Introduction to three-manifolds 2021-03-10

Lecture 8

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Last time:

- Sphere Theorem (statement)
- Normal maps and classes

Corollary (of sphere theorem): Suppose M^3 is closed, conn, oriented. If $\pi_2(M) \not\cong 1$ then either

(⊛) M is a connect sum or

(⊛) $M \cong \mathbb{RP}^3$ or $S^2 \times S^1$

Moral: π_2 is not so important in the theory of three-manifolds. π_1 really is important.

The disk theorem (which sort of follows from the techniques we've been developing) controls some of π_1.

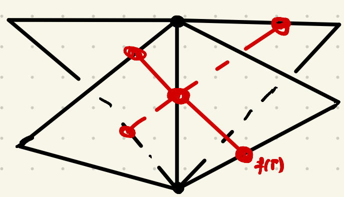
(I) PL Area: Suppose $f : S \longrightarrow (M, K)$ is a normal map

Define $\Gamma_f = f^{-1}(K^{(2)})$; is a finite graph in S.

Define $\ell(f) = \sum_{\gamma \in \Gamma^{(1)}} \text{length}(f(\gamma))$

Define $A(f) = (w(f), \ell(f))$ ordered lexicographically

This is the PL area



Theorem 5.3 [existence]

Fix (M, K). Suppose $\mathcal{G} \neq \emptyset$ is a normal class of maps

Then there is some $f \in \mathcal{G}$ that is normal and realises $\inf\{A(g) \mid g \in \mathcal{G}, g \text{ normal}\}$.

Proof: Set $A_\infty = \inf\{$ " " " $\}$. Define $g \in \mathcal{G}$

to be <u>straight</u> if g is normal and for every $\gamma \in \Gamma_g^{(1)}$ we have $g(\gamma)$ is straight in the containing $\Delta^2 \in K^{(2)}$. We restrict to such g.

Fix a <u>minimising sequence</u> $(g_n)_{n \in \mathbb{N}} \subset \mathcal{G}$ of straight maps with $A_n = A(g_n) \to A_\infty$ (from above). Note we have $W, L > 0$ so that

$$\left.\begin{array}{l} w(g_n) \leq W \\ \ell(g_n) \leq L \end{array}\right\} \text{ for all } n.$$

In any face Δ^2 of $K^{(2)}$ we see at most $\binom{W}{2}$ arcs of $g_n(S)$ (we are assuming Δ^2 embeds in M) and each such arc has length at most L.

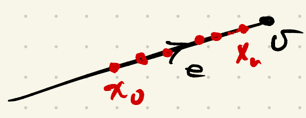
Define $\Gamma_n = \Gamma_{g_n}$. So $g_n | \Gamma_n$ has bounded combinatorics and length.

<u>Claim</u>: For some open set $U \subset M$, with $K^{(0)} \subset U$, we have $g_n(S) \cap U = \emptyset$ (for all n).

<u>Proof</u>: Suppose not. Then there is a vertex $v \in K^{(0)}$ an edge $e \in K^{(1)}$ and points $x_n \in g_n(S)$ so that

(*) $x_n \in e$ and

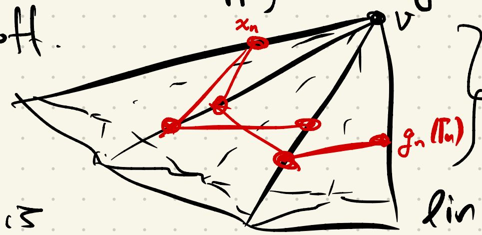
(*) $x_n \longrightarrow v$.



Recall g_n is normal so $S - \Gamma_n$ is a union of disks. So Γ_n is connected. So $g_n(\Gamma_n)$ is

- connected,
- contains x_n, and
- has at most $\binom{W}{2} \cdot (\# \text{ faces of } K^{(2)})$ edges.

Thus in the hyp metric on $K^{(2)} - K^{(0)}$ $g_n(\Gamma_n)$ has bounded diameter [Rmk: The metric on $K^{(2)} - K^{(0)}$ may not be complete ... but that is not a problem.] Apply the "dog on leash" principle: So for suff. large n, $g_n(\Gamma_n)$ is contained in star(v). So g_n is null-homotopic, a contradiction. // Claim



So there is some compact set $Q \subset K^{(2)}$ so that, for all n, we have $g_n(\Gamma_n) \subset Q$.

Pass to subsequences to arrange

(*) $\Gamma_n \cong \Gamma_m$ (isom as graphs) for all m, n

Say $\Gamma_n \cong \Gamma_\infty$ for all n.

Pass to further subseq so that

(*) for all $p \in \Gamma_\infty$, $g_n(p) \longrightarrow g_\infty(p)$ some point. [Suffices to deal with $p \in \Gamma_\infty^{(0)}$]

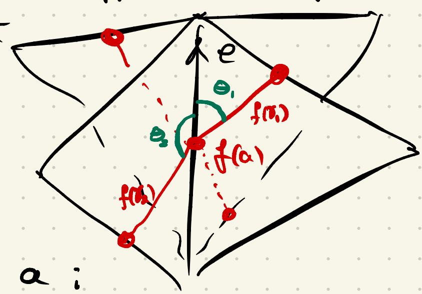
Thus we have $g_\infty : S \to (M, K)$ and for large n, $g_n \simeq g_\infty$ (homotopic) [use straight line homotopy]

So $g_\infty \in \mathcal{G}$

Finally, $A(g_\infty) = \lim_{n\to\infty} A(g_n) = A_\infty$ as desired.

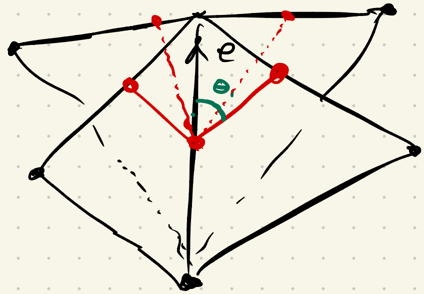
This proves Thm 5.3 [Existence] //

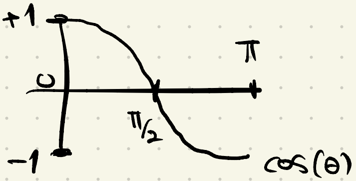
Lemma 5.4: Suppose $f: S \to (M, K)$ is straight and minimising in its homotopy class. Suppose $a \in \Gamma_f^{(0)}$ and $e \in K^{(1)}$ with $f(a) \in e$. Let γ_i be the edges of Γ_f meeting a. Let θ_i be the angle between e and $f(\gamma_i)$. Then f is balanced at a:



That is: $$\sum_i \cos(\theta_i) = 0$$

That is, we do not see a picture like this: where all $\theta_i < \pi/2$

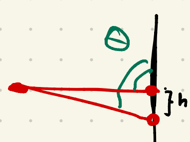




Proof: we use a variational argument:

Move $f(a)$ up by distance h (very small)

The length of $f(\gamma_i)$ increases by
$$-\cos(\theta_i)\cdot h + O(h^2)$$
[That is, curvature only contributes to length to second order ...]



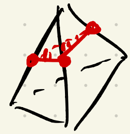
At the minimum length the first order terms cancel, so $\sum_i \cos(\theta_i) = 0$. //

(II) Faces of straight maps:

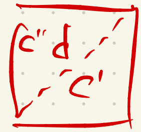
Sp. $f: S \to (M, K)$ straight. $\Gamma_f = f^{-1}(K^{(2)})$. Note

$C \subset S - \Gamma_f$ (component) is a disk. We homotope f to ensure

(*) if $|\partial C| = 3$ (triangle) then $f(C)$ is a linear triangle in Δ^3

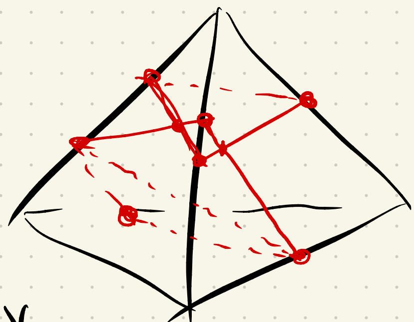


(*) if $|\partial C| = 4$ (quad) then we choose a diagonal $d \subset C$ and make $f(d)$ straight



let $C - d = C' \sqcup C''$ and require $f(C')$, $f(C'')$ be linear triangles.

Choose diagonals consistently. Now we have



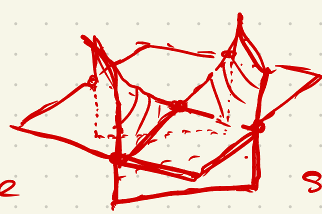
Corollary: Suppose $f: S \to (M, K)$, $g: T \to (M, K)$ are PL minimal. Suppose $a \in S$, $b \in T$ have $f(a) = g(b) \in e \in K^{(1)}$. Then there are open sets $U \subset \Gamma_f$, $V \subset \Gamma_g$ (*) with $a \in U$, $b \in V$ so that either

(*) $f(U) = g(V)$ or

(*) $g(V)$ meets both sides of $f(U)$.

[(*) Note difference from Casson's notes]

Picture

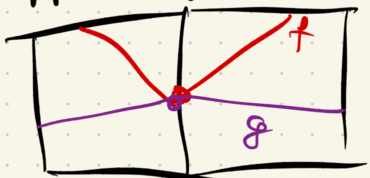


Proof: Let $\alpha_i \subset \Gamma_f$, $\beta_i \subset \Gamma_g$ be the adj edges to a, b in S, T respectively.

Let $e \in K^{(1)}$ be the edge containing $f(a)=g(b)$

Let θ_i, ϕ_i be the angles of α_i, β_i with e

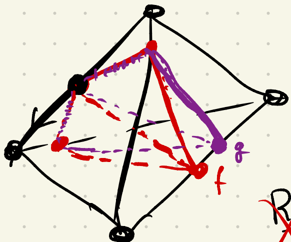
For a contradiction suppose f is above g at $f(a)=g(b)$. So



$\theta_i \le \phi_i$ for all i

and $\theta_i < \phi_i$ for some i

Thus $\sum_i \cos(\theta_i) > \sum_i \cos(\phi_i)$ $\divideontimes$



Choice of diagonals made once per quad type per tetrahedron.

No. See (*) on previous page.

~~Rmk: We realise (!) that the diagonals for f must be included in the one-skeleton Γ_f and so contribute to $\ell(f)$ (and also the angles θ_i etc).~~

III Disjointness of minimal surfaces

Picture of the exchange and round off trick

up one dimension: surfaces in 3-mfd

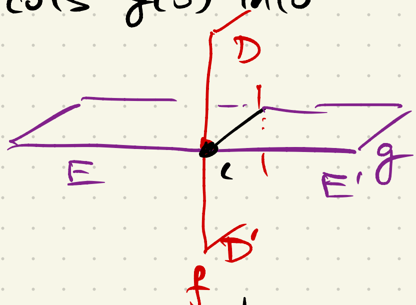
Def: A map $f: S^2 \longrightarrow M$ is essential if it is not null-homotopic.

Recall, any disk surgery of an ess. map f gives maps f', f'' at least one of which is ess.

Lemma 5.5: Suppose $f, g : S^2 \longrightarrow (M, K)$ are least area and essential. Suppose f, g both 1-1. Then either (a) $f(S^2) = g(S^2)$ or (b) $f(S^2) \cap g(S^2) = \phi$.

Proof

① Suppose $f \cap g \neq \phi$ and f transverse to g. Suppose also $f \cap g$ misses $K^{(1)}$. So fix C a compt of the intersection. C is simple so cuts $f(S)$ into disks D, D' and cuts $g(S)$ into disks E, E'.



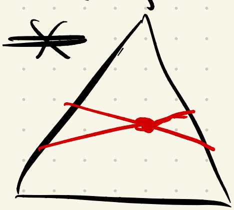
Define $A(D) = (w(D), \ell(D))$ just as for closed surfaces.

Suppose $A(D) = \min \left\{ \begin{matrix} A(D), A(D') \\ A(E), A(E') \end{matrix} \right\}$

Define $h, h' : S^2 \longrightarrow (M, K)$ immersions with
$$h(S^2) = D \underset{c}{\cup} E \ , \quad h'(S^2) = D \underset{c}{\cup} E'$$
Suppose h is ess. (at least one is, by above).
Normalise, surger, straighten h to obtain h^*.
If we reduce $w(h)$ then as $A(h) \leq A(f) = A(g)$ we have $A(h^*) < A(h)$ $\divideontimes$.

Exercise: $h(S^2) \cap \Delta^2$ has no loops.

Finally, if we straighten h in a face then $\ell(h^*) < \ell(h) \leq \ell(f) = \ell(g)$ $\divideontimes$



② Suppose $f \cap g$ meets $K^{(1)}$ or $f \cap g$ not transverse.