Lecture VII.

Cleaning Lemma: Suppose that μ is a complete marking of S. Then there exists at least 1 and at most $(K)^{\xi(S)}$ complete clean markings μ' so that ⊛ $\text{base}(\mu') = \text{base}(\mu)$

⊛ $\forall \alpha \in \text{base}(\mu)$ $d_\alpha(\mu, \mu') \leq 3$.

$d_\alpha \equiv d_{C(S^\alpha)}$

S^α = annular cover s.t. α lifts homeomorphically.

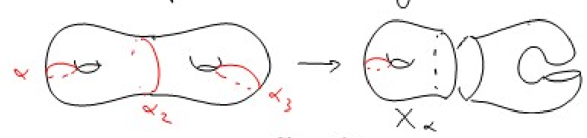
Pf: Exercise [Lemma 2.4 of MM II].//

Finiteness Lemma: $\mathcal{M}^\circ(S)/_{MCG(S)}$ is finite.

Pf: Step 1: Show that $P^\circ(S)/_{MCG(S)}$ is finite: Here $P^\circ(S) = \{\text{pants decomp}\}$ (up to isotopy)

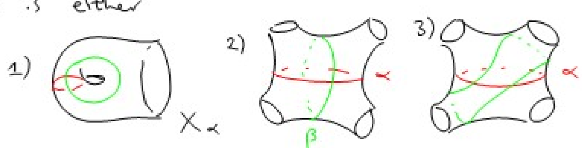
Exercise.

Step 2: If $\alpha \in \text{base}(\mu)$ define $X_\alpha \subseteq S \setminus (\text{base}(\mu) \setminus \{\alpha\})$ to be the component containing α.



or perhaps X_{α_2} = 

After Dehn twisting about α the clean transversal β is either

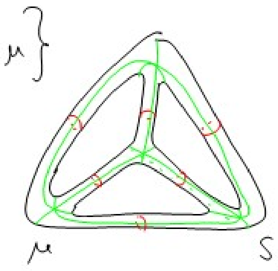


These differ by a half twist of X_α but not necess. by a homeo. of S.//

Def: $\text{Stab}(\mu) < MCG(S)$ is defined to be

$\{g \in MCG(S) \mid g(\mu) = \mu\}$

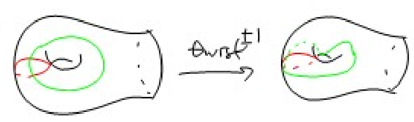
In the example $\text{Stab}(\mu) = \text{Sym}_4$

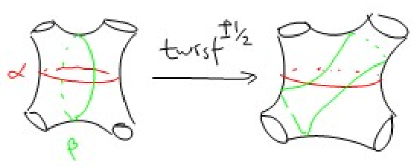


Lemma: $|\text{Stab}(\mu)| \leq K$ some uniform K.//

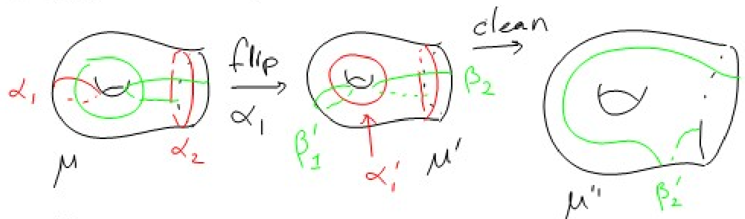
II Elementary moves on $\mathcal{M}^\circ(S)$

Twist





Flip (and clean)



Note we may choose $d_\alpha(\mu', \mu'') \leq 3$ $\forall \alpha$ in base(μ'). by cleaning lemma.

Flips also exist when $X_\alpha \cong S_{0,4}$. More complicated to draw [Do this!]

Call "twist", "flip and clean" elementary moves on $\mu \in \mathcal{M}^\circ(S)$. Define $\mathcal{M}(S)$ to be the marking graph with $\mathcal{M}^\circ(S)$ as vertices and elem. moves as edges.

Connectedness Lemma: $\mathcal{M}(S)$ is connected.

Pf: Step 1, $P(S)$ is connected [flips give edges] [Hatcher and Thurston give a 2-skeleton for $P(S)$ and prove $\pi_1(P^2(S)) = \{1\} \Rightarrow MCG(S)$ is fin. presented.]

Step 2: Any pair of clean transversals β, β' to $\alpha \in \text{base}(\mu)$ are connected by (half) twists. // We'll give another proof of 1) later.

Fix a base point $\mu \in \mathcal{M}^\circ(S)$.

Fix $X \subseteq MCG(S)$ a finite gen set.

Let $|g|_X$ be the length of g in the word metric.

Thm: $(MCG(S), d_X) \underset{\text{quasi isom}}{\cong} \mathcal{M}(S)$

$$g \longmapsto g(\mu)$$

Rmk. Also, $\forall \mu, \nu \in \mathcal{M}(S)$, $\forall g \in MCG(S)$

$$d_{\mathcal{M}}(\mu, \nu) = d_{\mathcal{M}}(g(\mu), g(\nu)).$$