Lecture 26, 8/3/2022

MA4J7. 9-2.

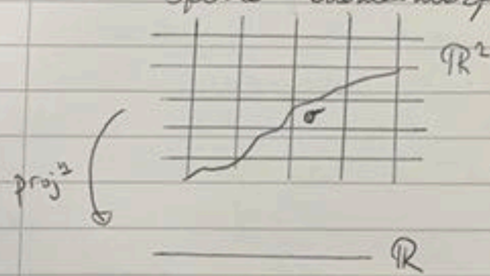
(4i) Patching.

~~Why? R - Comm' rg with~~

- We require R - commutative ring with discrete topology & unit 1_R.
- Consequence (exercise): $\mathbb{R}^n_R \cong \mathbb{R}^n \times R$, and all sections $\sigma: \mathbb{R}^n \to \mathbb{R}^n_R$ are "constant," i.e. $\pi_R \circ \sigma$ is constant.

  local trivy bundle

- In general, $M_R \to M$ is a covering space with fibre homeomorphic to R.



Remarks:

- the sections are not constant.
- In this fibre bundle, the fibre has topology of $\mathbb{R}$, so σ may be locally nonconstant.

- Thm (3.26): M - compact, connected without boundary.

  (a) Fix R. If M is R-orientable via

  $\sigma: M \to M_R$

  $x \mapsto (x, \alpha_x)$, then $\exists!$ fundamental class $\alpha_M = [M] \in H_n(M)$ s.t.

  $\Psi_{M,x}(\alpha_M) = \alpha_x$.

  Also, $\Psi_{M,x}$ is an isomorphism $\forall x$.

  (b) If M isn't R-orientable, then $\Psi_{M,x}$ is injective and $\operatorname{im}(\Psi_{M,x})$ is 2-torsion.

  (c) $H_k(M) \cong 0 \quad \forall k > n$.

- Exercise: Deduce Thm 3.26 from ...

... **Lemma (3.27)**: M^n – connected, without boundary.
Fix R – comm' ring with $1_R \in R$.
(a) Suppose $\sigma: M \to M_R$ is a section;
$\sigma(x) = (x, \alpha_x)$.
Suppose $A \subset M$ – compact. Then $\exists !\ \alpha_A \in H_n(M|A)$
s.t. $\Psi_{A,x}(\alpha_A) = \alpha_x \ \forall x \in A$ (localises "correctly").

(C) $H_k(M|A) \cong 0$ for $k > n$.

- Rks: (1) The lemma is a "semi-local" analogue, i.e. considering compact subsets.
(2) If M = not compact, eg. $M = \mathbb{R}^n$, we cannot find a fundamental class α_M.
The "local" classes α_A (for A-compact) are the next best thing.

- **Pf (3.27)**: (1) Suppose A is compact, and $A = B \cup C$, where $B, C, D = B \cap C$ all satisfy (a) and (C).
- We use relative Mayer-Vietoris for homology.
$D \subset B \subset A$, $D \subset C \subset A$ } viewing this via a diagram:

$$\cdots \to H_{k+1}(M|A) \to H_{k+1}(M|B) \oplus H_{k+1}(M|C) \to H_{k+1}(M|D) \to$$
$$\to H_k(M|A) \to H_k(M|B) \oplus H_k(M|C) \to H_k(M|D) \to \cdots$$

- If $k > n$, then $H_k(M|A)$ is trapped between zeros, proving C.
- Suppose $k = n$:

$$0 \to H_n(M|A) \xrightarrow{\Phi} H_n(M|B) \oplus H_n(M|C) \xrightarrow{\Psi} H_n(M|D)$$
$$a_A \longmapsto (a_B, -a_C)$$
$$(a, b_C) \longmapsto a_D + b_D.$$

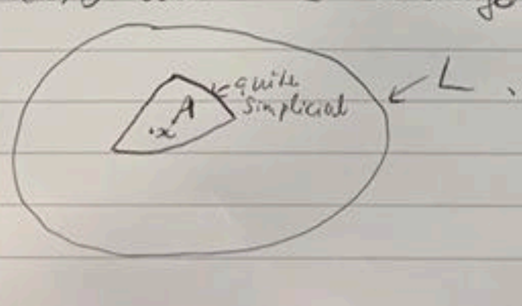
with $0 \cong H_{n+1}(M|D)$.

- Here, $a_B := \Phi_{A,B}(a_A)$, etc.
- Suppose $\sigma: M \to M_R$ is the given section. By hypothesis, we're given unique $\alpha_B, \alpha_C, \alpha_D$ st. $\Psi_{B,x}(\alpha_B) = \alpha_x$, and so on.

- By uniqueness and commutativity of localisation, $\Psi(\alpha_B, 0) = \alpha_D$, and $\Psi(0, \alpha_C) = \alpha_D$, so
$$\Psi(\alpha_B, -\alpha_C) = 0_D.$$
- By exactness of LES, we get a unique $\alpha_A \in H_n(M|A)$ s.t. $\Phi(\alpha_A) = (\alpha_B, -\alpha_C)$.
- Note: $\Psi_{A,x}(\alpha_A) = \Psi_{B,x} \circ \Psi_{A,B}(\alpha_A)$
$$= \Psi_{B,x}(\alpha_B) = \alpha_x \quad \forall x \in B.$$
(similarly for $x \in C$, so done part (1)).
- Uniqueness - suppose $\beta_A \in H_n(M|A)$ also has the property that $\Psi_{A,x}(\beta_A) = \alpha_x$ $(\forall x \in A)$.
- But consider $\Phi(\alpha_A - \beta_A) = (\alpha_B - \beta_B, -\alpha_C + \beta_C)$, and β_B, β_C have the (localisation) property, so $\alpha_B = \beta_B$ and $\alpha_C = \beta_C$, so we're done.

(2) Suppose A - compact. Then $\exists$ finite collection of compact sets $(A_i)_{i=1}^m$, & of charts $(U_i)_{i=1}^m$ s.t.
 (i) $U_i \cong \mathbb{R}^n$ [by def^n of chart];
 (ii) $A_i \subset U_i$;
 (iii) ~~A~~ $A = \bigcup_{i=1}^m A_i$;
 (iv) $\{U_i\}_{i=1}^m$ cover A.
- note: $A_i \cap A_j$ is compact and lies in U_i (say), so by (1), it suffices to prove ~~...~~ (a) and (c) for compact A inside a chart $U_i \cong \mathbb{R}^n$.

(3) ~~Suppose~~ From now on, assume $M = \mathbb{R}^n$. Suppose $A \subset \mathbb{R}^n$ is convex, i.e. if $x, y \in A$, the line segment $[x, y] \subset A$. Also, suppose A is compact.
- Fix $x \in A$ and L - a large ball s.t. $x \in A \subset L$.



- *Exercise:* $H_n(\mathbb{R}^n | L) \overset{r_{L,x}}{\cong} H_n(\mathbb{R}^n | x)$, and $H_n(\mathbb{R}^n | A)$ is trapped in between).
- *Remark:* $\partial L \simeq L \setminus \{x\}$
  $\partial \simeq L \setminus A$,
- → These + LES + 5-lemma + homotopy + excision give us what we want.
- What's being exploited is $H_n(L^n, \partial L) \cong R$ (apparently, probably, maybe)
- *Thus:* if A is a finite union of convex sets, we are done.
- However, most compact $A \subset \mathbb{R}^n$ are not such a union eg: $A = S^1$ – any segment (not a point) is not convex when $S^1 = A$ is embedded in $\mathbb{R}^2$.
- *Next time:* Tile S^1 with convex sets / round disks, with intersections, use relative hgy etc.