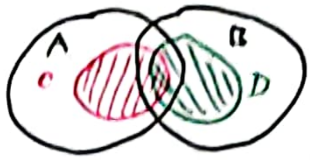
Lecture 14 2022-02-08

(25) Mayer-Victoris



We have :

$$0 \to C_*(A\cap B, C\cap D) \xrightarrow{\Delta} C_*(A,C)\oplus C_*(B,D) \xrightarrow{m} C_*^{\mathcal{U}}(X,Y) \to 0 \qquad (*)$$

Δ is the diagonal inclusion : $\Delta(c) = (c,c)$.

m is the codiagonal w/ sign change : $m(a,b) = a-b$.

Note : $m\circ\Delta = 0$

Claim: $(*)$ is SES, split

Pf : Exercise.

We dualise and obtain an exact triangle

$$\begin{tikzcd}
H^*(A\cap B, C\cap D) \arrow[dr, "\delta_{[+1]}"'] & & H^*(A,C)\oplus H^*(B,D) \arrow[ll, "\Delta^*"'] \\
& H^*_{\mathcal{U}}(X,Y) \arrow[ur, "m^*"'] &
\end{tikzcd}$$

$$H^*_{\mathcal{U}}(X,Y) \cong H^*(X,Y) \qquad \boxed{MV}$$

This completes our discussion of " dualising things from homology "

Q : What is an application of relative MV for H_* ?

A : Good question !

Q : If we use $\check{\Delta}(c) = (c,-c)$ and $\check{m}(a,b) = a+b$, we obtain another exact Δ. Does δ stay the same ? What is the action of R^* here ?

(26) Cup products

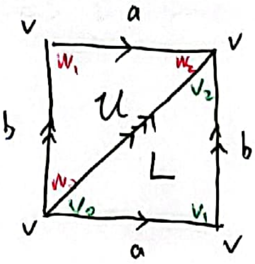
fix $X \in \mathbf{Top}$, $R \in \underline{\mathrm{CRing}}$, $\varphi \in C^k_{sing}(X;R)$, $\psi \in C^l_{sing}(X;R)$.

Suppose $\sigma^{k+l} : \Delta^{k+l} \to X$ is a ~~generator of~~ simplex in $C_{k+l}(X)$.

<u>Define</u> : $(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,v_{k+l}]})$,

and extend linearly.

<u>Example</u> : Take $X = \mathbb{T}^2$



$L|_{[v_0,v_1]} = a$

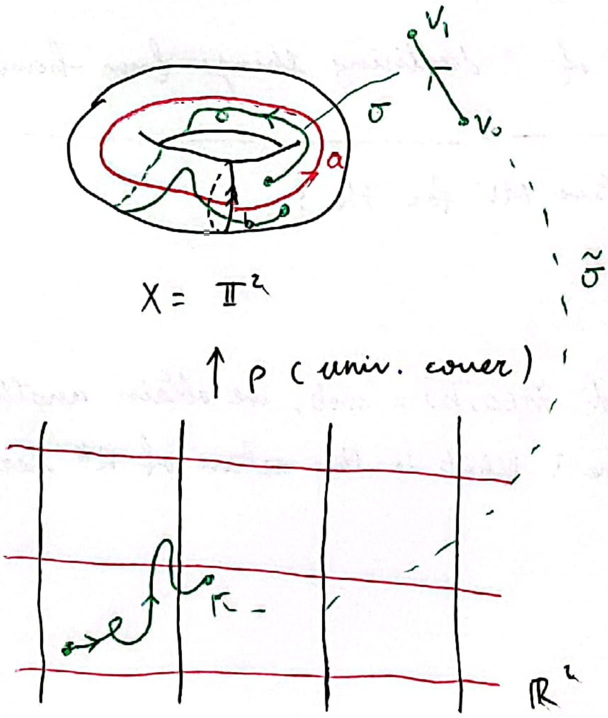
$L|_{[v_0,v_2]} = c$

$L|_{[v_1,v_2]} = b$.

Define $\alpha, \beta : C_1(X) \to \mathbb{R}$ to be the winding cocycles.

i.e. α measures change in x-coord,

β measures change in y-coord.



$\alpha(\sigma) = \tilde{\sigma}_x(1) - \tilde{\sigma}_x(0)$

$\beta(\sigma) = \tilde{\sigma}_y(1) - \tilde{\sigma}_y(0)$

⌈ We compute :

$(\alpha \cup \beta)(L) = 1$

$(\alpha \cup \beta)(U) = 0$

~~$(\alpha\cup\beta$~~

$(\beta \cup \alpha)(L) = 0$

$(\beta \cup \alpha)(U) = 1$ ⌋

e.g. Workings:

$$(\alpha \cup \beta)(L) = \alpha(L|_{[v_0, v_1]}) \cdot \beta(L|_{[v_1, v_1]})$$
$$= \alpha(a) \cdot \beta(b)$$
$$= 1 \cdot 1$$
$$= 1$$

$$(\alpha \cup \beta)(\mathcal{U}) = \alpha(\mathcal{U}|_{[w_0, w_1]}) \cdot \beta(\mathcal{U}|_{[w_1, w_2]})$$
$$= \alpha(b) \cdot \beta(a)$$
$$= 0 \cdot 0$$
$$= 0.$$

Similar for others

Define $Z = L - \mathcal{U}$,

$(\alpha \cup \beta)(Z) = 1$

$\cancel{(\alpha\cup)}$
$(\beta \cup \alpha)(Z) = -1$.

Working:

$$(\alpha \cup \beta)(Z) = (\alpha \cup \beta)(L) - (\alpha \cup \beta)(\mathcal{U})$$
$$= 1 - 0$$
$$= 1$$

$\cancel{(\alpha \cup \beta)}$
$$(\beta \cup \alpha)(Z) = (\beta \cup \alpha)(L) - (\beta \cup \alpha)(\mathcal{U})$$
$$= 0 - 1$$
$$= -1.$$