LEC 30

17/03/2022

47 POINCARÉ DUALITY

Theorem Suppose M^n is a connected, R-oriented n-manifold (via $\mu : \mathbb{R}^n \to \mathbb{R}^n_R$)

Then, the duality map

$$D_M : H^k_{cpt}(M) \longrightarrow H_{n-k}(M)$$

is an isomorphism.

Recall: $D_M = \varinjlim_{K\,cpt} D_{M|K}$

$$H^k_{cpt}(M) \cong \varinjlim_{K\,cpt} H^k(M|K)$$

Base Case: Take $M = \mathbb{R}^n$. So, $H_{n-k}(\mathbb{R}^n) \cong \begin{cases} R & k=n \\ 0 & o/w \end{cases}$

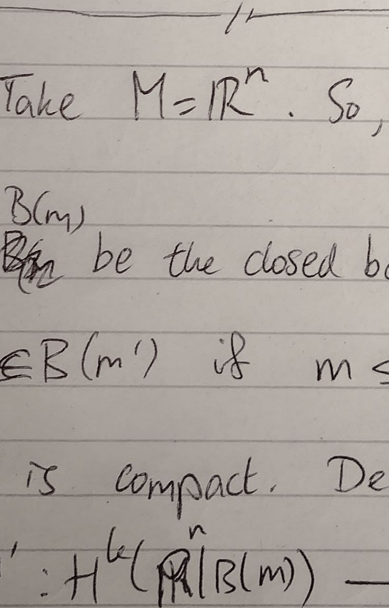
Also, let $B(m)$ be the closed ball of radius $m \in \mathbb{Z}_{\geq 0}$

Note, $B(m) \in B(m')$ if $m \leq m'$

And, $B(m)$ is compact. Define

$$\psi^{m,m'} : H^k(\mathbb{R}^n | B(m)) \longrightarrow H^k(\mathbb{R}^n | B(m'))$$

Picture:



Note the covariance of $\psi^{m,m'}$

Note: For all m

$$H^k(\mathbb{R}^n | 0) \cong H^k(\mathbb{R}^n | B(m))$$

$$\begin{array}{l} H^k(\mathbb{R}^n | 0) \\ \quad \| \\ H^k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\ \quad \wr\!\| \text{ excision} \\ H^k(B(1), B(1) \setminus \{0\}) \\ \quad \wr\!\| \text{ Long Exact Sequences} \\ H^k(\mathbb{B}^n, S^{n-1}) \cong \tilde{H}^k(S^n) \cong \begin{cases} R & k = n \\ 0 & \text{else} \end{cases} \end{array}$$

So, $\varphi^{m,m'}$ is an isomorphism for all $m' \geq m$

To be explizit:

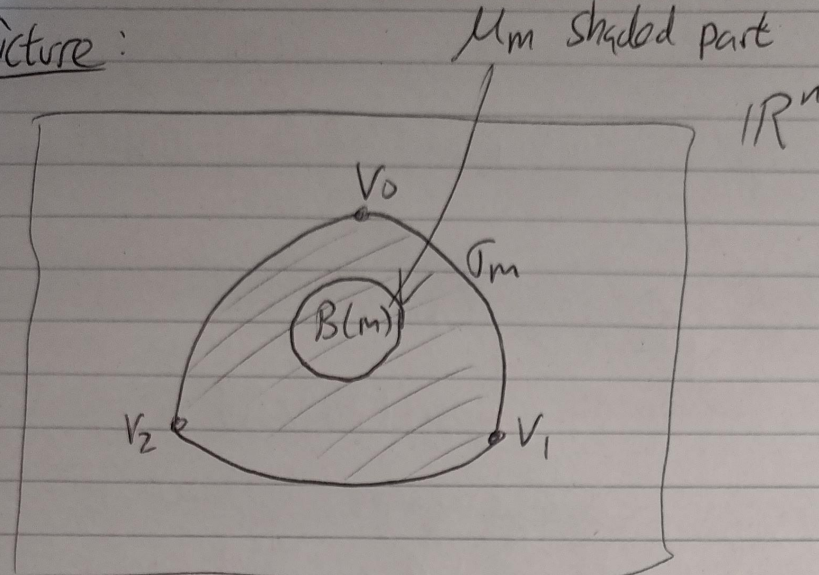
Let $\phi_m \in H^n(\mathbb{R}^n | B(m))$ be the resulting generator

Let $\mu_m \in H_n(\mathbb{R}^n | B(m))$ be the fundamental class given by the patching lemma.

Claim 1: $\varphi^{m,m'}(\phi_m) = \phi_{m'}$

Claim 2: $\mu_m \cap \phi_m = [pt]$

<u>Picture</u>:



A word about ϕ_m:

$$\phi_m \in H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B(m))$$

~~Coming from~~ Recall LES

$$\cdots \leftarrow \cancelto{0}{H^{n+1}(\mathbb{R}^n, \mathbb{R}^n \setminus B(m))}$$

$$\cancelto{0}{H^n(\mathbb{R}^n \setminus B(m))} \longleftarrow \cancelto{0}{H^n(\mathbb{R}^n)} \leftarrow H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B(m)) \xleftarrow{\delta}$$

$$H^{n-1}(\mathbb{R}^n \setminus B(m)) \leftarrow \cancelto{0}{H^{n-1}(\mathbb{R}^n)} \leftarrow \cdots$$

$$\cong ?$$

$$H^{n-1}(S^{n-1})$$

That is, $\phi_m = \delta e_m$ where e_m evaluates to 1 on correctly oriented $n-1$-dimensional spheres.

Now, compute

$$\mu_m \frown \phi_m = \mu_m \frown \delta\rho_m$$

$$= [\sigma_m] \frown \delta\rho_m$$

$$= [\delta\rho_m(\sigma_m|_{[v_0,\dots,v_n]})\,\sigma_m|_{[v_n]}]$$

$$= [\rho_m(\partial\sigma_m)\cdot pt]$$

$$\cancel{= \epsilon\varphi} = [1_R \cdot pt] = [pt] \checkmark$$

Set $\phi = \varinjlim_m \phi_m$

~~So $D_{\mathbb{R}^n}(\phi) = [pt]$ as desired.~~

Set ~~$\mu = \varinjlim_m \mu_m$~~ not strictly needed / somewhat makes no sense

Since $\mu_m \frown \phi_m = [pt]$ $\forall m$, we have

~~$\mu \frown \phi = [pt]$~~ Thus

$$D_{\mathbb{R}^n}(\phi) = [pt]$$

$$\varinjlim_m H_n(\mathbb{R}^n / B(m)) = H_n(\mathbb{R}^n)$$

doesn't make sense

These are "morally right" but makes no sense as this limit does not exist.

Convex Subsets

Plan of the rest of the proof:

1) Convex Subsets

2) Finite Union (using MV for H^*_{cpt}) ~~If $M = U \cup V$ and P.D holds for~~

Claim: If $M = U \cup V$ and P.D holds for U and V, then P.D. holds for ~~an~~ M.

3) Ascending Unions [use properties of direct Limits]

4) Covers by charts.

Read Hatcher

48 APPLICATIONS & EXERCISES

Corollary: Suppose M^n is closed and connected and n is odd. Then $\underset{\text{Euler Characteristic.}}{\chi}(M) = 0$

Special Case: Suppose $R = \mathbb{F}$ a field and M is R-orientable.

Then, $H_k(M) \underset{P.D.}{\cong} H^{n-k}(M) \underset{U.C.T}{\cong} H_{n-k}(M)$

e.g. $n=5$

k	0	1	2	3	4	5
$H_k(M)$	$H_0(M)$	$H_1(M)$	$H_2(M)$	$H_3(M)$	$H_4(M)$	$H_5(M)$

Exercise: If N^{n-1} is closed, connected, embeds in S^n, then N is $\mathbb{Z}$-orientable.

Pretty Formula:

$$\underset{C^l}{\psi}(\underset{C_{k+l}}{\alpha} \frown \underset{C^k}{\phi}) = \underbrace{(\phi \smile \psi)}_{C^{k+l}}(\alpha)$$

This yields an "intersection pairing"

$$\underset{\phi}{H^{2n}} \times \underset{\psi}{H^{2n}} \xrightarrow{(\phi \smile \psi)[M]} \mathbb{R}$$

when M is a closed $4n$-manifold. In ptic for M^4, this pairing almost determines the homeomorphism type of M.

When $\pi_1(M) = 0$ [Freedman]

NOTE: I use ϕ instead of φ
\phi \varphi

I used ρ instead of ϕ in the
\rho \phi

proof of claim 2.