WANYING ZHAO.

Lecture 27

2022-03-10.

(40). Finish proof of "3.21 Patching Lemma".

Given (i). M connected n-mfd.

(ii). $\sigma: M \to M_R$ section of homology bundle.

(iii). $A \subseteq M$ compact.

Goal: Construct $\alpha_A \in H_n(M|A)$ with $\psi_{A,x}(\alpha_A) = \alpha_x$

Remaining Case: (4) Suppose $A \subseteq U \cong \mathbb{R}^n$; Fix $z \in Z(M|A)$ a k-cycle.

So z is a finite formal sum of singular k-simplices.

Since $z \in Z_k(M, M-A)$, we get $\partial z \in C_{k-1}(M-A)$

Let $L = \bigcup_{\sigma_i \text{ in } \partial z} \text{image}(\sigma_i)$. This is compact and disjoint from A.

So there is some δ with $N_\delta(A) = \{x \in \mathbb{R}^n \mid \|x-a\| < \delta \text{ for some } a \in A\}$

[The δ-neighborhood]: is disjoint from L. $N_\delta(A) \cap L = \phi$

Thus we cover A by closed $\delta/2$ balls [$B_{\delta/2}(a) = \{x \in \mathbb{R}^n \mid \|x-a\| \leq \delta/2\}$]

Since A is compact, there is a finite subcover. Take the union to obtain K. Note

① K is finite union of convex balls.

② $K \cap L = \phi$

③. $A \subseteq \text{int}(K)$

$\Rightarrow z \in Z_k(\mathbb{R}^n|K)$

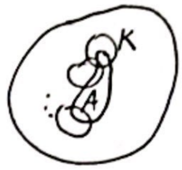
Now suppose $k > n$, WTS: $[z] \in H_k(M|A)$ is 0.

By (3), $H_k(\mathbb{R}^n, K) = 0$. So there is some $y \in C_{k+1}(\mathbb{R}^n|K)$ with $\partial y = z$. ~~$C_{k+1}(\mathbb{R}^n|K$~~

$$\begin{tikzcd} C_{k+1}(\mathbb{R}^n|K) \arrow[r, "\partial"] \arrow[d, "i_{k+1} := \psi_{K,A}"] & C_k(\mathbb{R}^n|K) \arrow[d, "i_k = \psi_{K,A}"] \\ C_{k+1}(\mathbb{R}^n|A) \arrow[r, "\partial"] & C_k(\mathbb{R}^n|A) \end{tikzcd}$$

so $[z]=0$ in $C_k(\mathbb{R}^n|A)$.

Suppose. $K=n$. Let N be a large ball around A, K



N

By (3). we are given α_N so that $\psi_{N,x}(\alpha_N) = \alpha_x \quad \forall x \in N$

Define: $\alpha_A = \psi_{N,A}(\alpha_N)$ and compute $\psi_{A,x} \circ \psi_{N,A}(\alpha_N)$
$= \psi_{N,x}(\alpha_N) = \alpha_x$ as desired.

Uniqueness: Suppose $\beta_A \in H_n(\mathbb{R}^n|A)$ also has the property
[Exercise]. $\psi_{A,x}(\beta_A) = \alpha_x$. Then pick a representative of $\alpha_A - \beta_A$. ~~and~~

Recap: If M connected; compact; ~~R~~ n-mfd; M is also R-orientable.
via a section $\sigma: M \to M_R$ Then there is a fundamental class
~~$[\alpha_m]$~~ $\alpha_m = [M] \in H_n(M, R)$ so that $\forall x \in M$
$\psi_{M,x}([M])$ is a generator for $H_n(M|x) \cong R$.
Thus $[M]$ generates $H_n(M; R)$. So. $H_n(M) \cong R$.

Compare to 0-degree: If X is path connected, $H_0(X, R) \cong R$.

Prop 3.29: If M connected, not compact n-mfd. Then $H_n(M; R) \cong 0$

Pf: See Hatcher.

(41) Cap Product

Fix X a top space. ; R · a comm Ring with 1_R. Ceffiants: $Q = R$.

Define: $C_{K+L}(X) \times C^K(X) \xrightarrow{\ \cap\ } C_L(X)$

$$(\sigma, \varphi) \longmapsto \sigma \cap \varphi = \varphi(\sigma|_{[v_0, \dots v_K]}) \cdot \sigma|_{[v_K, \dots v_{K+L}]}$$

and extend linearly to all chains in C_{K+L}.

Alt Defn: $C_{K+L}(X) \xrightarrow{\ \cap\ } \mathrm{Hom}_R(C^K(X), C_L(X))$

$$\sigma \longmapsto (\varphi \longmapsto \sigma \cap \varphi)$$

Lemma: (~~Leb~~ Leibnitz) $\partial(\sigma \cap \varphi) = (-1)^K((\partial\sigma) \cap \varphi - \sigma \cap (\delta\varphi))$

↳ Pf as exercise.

Cor: cap products induce:

$$Z_{K+L} \times Z^K \longrightarrow Z_L$$
$$B_{K+L} \times Z^K \longmapsto B_L$$
$$Z_{K+L} \times B^K \longrightarrow B_L$$
$$H_{K+L} \times H^K \longrightarrow H_L$$

With this Corollary, we have cap products defined on Homology

Naturality: Suppose $f: X \to Y$ is continous, consider $F(X) = H_{K+L}(X)$, $G(X) = \mathrm{Hom}_R(H^K(X), H_L(X))$

Then the cap product is a natural transformation from F to G.

$\eta: F \to G$

$$\begin{tikzcd}
H_{K+L}(X) \arrow[r, "f_*"] \arrow[d, "\cap"] \arrow[dr, phantom, "\circlearrowleft"] & H_{K+L}(Y) \arrow[d, "\cap"] \\
\mathrm{Hom}(H^K(X), H_L(X)) \arrow[r, "f_*"] & \mathrm{Hom}(H^K(Y), H_L(Y))
\end{tikzcd}$$

This can be generalised to a fully relative variant of cap product.

$$H_{k+l}(X, A \cup B) \longrightarrow \mathrm{Hom}_R(H^k(X,A), H_l(X,B)).$$