Lecture 28

Monday, 14 March 2022 12:07 PM

Poincaré Duality

Theorem: Let M be a compact, connected, n-manifold, R-orientable via $\mu : M \to M_R$.

Let $\mu_M = [M] \in H_n(M;R)$ be the resulting fundamental class.

Define the <u>duality map</u> $D_M : H^k(M;R) \to H_{n-k}(M;R)$

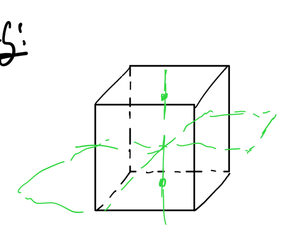
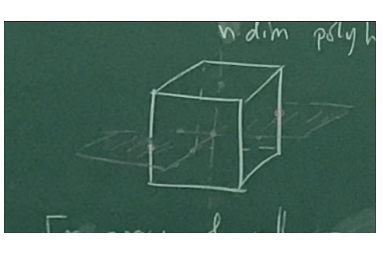
$$\varphi \longmapsto [M] \frown \varphi$$

then D_M is an isomorphism.

Exercise: $D_M([\varepsilon]) = [M]$

Morally, suppose M has a "nice" cell structure

Eg:

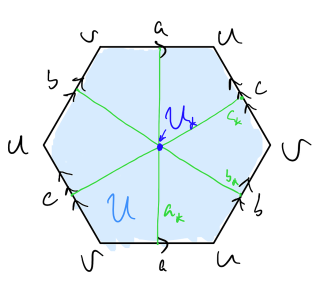
 

For every k-cell σ, build an $(n-k)$cell σ_* st

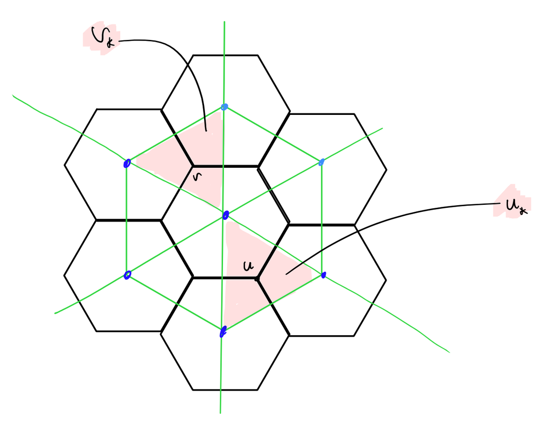
$\sigma \cap \sigma_*$ is a single point, the centre of σ

Actual intersection not cup.

Picture in dimension 2.



$\cong \mathbb{T}^2$



We get chain complexes

$$0 \to \underset{\langle U \rangle}{\mathbb{Z}} \to \underset{\langle a,b,c \rangle}{\mathbb{Z}^3} \to \underset{\langle u,v \rangle}{\mathbb{Z}^2} \to 0$$

and dually

$$0 \to \underset{\langle u_*, v_* \rangle}{\mathbb{Z}^2} \to \underset{\langle a_*, b_*, c_* \rangle}{\mathbb{Z}^3} \to \underset{\langle U_* \rangle}{\mathbb{Z}} \to 0$$

Careful: Some manifolds do not have cell structures

No example given at present.

Theorem: Suppose M is non-compact, connected, R-orientable via $\mu : M \to M_R$

Then there is a duality map $D_M : H^k_{cpt}(M;R) \to H_{n-k}(M,R)$

which is an isomorphism.

Thus, $H^n_{cpt}(\mathbb{R}^n;\mathbb{Z}) \cong \mathbb{Z}$

Cohomology with compact support(s)

Suppose X is a space, R commutative ring, $K \subset X$ compact.

Recall $D^k(X, X\backslash K;R) = \{ \varphi \in C^k(X;R) \mid$ if $\sigma \in C_k(X;R)$ has $\text{image}(\sigma) \subseteq X \backslash K$ then $\varphi(\sigma) = 0 \}$

$$= \text{image}\,(q^* : C^k(X, X\backslash K) \to C^k(X))$$

Def: $C^k_{cpt}(X) = \bigcup_{K\, cpt} D^k(X|K) \subset C^k(X)$

Exercise: $(C^k_{cpt}(X), \delta)$ is a cochain complex SEE Hatcher.

Use $(C^k(X|K), \delta) \cong (D^k(X|K), \delta)$ is also a chain complex

In fact $C^*_{cpt}(X)$ is the <u>direct limit</u> of the complexes $D^*(X|K)$

$$\varinjlim_K D^k(X|K) = C^*_{cpt}(X)$$

So we define B^*_{cpt}, Z^*_{cpt}, H^*_{cpt} in the usual way.

Exercise: $H^k_{cpt}(\mathbb{R}^n) \cong \begin{cases} R & n=k \\ 0 & \text{else} \end{cases}$

by direct computation from the definition.

Direct limits

Def: Suppose I is a set, $\le$ a relation on I.

$(I, \le)$ is a directed set iff,

i) $\le$ reflexive (i.e. $\alpha \le \alpha$ $\forall \alpha$)

ii) $\le$ antisymmetric ($\alpha \le \beta$, $\beta \le \alpha \Rightarrow \alpha = \beta$)

iii) $\le$ transitive ($\alpha \le \beta$, $\beta \le \gamma \Rightarrow \alpha \le \gamma$)

iv) Upper bounds $\forall \alpha, \beta \in I$, $\exists \gamma \in I$ st $\alpha, \beta \le \gamma$.

Eg: X a space, $(\{K \subseteq X \mid K \text{ compact}\}, \subseteq)$

$(\mathbb{N}, \le)$, $(\mathbb{R}, \le)$

Eg: Non example, $\{\alpha, \beta, \gamma\}$ st $\alpha \le \gamma$, $\alpha \le \beta$ but neither $\alpha \le \beta$ or $\beta \le \alpha$.

Def: A <u>directed system</u> on $(I, \le)$ is a covar functor from $(I, \le)$

Def: Suppose $(M_\alpha)_{\alpha \in I}$ is a directed system of R-modules,

st if $\alpha \le \beta$ we have $f_{\alpha\beta} : M_\alpha \to M_\beta$

Define the <u>direct limit</u> $\varinjlim_\alpha M_\alpha = \bigoplus M_\alpha \big/ {m \sim f_{\alpha\beta}(m)}$ for all $m \in M_\alpha$, $\beta \ge \alpha$.

Equivalently, $\varinjlim_\alpha M_\alpha = \coprod_\alpha M_\alpha \big/ \sim$ where $m \sim n$ iff $m \in M_\alpha$, $n \in M_\beta$ and $\exists \gamma$ st $\gamma \ge \alpha$, $\gamma \ge \beta$ st $f_{\alpha\gamma}(m) = f_{\beta\gamma}(n)$.

Eg: $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \cdots$

what is the direct limit?