Please let me know if any of the problems are unclear or have typos. Please let me know if you have suggestions for exercises. For some of the problems I have given a (very vague) level of difficulty.

Exercise 1.1. Prove that S^n is an n–manifold.

Exercise 1.2. Suppose that $M = M^n$ is a manifold. [You may assume $n > 1$ if you find reduced homology annoying.]

a) Recall the definition of orientability of M (given in lecture), in terms of (G, X)–structures. Define what it means for two orientations of M to be *compatible*.

b) Suppose that M is connected. Prove that M has either zero orientations or has exactly two (up to compatibility).

c) [Medium.] Prove that S^n is orientable.

Exercise 1.3.

a) Prove that B^n is an n–manifold with boundary.

b) Prove that $\partial B^n = S^{n-1}$.

Exercise 1.4. Suppose that M^n is an n–manifold with boundary.

a) Prove that ∂M is an $(n-1)$–manifold.

b) Prove that $\partial(\partial M)$ is empty.

Exercise 1.5. [Hard.] Classify closed, connected one-manifolds.

Exercise 1.6. Suppose that $M = M^n$ is an n–manifold.

a) Give (or look up) a definition of a k–dimensional *submanifold* $N = N^k$ in M, in terms of (G, X) structures.

b) Show that, if $k = n$ and M is orientable, then N is orientable.

Exercise 1.7. [Medium.] Prove that the open Möbius band $M = M^2$ is not orientable. Deduce that any surface containing a Möbius band is not orientable.

Exercise 1.8. Suppose that $M = M^n$ is an n–manifold.

a) Recall the definition of of a k–dimensional *foliation* $\mathcal{F}$ in M (given in lecture), in terms of (G, X)–structures.

b) Suppose that $x \in M$ is a point. We define the *leaf* L_x of $\mathcal{F}$ through x to be the subset of M reachable from x via paths that, in charts, remain in the $\mathbb{R}^k$ factor. Prove that L_x is a k–dimensional submanifold of M.

Exercise 1.9. Suppose that M is a manifold. Suppose that $\mathcal{F}$ is a foliation in M.

a) Define what it means for $\mathcal{F}$ to be *orientable*, in terms of (G, X)–structures.

b) Suppose that $\mathcal{F}$ is orientable and L is a leaf of $\mathcal{F}$. Prove that L is an orientable manifold.

c) Give an example of a non-orientable foliation in an orientable manifold.

Exercise 1.10.

a) Prove that $\mathbb{RP}^n$ is a closed manifold.

b) Prove that $\mathbb{CP}^1 \cong S^2$.

Exercise 1.11. [Medium.] Give an elegant drawing of $\mathbb{RP}^2$.

Exercise 1.12. [Medium.] Suppose that $S = S_g$ is the closed connected oriented surface of genus g. Prove that S admits a one-dimensional foliation if and only if S is the two-torus. [You may assume that the foliation $\mathcal{F}$ is smooth.]

Exercise 1.13. [Hard.] Sketch a classification, up to homeomorphism, of smooth one-dimensional foliations in the two-torus.

Exercise 1.14. [Hard.] Give an example of a two-dimensional foliation in the three-sphere.