orientable if and only if it does not contain a homeomorphic copy of the Möbius band M^2. [Here "direct" means "without appealing to the classification of surfaces".]

Exercise 5.4. Suppose that M is an orientable n–manifold. Suppose that $N \subset M$ is a closed connected $(n-1)$–dimensional sub-manifold which separates M. Prove that N is orientable.

Exercise 5.5. Suppose that $N \subset S^n$ is a closed connected $(n-1)$–dimensional sub-manifold. Prove that N is orientable. Using this, or otherwise, prove that the Klein bottle does not embed into $\mathbb{R}^3$.