5) (Proof of UCT) We need to determine the kernel of $h : H^k(C, Q) \twoheadrightarrow \mathrm{Hom}_R(H_k(C), Q)$.

$$h : [\varphi] \longmapsto ([z] \longmapsto \varphi(z))$$

We will need many names for the natural maps

$$B_k \overset{i_k}{\hookrightarrow} Z_k \overset{j_k}{\hookrightarrow} C_k$$

(should be a superscript C everywhere)

$$C_k \xrightarrow{\partial_k} C_{k-1}$$

$$Z_k \overset{q_k}{\twoheadrightarrow} H_k(C)$$

Recall that H_k is defined as a cokernel

$$0 \to B_k \overset{i_k}{\hookrightarrow} Z_k \overset{q_k}{\twoheadrightarrow} H_k \to 0 \qquad *$$

Since $B_k \subset Z_k \subset C_k$ and C_k is finitely generated and free, we know B_k, Z_k are also free.

So $(*)$ is a free res. of H_k. Dualise to obtain (for $k-1$)

$$0 \leftarrow (B_{k-1})^* \leftarrow (Z_{k-1})^* \leftarrow (H_{k-1})^* \leftarrow 0$$

thus by right exactness of Hom,

$$\mathrm{Hom}(H_{k-1}(C), Q) \underset{(q_{k-1})^*}{\cong} \ker((i_{k-1})^*) \qquad 5a$$

Also the cokernel of $(i_{k-1})^*$ is by 3.1 isomorphic to an Ext-group namely

$$\mathrm{Ext}^1_R(H_{k-1}(C), Q) \underset{\Phi}{\cong} \mathrm{coker}((i_{k-1})^*) \qquad 5b$$

- Very clever trick

Define

$$Z_* = (Z_k, 0)_{k \in \mathbb{Z}} \,, \qquad \begin{matrix} B_*[-1] \\ \parallel \\ B_{*-1} \end{matrix} = (B_{k-1}, 0)_{k \in \mathbb{Z}}$$

Remark. Note the degree shift in $B_*[-1]$ by -1.

Also,

$$0 \to Z_* \xrightarrow{j_*} C_* \xrightarrow{\partial_*} B_*[-1] \to 0$$

is a SES of chain complexes (exercise to check). $B_*[-1]$ is free so this SES is ~~free~~ split and by the splitting lemma II

$$0 \leftarrow (Z_*)^* \xleftarrow{(j_k)^*} (C_k)^* \xleftarrow{(\partial_k)^*} (B_{k-1})^* \leftarrow 0$$

(Note $(\partial_k)^*$ is the dual of ∂_k but not δ^k!) is a short exact split sequence.

So we obtain a LES of cohomology groups

$$\begin{tikzcd}
(Z_k)^* & H^k(C) \arrow[l, "{[j_k^*]}"'] & (B_{k-1})^* \arrow[l, "{[\partial_k^*]}"'] \\
(Z_{k-1})^* & H^{k-1}(C) \arrow[l, "{[j_{k-1}^*]}"] & (B_{k-2})^* \arrow[l, "{[\partial_{k-1}^*]}"]
\end{tikzcd}$$

We want to understand the connecting homomorphism.

Claim. The connecting homomorphism is $(i_k)^*$.

Proof. Go through proof of snake lemma and observe.

$$\begin{tikzcd}
0 & Z_k^* \arrow[l] & C^k \arrow[l, "(j_k)^*"'] & B_{k-1}^* \arrow[l, "(\partial_k)^*"'] & 0 \arrow[l] \\
0 & Z_{k-1}^* \arrow[l] \arrow[u, "0"] & C^{k-1} \arrow[l] \arrow[u, "\delta^k"] & B_{k-2}^* \arrow[l] \arrow[u, "0"] & 0 \arrow[l]
\end{tikzcd}$$

Fix $\zeta \in (Z_{k-1})^*$. Note $(j_{k-1})^*$ is surjective so choose $\gamma \in C^{k-1}$ with $(j_{k-1})^*(\gamma) = \zeta$. So $\gamma \circ j_{k-1} = \zeta$, that is, $\gamma | Z_{k-1} = \zeta$.

Note $(j_k)^*(\delta\gamma) = 0$ (check square) so there is $\beta \in (B_{k-1})^*$ such that $(\partial_k)^*(\beta) = \delta\gamma$. Thus the image of ζ under connecting homomorphism is β. Exercise: $\beta = (i_{k-1})^*(\zeta)$ 5c

Exercise: $(q_k)^* \circ h = (j_k)^*$ 5d

Now for a diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[dl] \\
 & & \mathrm{Ext}^1_R(H_{k-1}(C),Q) \arrow[d] \arrow[r, phantom, "\overset{\Phi}{\cong}"] & \mathrm{coker}(i_{k-1}^*) \arrow[dl] & \\
(B_k)^* & (Z_k)^* \arrow[l, "{[i_k^*]}"'] & H^k(C,Q) \arrow[l, "{[j_k^*]}"'] \arrow[dl, "{[j_k^*]}"'] \arrow[d, "h"] \arrow[d, phantom, shift right=8, "5d"] & (B_{k-1})^* \arrow[l, "{[\partial_k^*]}"'] \arrow[u] & (Z_{k-1})^* \arrow[l, "{[i_{k-1}^*]}"'] \\
 & \ker(i_k^*) \arrow[u, hook] \arrow[r, phantom, "\underset{q_k^*}{\cong}"] \arrow[dl] & \mathrm{Hom}(H_k(C),Q) \arrow[d] & & \\
0 & & 0 & &
\end{tikzcd}$$