Lecture 6 [10/18-20, 21]

The def. theorem

Suppose M is compact connected. Suppose $F \subset \partial M$ is a compact connected, and $F \neq \emptyset$. Suppose $H \subset \pi_1(F)$. Suppose

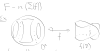
$f : (\Sigma, \partial\Sigma) \longrightarrow (M, F)$ is a general position map with $[f(\partial\Sigma)] \notin H$.

Thm (?) There is a proper curve $g : (D, \partial D) \longrightarrow (M, F)$ s.t.
(1) $[g(\partial D)] \notin H$
(2) g is made from the sectors of f. [with real-analytic curves]

Lemma (Simple)
Given M, F, H, f as above there is a general position map $f : (D, \partial D) \longrightarrow (M, F)$ which has ≤ 1 sector, and f is an immersion.
[Note: $\chi(f) = 0$.]

Ex Exercise Take homotopic simple loops γ_0, γ_1 on $\partial M \cong$ a surface with $\chi(\partial M) < 0$. Prove that if $\gamma_0 \neq \gamma_1$ then $\gamma_0 \not\simeq \gamma_1$.

Recall $f : (\Sigma, \partial\Sigma) \to (M, F)$ gives rise to $\Sigma(f) = \{x \in \Sigma \mid |f^{-1}(f(x))| > 1\}$.
sectors of f are components of $F - \pi(\Sigma(f))$.
Ex



three sectors

Lemma (Double) Suppose M, F, H, f are as above with $\chi(\partial M) = \chi(F) = 0$. Then [the theorem holds]. There is a curve $g : D \to M$ as desired.

Ex Exercise Idea: Perform surgery to reduce $\#(f(\partial\Sigma) \cap \partial M)$ [or use covers]

Note! Triple points must be treated with greater respect!

The tower construction [Step 1]

We assume $\#(f) > 0$ [$\leq$ pf.]
Set $M_1 = M$
$F_1 = F$
$f_1 = f$
Define $N_1 = N(f(\partial\Sigma))$
Picture

Define $G_1 = F_1 \cap N_1$ $N_1 = N(G_1)$
[Note G_1 contains the loop $f(\partial D)$.]
Define $\iota_1 : (N_1, G_1) \hookrightarrow (M, F)$
Define $H_1 = \iota_1^{-1}(H) \subset \pi_1(G_1)$

$(D, \partial D) \xrightarrow{f_1} (M_1, F_1) \cong (N_1, G_1)$
Note $[f_1(\partial D)] \notin H_1$.

Step k We have
$M_k, F_k, f_k, N_k, G_k, H_k$
as usual.

Case (1) Suppose N_k has no connected double cover. Then set rank and the construction of the tower is complete.

Case (2) [Otherwise] Suppose
Then $M_k \xleftarrow{\alpha} N_k$ is a connected double cover.
Define $F_{k1} = \tilde{\pi}_{k-1}(G_k)$ Since $f_k : (D, \partial D) \to (N_k, G_k)$ is a map and $\pi_1(D) = 1$ we choose a lift
$f_{k1} : (D, \partial D) \to (M_{k1}, F_{k1})$
of f_k to $\tilde{F}_{k1}$ [Choose one of the two possible lifts.]
Define $N_{k1} = N(f_{k1}(D)) \subset M_{k1}$
$G_{k1} = F_{k1} \cap N_{k1}$
Define $H_{k1} = (\alpha\iota_k)^{-1}(\pi_1(N_k))$
[Here $\iota_{k1} : (N_k, G_{k1}) \hookrightarrow (M_k, F_k)$]
Picture $\alpha : (M_{k1}, F_{k1}) \to (M_k, G_k)$ are covers of the same thing
$(D, \partial D) \xrightarrow{f_{k1}} (M_{k1}, F_{k1}) \xrightarrow{\alpha} (N_k, G_k)$
[Rank. We are sort of following Hempel's notation.]

Lemma (Tower) $\chi(F_{k1}) < \chi(F_k)$
Ex Consider f_{k1} and f_{k2} be two lifts of f_k to M_{k1}.
Check $f_{k1}(D) \cap f_{k2}(D) \neq \emptyset$
So they must be at least a double arc in a triple point.
Thus eventually N_k has no double cover [or we reach the top of the tower].

Part Two

View from the top of the tower

Question: Why is $f_{\text{ker}}(D) \cap \int_{\text{im}} (D)$ non-empty?

Answer: N_k def retracts to $\int_k(D)$

So take any PL α from $f_{\text{ker}}(D)$ to $f_{\text{im}}(D)$ and apply the (LLL-f) the def retraction

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Back to the view: N_k has no double cover. Thus, by crushing to $[\alpha^+ \alpha^-]$ N_{α^+} is orientable and all components of ∂N_{α^+} are two spheres. Let $S_0 \subseteq \partial N_{\alpha^+}$ be the two sphere containing G_{α^+}

Picture:

Rmk: G_{α^+} is thus cover and planar

So $\pi_1(G_{\alpha^+})$ is generated by the (finitely) classes $\{[\partial D]\}_{D \subseteq \partial N_{\alpha^+}}$

Note: that $[\partial \partial D] \neq H_{\alpha^+} \triangleleft \pi_1(G_{\alpha^+})$

So, some $D \subseteq \partial G_{\alpha^+}$ also lies outside $\cdot$ $(D) \notin H_{\alpha^+}$

JCT: γ bounds a disk $E \subset \overline{S_0 - G_{\alpha^+}}$

Note: After isotopy. [check!] we find a proper embedding

$g_{\alpha^+}: (D,\partial D) \longrightarrow (N_{\alpha^+}, \gamma)$

where: (i) $[g_{\alpha^+}|_{\partial D}] = [D] \notin H_{\alpha^+} \triangleleft \pi_1(G_{\alpha^+})$

(ii) g_{α^+} is a union of sectors of $\int_{\alpha^+}$ [at most one of each]

[Because] [Idea: Apply def retraction $N_{\alpha^+} \to E(D) \cong E$]

Climb down: Suppose we have fixed $g_k: (D,\partial D) \xrightarrow{\text{proper}} (N_k, G_k)$ s.t. (i) $[g_k|_{\partial D}] \notin H_k$

(ii) g_k made from sectors of $\int_k$

Check: $\alpha \circ g_k$ has the correct properties wrt M, F_k

Note: $p_k \circ i_k \circ g_k = h_{k+1}$

then h_{k+1} has $\leq$ as many double arcs and curves. Tie form arcs to h_{k+1} to obtain g_{k+1}, an embedding [Lemma (Double)]

Then $i_\alpha \circ g_\alpha$ is the desired emb.

Exercises: Use disk thm to prove

Disk

① Characterization of $\mathbb{U}$ [$h \cong \mathbb{U}$ iff $\pi_1(N) \cong \mathbb{Z}$]

② Characterization of all handlebodies [M is any, simply conn, irreducible 3-mfd with $\chi(M) \cong 2\mathbb{Z}_2(g-1)$ then $M \cong V_g$ ($g \geq 0$)] [$\mathbb{U} = V_0$]

③ Dehn's Lemma: If $\alpha \subset \partial M$ is a simple closed curve [M is a 3-mfd] and $[\alpha] = 1 \in \pi_1(M)$ then α bounds a proper emb disk $D \subseteq M$

§1 Incompressible surfaces

Def: Suppose $(F, \partial F) \to (M, \partial M)$ is proper emb

Def: $N(F) \cong$ be a reg nbh. So $N = N(F)$ is an I-bundle

Define: $\partial_1 N = N|_{\partial F = F \times \{0,1\}}$

$\partial_2 N = \overline{\partial N - \partial_1 N}$ [∂_2 is called frontier]

Contour: [$I \to N \to F$ is an I-bundle]

Def: F a two-sided surface in M if any (thus all) of the following

(i) $\partial_1 N \cong F \times \{-1,1\}$

(ii) $N \cong I \times F$ (as bundles!)

(iii) F separates N

(iv) ∂F separates ∂N

Basic

Ex: $P^2 \setminus \{pt\} \subseteq P^2 \times S^1$ is two-sided

Three properties "fall" to each other

(i) orientability of F

(ii) " " M

(iii) one curve two intersections

[Exercise: Which selections of these determine the remainder]

Ex: If $F \subset S^3$ is closed embedded then F is orientable and 2-sided