Lecture 5

Exercise: Suppose $K \subseteq S^3$ is a knot and $X_K = S^3 - n(K)$ is the knot exterior.

Prove: $K \cong U$ (isotopic) if and only if $\pi_1(X_K) \cong \mathbb{Z}$

Business ① The registration page for the Minsky Lectures 3 now up

② The paper report topics should be papers (not books) and should be posted on the Google group.

[Soft deadline of next week]

Ⅸ General position and maps

Suppose M^3 is connected and compact. Suppose F is a conn. compact surface. Say

$$f: (F, \partial F) \longrightarrow (M, \partial M)$$

is proper if $f^{-1}(\partial M) = \partial F$

We need tools to "improve" f

Eg f has nice algebraic properties and we'd like to modify f, keeping the algebraic properties, and making f an embedding

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Def: We say f is a general position map if f is smooth and (i) f is an immersion away from a finite collection of simple branch points

(ii) f is injective away from a finite collection of double arcs and curves

(iii) f is a most two-to-one away from a finite collection of triple points

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Thus: For any $x \in F$

$$|f^{-1}(f(x))| \leq 3$$

Define the singular set $\Sigma(f) \subset F$ by

$$\Sigma_1(f) = \{x \in F \mid |f^{-1}(f(x))| \geq 2\}$$

This set measures the failure of f to be an embedding

Define: for f in general position the complexity of f is

$$c(f) = (s(f), t(f), d(f))$$

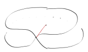
where $s = \#$ of simple branch pts

$t = \#$ " triple "

$d = \#$ double arcs and curves

we compare $c(f)$ to $c(f')$ lexicographically

Pictures

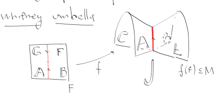


This is the model for a simple branch point. It is the graph of

$$f: D \longrightarrow D \times \mathbb{R}$$

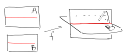
$$z \longmapsto (z^2, \mathrm{Im}(z))$$

This gen pos map is also called a Whitney umbrella

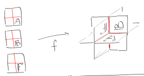


Since f is an immersion (away from the simple branch points, the failure of injectivity is (almost!) a one manifold

Picture



Finally general position ensures that f is at worst three-to-one



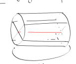
Thm: Any proper $f: (F, \partial F) \to (M, \partial M)$ is properly homotopic to a general position map

Pf [See Hempel's book or Bing's book for a discussion of the PL case] //

Part Two

Examples of general position maps

(1)



$M = D^2 \times S^1$

$F = A^2 \cup A^2$

$c(f) = (0,0,1)$

Exercise: Compute M, F, f for the remaining 7 possible gluings.

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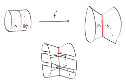
(2) $M = B^3$, $F = A^2$, $c(f) = (2,0,1)$



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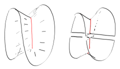
(3) What is F here?



Glue pair of umbrellas with $90°$ rotation

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(4)

(5)

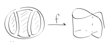
$c(f) = (0,0,1)$

Swaps and sectors

Define the sectors of f to be the components of

$$F - n(\Sigma(f))$$

Eg



three sectors

Suppose $c(f) = (0,0,d)$. Suppose α is either a double arc or a " curve with rotation being zero or $\frac{1}{2}$ (well, do the exercise of 7 things)

Then we may perform a swap

Ie cut along $n(\alpha)$ and reglue the sectors to reduce complexity.

or the other swap, there are two

Note that a swap may change F to F' of different topological type!

So we get

$$f : F \longrightarrow M^3$$
$$\Downarrow \text{swap}$$
$$f' : F' \longrightarrow M^3$$

with $c(f') = (0,0,d-1) < (0,0,d) = c(f)$

X Papakyriakopoulos' Disk Theorem

Disk Theorem Suppose M is conn. compact. Suppose $F \subseteq \partial M$ is conn. compact and $\neq \emptyset$. Suppose $N \triangleleft \pi_1(F)$. Suppose $D \cong \mathbb{D}^2$

$$f : (D, \partial D) \longrightarrow (M, F)$$ is proper and general position map.

Suppose that $[f|\partial D] \notin N$.

Then. There is an embedding [proper]

$$g : (E, \partial E) \longrightarrow (M, F)$$ with

(i) $E \cong \mathbb{D}^2$

(ii) $[g|\partial E] \notin N$

(iii) g is obtained by glue sectors (at most one of each) of f

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Corollary [Dehn's lemma]

Suppose $\alpha \subset \partial M$ is simple closed curve and $[\alpha] = \mathbb{1} \in \pi_1(M)$. Then α bounds an emb. disk E in M.

Exercise Disk Thm $\Rightarrow$ Dehn's Lemma

Exercise Suppose M^3 is conn. cpt, oriented, irreducible and $\pi_1(M) \cong \mathbb{F}_g$ (free group). Prove $M \cong V_g$ (handlebody)
