Introduction to three-manifolds: Saul Schleimer

Lecture 10

2021-03-24

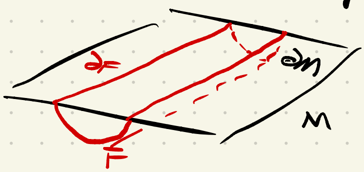
Last time

- Disjointness of PL minimal surfaces.
- The Meeks-Yau trick
- Tower construction – proof of sphere theorem.

(I) Non-peripheral surfaces

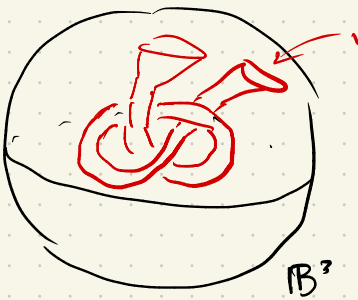
Definition: Suppose $f_0 : (F^2, \partial F) \longrightarrow (M^3, \partial M)$ is a map of pairs. We call f_0 peripheral if there is a homotopy of pairs of f_0 to f_1 with $f_1(F)$ contained in a collar neighborhood of ∂M.

Picture:



F peripheral in M

Example:



Rmk: A surface $(F, \partial F) \to (M, \partial M)$ is "interesting" if it is essential and non-peripheral.

(II) Atoroidal:

Definition: A manifold M^3 is geometrically atoroidal if every embedded essential (π_1-injective) two-torus $T \subset M$ is peripheral.

Definition: A manifold M^3 is algebraically atoroidal if for every $\mathbb{Z}^2 \cong \Gamma < \pi_1(M)$ and for every map $f: T^2 \longrightarrow M$ with $f_*(\pi_1(T)) = \Gamma$ we have f peripheral.

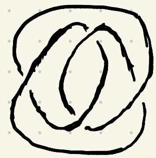
Rmk: Alg atoroidal $\Rightarrow$ geom. atoroidal.

Rmk: The converse is false. [Exercise: Give example of such M^3] but all counterexamples are "special". see below.

Example: (1) The solid torus $U = S^1 \times D^2$ is atoroidal b/c $\pi_1(M)$ contains no $\mathbb{Z}^2$.

(2) The figure-eight knot complement is atoroidal.

That is let F be the knot

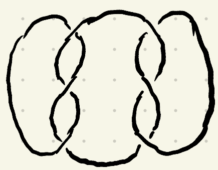


in S^3

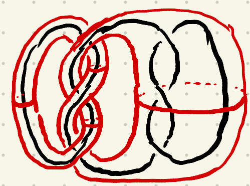
Set $X_F = S^3 - n(F)$. Note $\partial X_F \cong T^2$.

Exercises
- Prove $\partial X_F \hookrightarrow X_F$ is π_1-inj
- Prove X_F is atoroidal. [Give "direct proof"]

(3) Let K be the square knot $K = \text{Tref} \# \text{Tref}$



Then $X_K = S^3 - n(K)$ is toroidal



swallow-follow torus.

[Exercise: Suppose H, H' are nontrivial knots Then $K = H \# H'$ is toroidal

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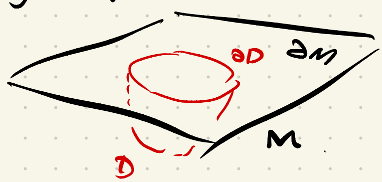
Definition: Suppose M irreducible. We call M boundary irreducible if any prop. emb. disk $(D, \partial D) \subset (M, \partial M)$ is peripheral.

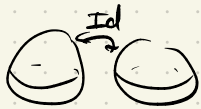


Examples
- $S^1 \times D^2$ — No.
- X_F = fig 8 knot cmp. — YES.

Question: Suppose M irreducible. Is M ∂-irred iff the double $D(M)$ is irreducible?

Def: $D(M) = M \times \{0\} \sqcup M \times \{1\} \Big/ (x,0) \sim (x,1)$ if $x \in \partial M$.

Example: $D(B^3) \cong S^3$.

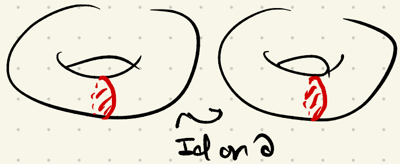


gives S^3.

Exercise: What is $D(S^1 \times D^2)$?

Ans: $S^1 \times S^2$.

Picture



gives $S^1 \times S^2$.

Def: Suppose M is irred, boundary irred, atoroidal. We call M __acylindrical__ if all essential, prop emb annuli $(A, \partial A) \subset (M, \partial M)$ are peripheral.

III Geometrization of knots in S^3 [Thurston]

Flowchart: Suppose $K \subset S^3$ is a knot.

Set $X = X_K = S^3 - n(K)$. __Thm__ [Alex] X is irreducible.

$\boxed{\text{Is } X \text{ boundary reducible?}}$ $\xrightarrow{\text{YES}}$ K is unknot, $\pi_1(X) \cong \mathbb{Z}$.

$\downarrow$ No

$\boxed{\text{Is } X \text{ toroidal?}}$ $\xrightarrow{\text{YES}}$ K is a __satellite knot__ [$\mathbb{Z}^2 < \pi_1(X_K)$ not conj to $\pi_1(\partial X_K)$.]

$\downarrow$ No

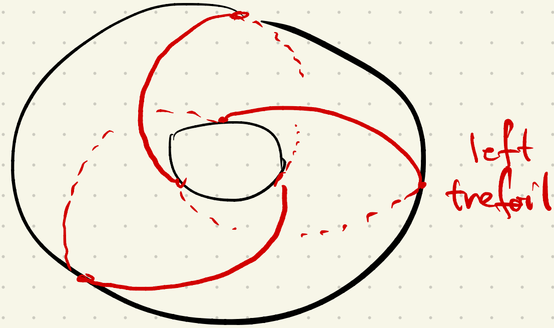
$\boxed{\text{Is } X \text{ cylindrical?}}$ $\xrightarrow{\text{YES}}$ K is a __torus knot__. [Iff $\pi_1(X_K)$ has non-triv centre [$\mathbb{Z}$]]

$\downarrow$ No

X admits a finite volume $\mathbb{H}^3$ structure i.e. K is a __hyperbolic knot__.

Definitions: Consider $T^2 \subset S^3$ the standard torus [also called the Clifford torus] Picture

If $K \subset T$ is a knot in S^3 we call K a torus knot.



Definition: Suppose C is a knot in S^3, called the companion. Suppose P is a knot in $S^1 \times D^2$, called the pattern. Fix a homeomorphism $f: S^1 \times D^2 \longrightarrow N(C)$, the framing. Set $K = f(P) \subset S^3$. This is a satellite knot.

Exercise: • Connect sums are satellites.
• satellites are toroidal.
[Need P, C nontrivial]

IV) JSJ decompositions:

Theorem [Jaco-Shalen, Johannson 1979] Suppose M is irreducible, connected, oriented three-manifold, $\partial M = \sqcup$ tori. Then there is a collection $(T_i)_{i=1}^{n}$ of disjoint emb. ess. non-peripheral tori so that the components of

$M - n(\sqcup T_i)$ are (i) Seifert fibered or
(ii) algebraically atoroidal.

Furthermore any two <u>minimal</u> such collections are isotopic.

<u>Rmk</u>: A major step in the proof of this is the <u>torus theorem</u>: this can be proved using PL minimal surfaces and covering arguments---

<u>Question</u>: What about cutting using higher genus surfaces?

<u>Answer</u>: This <u>is</u> done (Heeg. splittings, fibrings--) but it is not necessary for geometrisation and it is <u>rarely</u> canonical.

V Geometrisation:

<u>Theorem</u> [Perelman] Suppose M is irred, conn, oriented alg. atoroidal, and $\partial M = \sqcup$ tori. Then M has S^3 or $\mathbb{H}^3$ geometry.

[This space left intentionally blank]

A flowchart

Suppose M is compact, oriented, $\partial M = \sqcup$ tori

Is M not connected? $\xrightarrow{\text{YES}}$ take components.

↓ NO

Is M a connect sum? $\xrightarrow{\text{YES}}$ take connect summands

↓ NO

Is M reducible? $\xrightarrow{\text{YES}}$ $M \cong \boxed{S^1 \times S^2}$.

↓ NO

Is $\pi_1(M)$ finite? $\xrightarrow{\text{YES}}$ M has $\boxed{S^3}$ geometry.

↓ NO

Is M geom. toroidal? $\xrightarrow{\text{YES}}$ Is M a torus (semi-) bundle? $\xrightarrow{\text{NO}}$ Cut along maximal system of disjoint, embedded, essential tori

Is M a torus (semi-) bundle? $\xrightarrow{\text{YES}}$ M has $\boxed{\mathbb{E}^3}$, $\boxed{\text{Nil}}$, $\boxed{\text{Solv}}$ geom.

↓ NO

Is M alg toroidal? $\xrightarrow{\text{YES}}$ M has $\boxed{\mathbb{H}^2 \times \mathbb{R}}$ or $\boxed{\widetilde{PSL}}$ geom.
"Small Seifert fibered spaces"

↓ NO

M has $\boxed{\mathbb{H}^3}$ geom.