Intro to three-mfds [Lecture 7 2021-03-03
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Last time: Existence and uniqueness of connect sum decompositions of 3-mfds

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Question: what about prime decomp in dim 4?
Answer: $(S^2 \times S^2) \# \overline{\mathbb{CP}^2} \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \# \overline{\mathbb{CP}^2}$

(I) Matveev complexity: Suppose M is a cpt conn three-manifold.

Def: The Matveev complexity of M is
$c(M) = \min \{ n \in \mathbb{N} \mid$ there is some triang K of M with n tetrahedra $\}$

Example: $c(S^3) = 1$. [Rmk: Orig. definition instead counts # of vertices in special spine for M so $c(S^3) = c(\mathbb{RP}^3) = c(L(3,1)) = 0$ for all other irred mfds def's agree.]

Rmk: Suppose $M \not\cong S^3$. Define $M_0 = S^3$
$M_{n+1} = M_n \# M$.

Exercise: $c(M_n)$ is linear in n.
[give upper and lower bounds]

Open: Compute, for p, q, the complexity of $L(p,q)$

Exercise: Suppose S_g is the surface of genus g.
(a) Show $S_g \times S^1$ is irred.
(b) Show $c(S_g \times S^1)$ is linear in g.

Exercise: We call M (closed, conn) ~~toroidal~~ toroidal if $\mathbb{Z}^2 < \pi_1(M)$. Find irred, ator. mfds M_n so that $c(M_n)$ grows linearly with n.

II) Sphere theorem:

Thm 6.1: Suppose M^3 is a closed conn 3-mfd. Suppose $\pi_2(M) \neq 1$. Then, either M^3 contains an emb. $\mathbb{RP}^2$ or M is not prime.

Question: what about dimension four?

Exercise: Show $X = S^2 \times S^2$ is irreducible But $\pi_3(X) \cong \mathbb{Z}^2$.

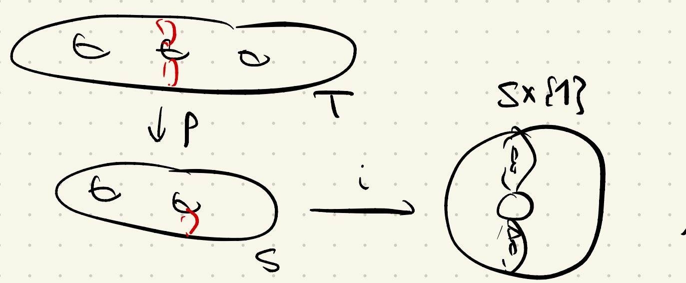
III) PL minimal surfaces:

Given $f: S \to M$ we'd like to "improve" f to make it be in "good position": namely so it is self transverse.

Remark: We cannot hope to improve f to become an embedding. Eg consider a finite covering $p: T \to S$ of surfaces, define $i: S \to S \times S^1$ and define

$$x \longmapsto (x,1)$$

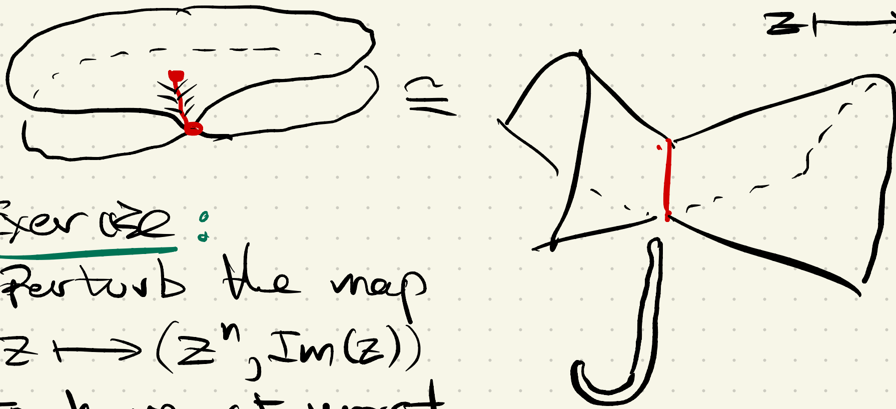
$f: T \to S \times S^1$ given by $f = i \circ p$



we should try to homotope f so self intersections are transverse

Definition: A Whitney umbrella (branch point) is a map modelled on $f:\mathbb{C}\to\mathbb{C}\times\mathbb{R}$

$$z\mapsto (z^2, \mathrm{Im}(z))$$



Not so good in the rain.

Exercise: Perturb the map $z\mapsto (z^n, \mathrm{Im}(z))$ to have at worst simple branch points.

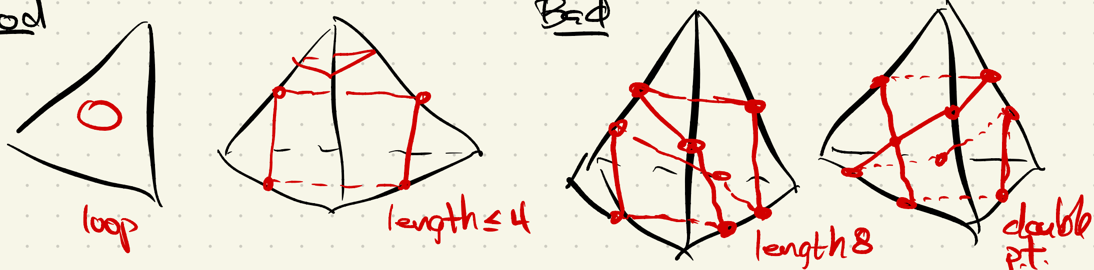
Lemma 5.1: Suppose (M,K) is triang. 3-mfd. Suppose S is a closed surface. Fix $f:S\to M$. Via homotopy we may ensure

(0) f is transverse to K.

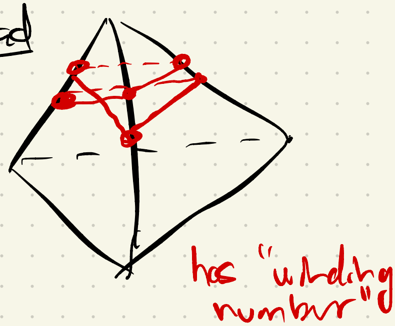
(1) f is least weight in homotopy class
$$\left(\mathrm{wt}(f)=w(f)=|f^{-1}(K^{(1)})|\right)$$

(2) for each tet Δ^3 of K, for each component c of $f^{-1}(\partial\Delta)$, the curve $f(c)\subset\partial\Delta^3$ is either (i) a loop in a face (simple closed curve)
(ii) normal curve of length ≤ 4.

Good | Bad

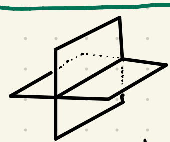


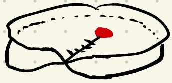
Bad

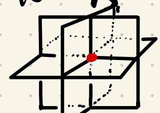


Proof: Homotope $f: S \to M$ to arrange (0) and (1). We also arrange that the self intersections of f (lie in one-submfd of M) are transverse to K.

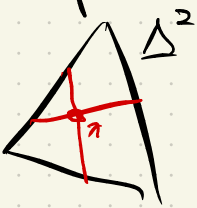
Pictures:


double pts


simple branch points

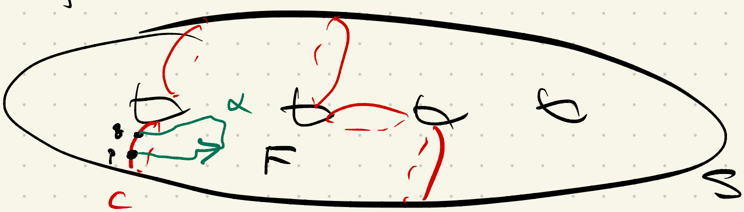

triple point.

We now must eliminate double points in faces coming from curves of the following kind: Fix Δ^3. Let C be a curve of $f^{-1}(\partial \Delta^3)$. Suppose $f|C$ is not injective. Say $p, q \in C$ map to $x = f(p) = f(q)$ in a face Δ^2 of Δ^3.



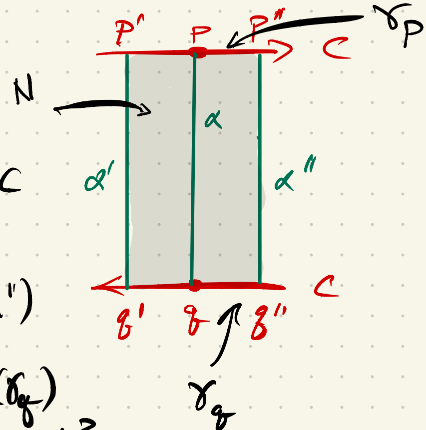
Fix $F \subset S - f^{-1}(\partial \Delta^3)$ the component meeting C and so that $f(F) \subset \Delta^3$.

Cartoon



Rmk: F is a subsurface of S by transversality.
Fix α prop. emb in F, an arc connecting p to q. Let $N = N(\alpha)$ be a very small product neigh of α in F.

local picture



Def: α', α'' are the compts of $\partial N - C$

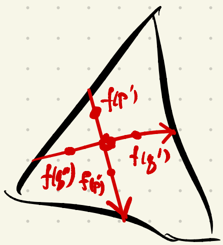
Let γ_p, γ_q be the compts of $\partial N - (\alpha' \cup \alpha'')$

γ_p runs from p' to p''

γ_q runs from q'' to q'

Arrange for $f(\gamma_p), f(\gamma_q)$ strictly contained in Δ^2.

In Δ^2



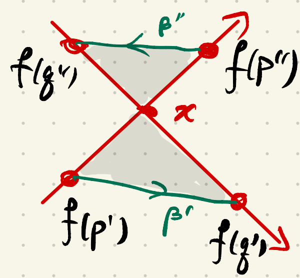
In Δ^3

Cartoon of $f(N)$

$f(N) \subset \Delta^3$

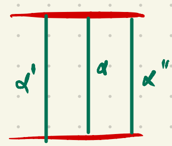
Pick arc β' in Δ^2 conn. $f(p')$ to $f(q')$ and also
" β'' " " " $f(p'')$ " $f(q'')$.

Pick them close to x to ensure
$$(\beta' \cup \beta'') \cap f(S) = \partial\beta' \cup \partial\beta''.$$

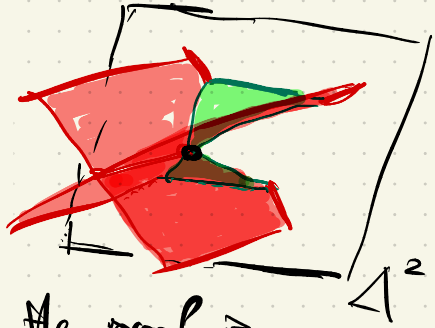


Homotope f (supp. in small neigh of N) to make $f(N)$ lie in Δ^2, with $f(\alpha') = \beta'$, $f(\alpha'') = \beta''$

Now push (image of) N out of Δ^3.



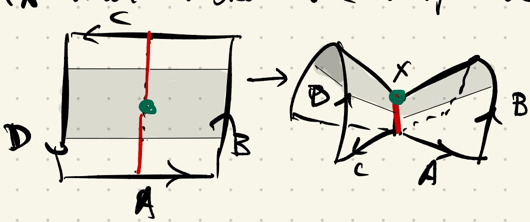
Picture after homotopy



That is, we remove a double point in a face at the cost of producing a Whitney umbrella in a tetrahedron. The rest of the proof is similar to normalising sphere systems.
[that is get rid of bent arcs, long normal curves via weight reducing homotopies] // 5.1

Question: In your final picture $f(\alpha)$ is a point?

Answer: on right; fix that inside the neigh. tetra.



Definition: M, K, S as before. A map

$f : S \to (M, K)$ is normal if

(i) $f \pitchfork K$ (transverse)

(ii) all curves of $f \cap \partial\Delta^3$ normal

(iii) all cpts of $f \cap \Delta^3$ are disks.

[That is, the cpts of $f^{-1}(\Delta^3)$ are disks D_i and $f|D_i$ is embedding]

Def: A collection $\{f_\alpha : S_\alpha \to (M, K)\}_\alpha = \mathcal{G}$ of maps is a normal class if

(i) $\mathcal{C}$ is closed under homotopy

(ii) $\mathcal{C}$ does not contain a null-homotopic map

(iii) $\mathcal{C}$ is closed under disk surgery along null homotopic (in M) separating (in S) simple closed curves (in S).

$f|c$ is null homotopic via $H: D \to M$

glue in two copies of D and extend $f|S - n(c)$ via two copies of H.

Example: Suppose $F \xrightarrow{f} M \to S^1$ is a surface bundle. Then the homotopy class of f is a normal class. [Exercise]

Exercise: Call $f: F \to M$ π_1-injective if the induced homomorphism $f_*: \pi_1(F) \to \pi_1(M)$ is injective. Show the homotopy class of f is a normal class.

Lemma 5.2: Suppose $\mathcal{C}$ is a normal class. Then for any triang K of M; $\mathcal{C}$ contains a normal map. Proof: Exercise. //

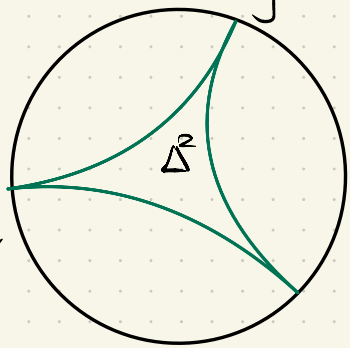
(IV) Hyperbolic faces: Suppose (M, K) is triang. 3-mfd. We place a metric on $K^{(2)} - K^{(0)}$ as follows.

(1) edges isometric to $\mathbb{R}$

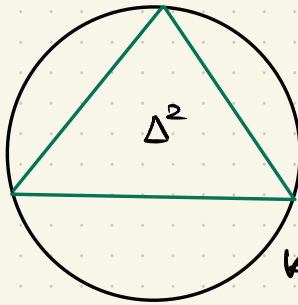
(2) faces isometric to ideal triangle in $\mathbb{H}^2$ (glued to edges via isometries).

[Rmk: There are choices here: and the metric on $K^{(2)} - K^{(0)}$ may not be complete.]

Pictures



Poincaré



Klein

We use the Klein model

(i) Geodesics are $\mathbb{E}^2$ line segments

(ii) vertices are inf far away.

We arrange matters so that the affine structures coming from (a) Klein model, (b) model tetrahedra } agree.

(V) PL area (after Jaco and Rubinstein)

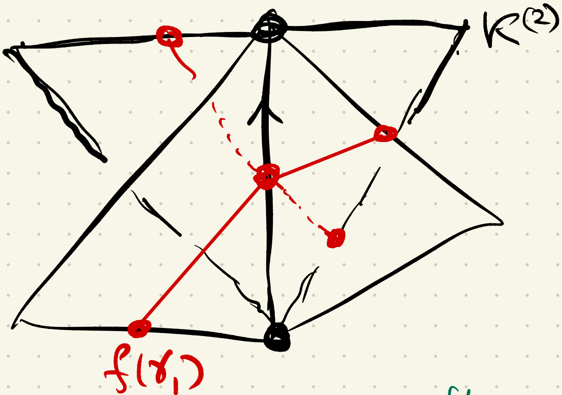
Definition: Suppose $f: S \to (M, K)$ is a normal map. So $\Gamma = \Gamma_f = f^{-1}(K^{(2)})$ is a finite graph in S. Let $\{\gamma_i\}$ be the edges of Γ. So $f(\gamma_i)$ is an arc in some face Δ of $K^{(2)}$.

Define $\ell(f) = \text{length}(f) = \sum_i \text{length}(f(\gamma_i))$

The **PL area** of f is

$$A(f) = (w(f), \ell(f)) \text{ ordered lexicograp.}$$

Picture



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