Please let me know if any of the problems are unclear or have typos. Please let me know if you have suggestions for exercises. For some of the problems I have given a (very vague) level of difficulty. Finally, if you want to do just part of a problem, let me know.

The first two exercises focus on details, omitted in lecture, of the proof of Alexander's theorem. For these exercises we adopt the following notation. Suppose that $S \subset S^3$ is a smoothly embedded two-sphere. Let $h\colon \mathbb{R}^3 \to \mathbb{R}$ be the projection onto the third coordinate. For $c \in \mathbb{R}$, the *level set* $L_c = h^{-1}(c)$ is a horizontal plane. Define $H = h|S$ and suppose that H is *Morse*. Thus S is transverse to the foliation $\{L_t\}_t$ except at finitely many *critical points*, all at distinct heights. These critical points are locally modelled on those of quadratic polynomials. Set $S_c = H^{-1}(c) = S \cap L_c$. We define the *slabs* between a and b in $\mathbb{R}$ as follows.

$$L_{[a,b]} = h^{-1}([a,b]) \quad \text{and} \quad S_{[a,b]} = H^{-1}([a,b])$$

Exercise 4.1. Suppose that c is a regular value of H. Suppose that ϵ is sufficiently small. Take $a = c - \epsilon$ and $b = c + \epsilon$. Show that every component A of $S_{[a,b]}$ is an annulus properly embedded in $L_{[a,b]}$. Show that A cuts a three-ball off of $L_{[a,b]}$.

Exercise 4.2. Suppose that c is a critical value of H. Suppose that ϵ is sufficiently small. Take $a = c - \epsilon$ and $b = c + \epsilon$. Classify the possible homeomorphism types components A of $S_{[a,b]}$, including how they embed in $L_{[a,b]}$. Show that A cuts a three-ball or a solid torus off of $L_{[a,b]}$.

Exercise 4.3. Suppose that $S \subset S^3$ is a smoothly embedded two-sphere. Find an ambient isotopy of S to the equatorial two sphere. [Finding a diffeotopy is harder.]

Exercise 4.4. Prove that every smooth curve in S^2 is ambient isotopic to the equator. [Hint: Copy the proof of Alexander's theorem.]

Exercise 4.5. [Hard.] Compute the homotopy type of $\mathrm{Emb}(S^2, S^3)$, the space of smooth embeddings of the two-sphere into the three-sphere.

Exercise 4.6. Suppose that $T \subset S^3$ is a smoothly embedded two-torus. Show that T bounds a solid torus on at least one side.

Exercise 4.7. Give an example of a (smooth) genus two surface F embedded in S^3 so that neither component of $S^3 - F$ is homeomorphic to a genus two handlebody.

Exercise 4.8. Suppose that M is a three manifold and suppose that $p\colon \widetilde{M} \to M$ is the universal covering. Show that if $\widetilde{M}$ is irreducible then so is M. Deduce that $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ is irreducible.

Exercise 4.9. [Alexander trick.] Suppose that $h\colon \mathbb{B}^3 \to \mathbb{B}^3$ is a homeomorphism. Suppose that $h|S^2 = \mathrm{Id}$. Show that h is isotopic to the identity on $\mathbb{B}^3$, relative to the boundary.

Exercise 4.10. Suppose that M is a three-manifold. Prove that $M \# S^3 \cong M$.

Exercise 4.11. Supose that M is prime. Prove that M either is irreducible or is a two-sphere bundle over the circle.