Please let me know if any of the problems are unclear, have typos, or have mistakes. Please turn in your solution to Exercise 4.5 on Friday (2022-03-04) by noon, on Moodle. Below, if coefficients are not given, they are assumed to be $\mathbb{Z}$.

**Exercise 4.1.** Give $\Delta$–complex structures for the following surfaces: $S^2$ (the two-sphere), $P^2$ (the real projective plane), $T^2$ (the two-torus), and $K^2$ (the Klein bottle). (Finding the smallest possible structure in each case will greatly simplify the next five exercises.)

**Exercise 4.2.** For each surface $M$ in Exercise 4.1 use its $\Delta$–complex structure, and the Seifert–van Kampen theorem, to compute the fundamental group $\pi_1(M)$.

**Exercise 4.3.** Suppose that $R = \mathbb{Z}$ is the coefficient ring. For each surface $M$ in Exercise 4.1 give the simplical chain complex associated to its $\Delta$–complex structure. Compute the resulting homology groups $H_k(M)$ as well the Euler characteristic $\chi(M)$.

**Exercise 4.4.** For each surface $M$ in Exercise 4.1 give the simplical cochain complex associated to its $\Delta$–complex structure, using $Q = R = \mathbb{Z}$ as the coefficient module. Compute the resulting cohomology groups $H^k(M; \mathbb{Z})$.

**Exercise 4.5.** For each surface $M$ in Exercise 4.1 compute the cup product structure (and thus the $\mathbb{Z}$–algebra structure) on $H^*(M; \mathbb{Z})$.

**Exercise 4.6.** Let $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. For each surface $M$ in Exercise 4.1 compute the cup product structure (and thus the $\mathbb{F}_2$–algebra structure) on $H^*(M; \mathbb{F}_2)$.

**Exercise 4.7.** In this exercise, all modules are $\mathbb{Z}$–modules, and all tensor products are over $\mathbb{Z}$. Recall that if $a \in A$ and $b \in B$ are elements, then the equivalence class $a \otimes b \in A \otimes B$ is called a *pure tensor*.

- Find a pair of modules $A$ and $B$ and an element $c \in A \otimes B$ which is not pure.
- Find a pair of modules $A$ and $B$ and distinct elements $a, a' \in A$ and $b, b' \in B$ so that $a \otimes b = a' \otimes b'$.

**Exercise 4.8.** In this exercise, all modules are $\mathbb{Z}$–modules, and all tensor products are over $\mathbb{Z}$.

- Show that $A \otimes B \cong B \otimes A$.
- Show that $(\oplus_i A_i) \otimes B \cong \oplus_i (A_i \otimes B)$.
- Show that $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$.
- Show that $A \otimes \mathbb{Z} \cong A$.
- Show that $A \otimes \mathbb{Z}/n\mathbb{Z} \cong A/nA$.

- Show that if $f\colon A \to A'$ and $g\colon B \to B'$ are homomorphisms then there exists a unique homomorphism $f \otimes g\colon A \otimes B \to A' \otimes B'$ so that $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$.
- Show that if $\phi\colon A \times B \to C$ is bilinear then there exists a unique homomorphism $\bar{\phi}\colon A \otimes B \to C$ so that $\bar{\phi}(a \otimes b) = \phi(a, b)$.

**Exercise 4.9.** Suppose that $R$ is a commutative ring (with unit).

1. Prove that $R \otimes_R R \cong R$ as $R$–modules.
2. Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ as $\mathbb{Z}$–modules.
3. Suppose that $R = \mathbb{Q}(\sqrt{2})$. What is $R \otimes_{\mathbb{Q}} R$ as a $\mathbb{Q}$–module?
4. [Harder.] Prove that $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}$ as $\mathbb{Z}$–modules.