Please let me know if any of the problems are unclear, have typos, or have mistakes. Please turn in your solution to Exercise 3.4 on Friday (2020-02-14) before noon.

Exercise 3.1. Suppose that F is a vector space over $\mathbb{R}$. Suppose that $\pi\colon E \to B$ is a *vector bundle* with fibre F. Compute the homology groups $H_*(E)$ in terms of $H_*(B)$.

Exercise 3.2. [Challenge] Here is a hands-on definition of $\mathrm{UT}(S^n)$, the unit tangent bundle to the n–sphere.

$$\mathrm{UT}(S^n) = \{(u, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : |u| = |v| = 1, \langle u, v\rangle = 0\}$$

Here $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^{n+1}$. Compute the homology groups $H_*(\mathrm{UT}(S^n))$.

Exercise 3.3. A natural transformation $\delta\colon F \to G$ is called a *natural isomorphism* if there is another natural transformation $\epsilon\colon G \to F$ so that both $\delta \circ \epsilon$ and $\epsilon \circ \delta$ are identities. Now suppose that (X, Y) is a pair of spaces and Q is an R–module. Fix $k \in \mathbb{Z}$. Show that there is a natural isomorphism between the functors C and D where

$$(X, Y) \overset{C}{\mapsto} C^k(X, Y; Q) = \mathrm{Hom}_R(C_k(X, Y); Q)$$

and

$$(X, Y) \overset{D}{\mapsto} D^k(X, Y; Q) = \ker(C^k(X; Q) \to C^k(Y; Q))$$

Exercise 3.4. Suppose that X is a space and R is a commutative ring with unit. Let $\epsilon \in C^0(X, R)$ be the *augmentation homomorphism*: for all singular zero-simplices σ^0 we have $\epsilon(\sigma^0) = 1_R$. Now prove that the cup product at the level of cochains is:

- R–linear in both coordinates,
- associative, and
- has $\epsilon \in C^0(X, R)$ as its identity element.

Show, by means of an example, that the cup product at the level of cochains is not graded commutative.

Exercise 3.5. Set $R = \mathbb{R}$.

- Prove that $H_2(S^1) \cong 0$.
- Fix $\omega \in H^1(S^1; \mathbb{R})$ to be the homology class of the winding number cocycle, as defined in lecture. Prove that $\omega \cup \omega = 0$.
- [Challenge] Give *direct* proofs of the above: that is, from the definitions while not using various huge theorems.
- [Challenge] More generally, give a direct proof that $H_k(S^1)$ vanishes for $k > 1$.