Class notes written by Simon Thomas These notes are transcribed from class notes written by Professor Simon Thomas. The notes follow the notation of Enderton's *A mathematical introduction to logic* but may be read independently.

1 A few introductory remarks

In mathematical reasoning, logical arguments are used to deduce the consequences (called *theorems*) of basic assumptions (called *axioms*).

Question 1.1. What does it mean for one sentence to "follow logically" from another sentence?

Question 1.2. Suppose that a sentence σ does *not* follow logically from the set T of axioms. How can we prove that this is so?

We will begin the course by studying some basic set theory.

Motivation:

1. We will need this material in our study of mathematical logic.
2. Set theory is a foundation for all of mathematics.
3. Set theory is beautiful.

Remark 1.3. In a couple of weeks we will come across a natural set-theoretic statement, the Continuum Hypothesis, which can neither be proved nor disproved using the classical axioms of set theory.

2 Basic Set Theory

Notation: $\{2,3,5\} = \{2,5,5,2,3\}$.
$\{0,2,4,6,\ldots\} = \{x \mid x \text{ is an even natural number}\}$.
$x \in A$ means "x is an element of A".
$\emptyset$ is the empty set.
$\mathbb{N} = \{0,1,2,3,\ldots\}$ is the set of natural numbers.
$\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$ is the set of integers.
$\mathbb{Q} = \{a/b \mid a,b \in \mathbb{Z},\ b \neq 0\}$ is the set of rational numbers.
$\mathbb{R}$ is the set of real numbers.

Axiom of Extensionality: Suppose that A, B are sets. If for all x,

$$x \in A \quad \text{iff} \quad x \in B$$

then $A = B$.

Definition 2.1. Suppose that A, B are sets. Then A is a *subset* of B, written $A \subseteq B$, iff for all x,

$$\text{iff} \quad x \in A \quad \text{then} \quad x \in B.$$

Example 2.2.

1. $\mathbb{N} \subseteq \mathbb{Z}$
2. If A is any set then $\emptyset \subseteq A$.

Proposition 2.3. *If $A \subseteq B$ and $B \subseteq A$, then $A = B$.*

Proof. Let x be arbitrary. Since $A \subseteq B$, if $x \in A$ then $x \in B$. Since $B \subseteq A$, if $x \in B$ then $x \in A$. Hence $x \in A$ iff $x \in B$. By the Axion of Extensionality, $A = B$. □

Definition 2.4. Let A, B be sets. The *union* of A and B, written $A \cup B$, is the set defined by

$$x \in A \cup B \quad \text{iff} \quad x \in A \quad \text{or} \quad x \in B.$$

Proposition 2.5. $A \cup (B \cup C) = (A \cup B) \cup C$

Proof. Let x be arbitrary. Then $x \in A \cup (B \cup C)$
iff $x \in A$ or $x \in B \cup C$
iff $x \in A$ or $(x \in B$ or $x \in C)$
iff $x \in A$ or $x \in B$ or $x \in C$
iff $(x \in A$ or $x \in B)$ or $x \in C$
iff $x \in A \cup B$ or $x \in C$
iff $x \in (A \cup B) \cup C$. □

Definition 2.6. Let A, B be sets. The *intersection* of A and B, written $A \cap B$ is the set defined by

$$x \in A \cup B \quad \text{iff} \quad x \in A \quad \text{and} \quad x \in B.$$

Exercise 2.7. Prove that $A \cap (B \cap C) = (A \cap B) \cap C$.

Proposition 2.8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof. Let x be arbitrary. Then $x \in A \cap (B \cup C)$
iff $x \in A$ and $x \in B \cup C$
iff $x \in A$ and $(x \in B$ or $x \in C)$
iff $(x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$
iff $(x \in A \cap B)$ or $(x \in A \cap C)$
iff $x \in (A \cap B) \cup (A \cap C)$. □

Exercise 2.9. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Definition 2.10. Let A, B be sets. The *set theoretic difference* of A and B, written $A \smallsetminus B$, is the set defined by

$$x \in A \smallsetminus B \quad \text{iff} \quad x \in A \quad \text{and} \quad x \notin B.$$

$\{1,2,3\} \smallsetminus \{3,4,5\} = \{1,2\}$.
$\mathbb{N} \smallsetminus \mathbb{Z} = \emptyset$.
$\mathbb{Z} \smallsetminus \mathbb{N} = \{-1,-2,-3,\ldots\}$.

Proposition 2.11. $A \smallsetminus (B \cup C) = (A \smallsetminus B) \cap (A \smallsetminus C)$

Proof. Let x be arbitrary. Then $x \in A \smallsetminus (B \cup C)$
iff $x \in A$ and $x \notin B \cup C$
iff $x \in A$ and not ($x \in B$ or $x \in C$)
iff $x \in A$ and ($x \notin B$ and $x \notin C$)
iff $x \in A$ and $x \notin B$ and $x \notin C$
iff $x \in A$ and $x \notin B$ and $x \in A$ and $x \notin C$
iff ($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$)
iff $x \in A \smallsetminus B$ and $x \in A \smallsetminus C$
iff $x \in (A \smallsetminus B) \cap (A \smallsetminus C)$. □

Exercise 2.12. Prove that $A \smallsetminus (B \cap C) = (A \smallsetminus B) \cup (A \smallsetminus C)$.

3 Functions

Provisional Definition:

Let A, B be sets. Then f is a *function* from A to B, written $f\colon A \to B$, iff f assigns a unique element $f(a) \in B$ to each $a \in A$.

What is the meaning of "assigns"? To illustrate our earlier comments on set theory as a foundation for mathematics, we shall reduce the notion of a function to the language of basic set theory.

Basic idea

For example, consider $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Then the graph of f is a subset of $\mathbb{R}^2$. We shall identify f with its graph.

To generalize this idea to arbitrary functions, we first need to introduce the idea of an *ordered pair*; ie a mathematical object $\langle a,b \rangle$ such that

$$\langle a,b \rangle = \langle c,d \rangle \quad \text{iff} \quad a = c \quad \text{and} \quad b = d. \qquad (*)$$

Definition 3.1. Let A and B be sets. Then the *Cartesian product* of A and B is the set

$$A \times B = \{\langle a,b \rangle \mid a \in A,\ b \in B\}.$$

Definition 3.2. f is a function from A to B iff the following conditions hold:

1. $f \subseteq A \times B$
2. For each $a \in A$, there is a unique $b \in B$ such that $\langle a, b \rangle \in f$.

In this case, the unique such b is said to be the value of f at a and we write $f(a) = b$.

In order to reduce the notion of a function to basic set theory, we now only need to find a purely set theoretic object to play the role of $\langle x, y \rangle$.

Definition 3.3. $\langle x, y \rangle = \{\{x\}, \{x, y\}\}$.

Finally, we must prove that with this definition, the set $\langle x, y \rangle$ satisfies $(*)$.

Theorem 3.4. *$\langle a, b \rangle = \langle c, d \rangle$ iff $a = c$ and $b = d$.*

Proof. ($\Leftarrow$): Clearly if $a = c$ and $b = d$ then $\langle a, b \rangle = \langle c, d \rangle$.
($\Rightarrow$): Conversely, suppose that $\langle a, b \rangle = \langle c, d \rangle$; ie

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}.$$

We split our analysis into three cases.

Case 1

Suppose that $a = b$. Then $\{\{a\}, \{a, b\}\}$ equals

$$= \{\{a\}, \{a, a\}\}$$
$$= \{\{a\}, \{a\}\}$$
$$= \{\{a\}\}$$

Since

$$\{\{c\}, \{c, d\}\} = \{\{a\}\}$$

it follows that

$$\{c\} = \{c, d\} = \{a\}.$$

This implies that $c = d = a$. Hence $a = c$ and $b = d$.

Case 2

Similarly, if $c = d$, we obtain that $a = c$ and $b = d$.

Case 3

Finally suppose that $a \neq b$ and $c \neq d$. Since

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$$

we must have that $\{a\} = \{c\}$ or $\{a\} = \{c, d\}$. Since $c \neq d$ the second option is impossible. Hence $\{a\} = \{c\}$ and so $a = c$.

Also $\{a, b\} = \{c\}$ or $\{a, b\} = \{c, d\}$. Clearly the first option is impossible and so $\{a, b\} = \{c, d\}$. Since $a = c$, we must have $b = d$. $\square$

Important remark When working with functions, it is almost never necessary to remember that a function is literally a set of ordered pair as above.

Definition 3.5. The function $f\colon A \to B$ is an *injection* (one-to-one) iff

$$a \neq a' \quad \text{implies} \quad f(a) \neq f(a').$$

Definition 3.6. The function $f\colon A \to B$ is a *surjection* (onto) iff for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$.

Definition 3.7. If $f\colon A \to B$ and $g\colon B \to C$ are functions, then their *composition* is the function $g \circ f\colon A \to C$ defined by $(g \circ f)(a) = g(f(a))$.

Proposition 3.8. *If $f\colon A \to B$ and $g\colon B \to C$ are surjections then $g \circ f\colon A \to C$ is also a surjection.*

Proof. Let $c \in C$ be arbitrary. Since g is surjective, there exists a $b \in B$ such that $g(b) = c$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$. Hence $(g \circ f)(a) =$

$$\begin{aligned} &= g(f(a)) \\ &= g(b) \\ &= c \end{aligned}$$

Thus $g \circ f$ is surjective. □

Exercise 3.9. If $f\colon A \to B$ and $g\colon B \to C$ are injections then $g \circ f\colon A \to C$ is also an injection.

Definition 3.10. The function $f\colon A \to B$ is a *bijection* iff f is both an injection and a surjection.

Definition 3.11. If $f\colon A \to B$ is a bijection, then the *inverse* $f^{-1}\colon B \to A$ is the function defined by

$$f^{-1}(b) \quad \text{equals the unique} \quad a \in A \quad \text{such that} \quad f(a) = b.$$

Remark 3.12.

1. It is easily checked that $f^{-1}\colon B \to A$ is also a bijection.
2. In terms of ordered pairs:

$$f^{-1} = \{\langle b, a\rangle \mid \langle a, b\rangle \in f\}.$$

4 Equinumerosity

Definition 4.1. Two sets A and B are *equinumerous*, written $A \sim B$, iff there exists a bijection $f\colon A \to B$.

Example 4.2. Let $\mathbb{E} = \{0, 2, 4, \ldots\}$ be the set of even natural numbers. Then $\mathbb{N} \sim \mathbb{E}$.

Proof. We can define a bijection $f\colon \mathbb{N} \to \mathbb{E}$ by $f(n) = 2n$. $\square$

Important remark It is often extremely hard to explicitly define a bijection $f\colon \mathbb{N} \to A$. But suppose such a bijection exists. Then letting $a_n = f(n)$, we obtain a *list* of the elements of A

$$a_0, a_1, a_2, \dots, a_n, \dots$$

in which each element of A appears exactly once. Conversely, if such a list exists, then we can define a bijection $f\colon \mathbb{N} \to A$ by $f(n) = a_n$.

Example 4.3. $\mathbb{N} \sim \mathbb{Z}$

Proof. We can list the elements of $\mathbb{Z}$ by

$$0, 1, -1, 2, -1, \dots, n, -n, \dots$$ $\square$

Theorem 4.4. $\mathbb{N} \sim \mathbb{Q}$

Proof. **Step 1** First we prove that $\mathbb{N} \sim \mathbb{Q}^+$, the set of positive rational numbers. Form an infinite matrix where the $(i,j)^{\text{th}}$ entry is j/i.

Proceed through the matrix by traversing, alternating between upward and downward, along lines of slope one. At the $(i,j)^{\text{th}}$ entry add the number j/i to the list if it has not already appeared.

Step 2 We have shown that there exists a bijection $f\colon \mathbb{N} \to \mathbb{Q}^+$. Hence we can list the elements of $\mathbb{Q}$ by

$$0, f(0), -f(0), f(1), -f(1), \dots$$ $\square$

Definition 4.5. If A is any set, then its *powerset* is defined to be

$$\mathcal{P}(A) = \{B \mid B \subseteq A\}.$$

Example 4.6.

1. $\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.
2. $\mathcal{P}(\{1,2,\dots n,\})$ has size 2^n.

Theorem 4.7. *(Cantor)* $\mathbb{N} \nsim \mathcal{P}(\mathbb{N})$

Proof. (The diagonal argument) We must show that there does *not* exist a bijection $f\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$. So let $f\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be any function. We shall show that f isn't a surjection. To accomplish this we shall define a subset $S \subseteq \mathbb{N}$ such that $f(n) \neq S$ for all $n \in \mathbb{N}$. We do this via a "time and motion study". For each $n \in \mathbb{N}$, we must perform:

1. the n^{th} decision: is $n \in S$?

2. the n^{th} task: we must ensure that $f(n) \neq S$.

We decide to accomplish the n^{th} task with the n^{th} decision. So we decide that

$$n \in S \quad \text{iff} \quad n \notin f(n)$$

Clearly S and $f(n)$ differ on whether they contain n and so $f(n) \neq S$. Hence f is not a surjection. $\square$

Discussion Why is this called the "diagonal argument"?

Definition 4.8. A set A is *countable* iff A is finite or $\mathbb{N} \sim A$. Otherwise A is *uncountable*.

eg $\mathbb{Q}$ is countable
$\mathcal{P}(\mathbb{N})$ is uncounable.

Theorem 4.9. *(Cantor) If A is any set, then $A \nsim \mathcal{P}(A)$.*

Proof. Supose that $f\colon A \to \mathcal{P}(A)$ is any function. We shall show that f isn't a surjection. Define $S \subseteq A$ by

$$a \in S \quad \text{iff} \quad a \notin f(a).$$

Then S and $f(a)$ differ on whether they contain a. Thus $f(a) \neq S$ for all $a \in A$. $\square$

Definition 4.10. Let A, B be sets.

1. $A \preceq B$ iff there exists an injection $f\colon A \to B$.

2. $A \prec B$ iff $A \preceq B$ and $A \nsim B$.

Corollary 4.11. *If A is any set, then $A \prec \mathcal{P}(A)$.*

Proof. Define $f\colon A \to \mathcal{P}(A)$ by $f(a) = \{a\}$. Clearly f is an injection and so $A \preceq \mathcal{P}(A)$. Since $A \nsim \mathcal{P}(A)$, we have $A \prec \mathcal{P}(A)$. $\square$

Corollary 4.12. $\mathbb{N} \prec \mathcal{P}(\mathbb{N}) \prec \mathcal{P}(\mathcal{P}(\mathbb{N})) \prec \dots$ $\square$

Having seen that we have a nontrivial subject, we now try to develop some general theory.

5 Cantor-Bernstein Theorem

Theorem 5.1. *Let A, B, C be sets.*

1. $A \sim A$

2. If $A \sim B$, then $B \sim A$.

3. If $A \sim B$ and $B \sim C$, then $A \sim C$.

$\square$

Exercise 5.2. If $A \preceq B$ and $B \preceq C$, then $A \preceq C$.

Theorem 5.3. *(Cantor-Bernstein) If $A \preceq B$ and $B \preceq A$, then $A \sim B$.*

Proof delayed

Theorem 5.4. *If A, B are any sets, then either $A \preceq B$ or $B \preceq A$.*

Proof omitted

This theorem is equivalent to:

Axiom of Choice If $\mathcal{F}$ is a family of nonempty sets then there exists a function f such that $f(A) \in A$ for all $A \in \mathcal{F}$. (Such a function is called a *choice function*.)

6 The Cantor-Bernstein Theorem (continued)

Some applications of the Cantor-Bernstein theorem

Theorem 6.1. $\mathbb{N} \sim \mathbb{Q}$.

Proof. First define a function $f\colon \mathbb{N} \to \mathbb{Q}$ by $f(n) = n$. Clearly f is an injection and so $\mathbb{N} \preceq \mathbb{Q}$.

Now define a function $g\colon \mathbb{Q} \to \mathbb{N}$ as follows. First suppose that $0 \neq q \in \mathbb{Q}$. Then we can *uniquely* express

$$q = \epsilon \tfrac{a}{b}$$

where $\epsilon = \pm 1$ and $a, b \in \mathbb{N}$ are positive and relatively prime. Then we define

$$g(q) = 2^{\epsilon+1} 3^a 5^b.$$

Finally define $g(0) = 7$. Clearly g is an injection and so $\mathbb{Q} \preceq \mathbb{N}$.

By Cantor-Bernstein, $\mathbb{N} \sim \mathbb{Q}$. $\square$

Theorem 6.2. $\mathbb{R} \sim \mathcal{P}(\mathbb{N})$.

We shall make use of the following result.

Lemma 6.3. $(0,1) \sim \mathbb{R}$.

Proof of Lemma 6.3. By Calc I, we can define a bijection $f\colon (0,1) \to \mathbb{R}$ by $f(x) = \tan(\pi x - \pi/2)$. $\square$

Proof of Theorem 6.2. By the lemma, it is enough to show that $(0,1) \sim \mathcal{P}(\mathbb{N})$. We shall make use of the fact that eact $r \in (0,1)$ has a *unique* decimal expansion

$$r = 0.r_1 r_2 r_3 \dots r_n \dots$$

so that

1. $0 \leq r_n \leq 9$
2. the expansion does not terminate with infinitely many 9s. (This is to avoid two expansions such as $0.5000\dots = 0.4999\dots$)

First we define $f\colon (0,1) \to \mathcal{P}(\mathbb{N})$ as follows. If

$$r = 0.r_0 r_1 r_2 \dots r_n \dots$$

then

$$f(r) = \{2^{r_0+1}, 3^{r_1+1}, \dots, p_n^{r_n+1}, \dots\}$$

where p_n is the n^{th} prime. Clearly f is an injection and so $(0,1) \preceq \mathcal{P}(\mathbb{N})$.

Next we define a function $g\colon \mathcal{P}(\mathbb{N}) \to (0,1)$ as follows: If $\emptyset \neq S \subseteq \mathbb{N}$ then

$$g(S) = 0.s_0 s_1 s_2 \dots s_n \dots$$

where

$$s_n = 0 \text{ if } n \in S$$
$$s_n = 1 \text{ if } n \notin S.$$

Finally, $g(\emptyset) = 0.5$. Clearly g is an injection and so $\mathcal{P}(\mathbb{N}) \preceq (0,1)$.

By Cantor-Bernstein, $(0,1) \sim \mathcal{P}(\mathbb{N})$. □

The following result says that "$\mathbb{N}$ has the smallest infinite size."

Theorem 6.4. *If $S \subseteq \mathbb{N}$, then either S is finite or $\mathbb{N} \sim S$.*

Proof. Suppose that S is infinite. Let

$$s_0, s_1, s_2, \dots, s_n, \dots$$

be the increasing enumeration of the elements of S. This list witnesses that $\mathbb{N} \sim S$. □

The Continuum Hypothesis (CH) If $S \subseteq \mathbb{R}$, then either S is countable or $\mathbb{R} \sim S$.

Theorem 6.5. *(Godel 1930s, Cohen 1960s) If the axioms of set theory are consistent, then CH can neither be proved nor disproved from these axioms.*

Definition 6.6. $\mathrm{Fin}(\mathbb{N})$ is the set of all finite subsets of $\mathbb{N}$.

Theorem 6.7. $\mathbb{N} \sim \mathrm{Fin}(\mathbb{N})$.

Proof. First define $f\colon \mathbb{N} \to \mathrm{Fin}(\mathbb{N})$ by $f(n) = \{n\}$. Clearly f is an injection and so $\mathbb{N} \preceq \mathrm{Fin}(\mathbb{N})$. Now define $g\colon \mathrm{Fin}(\mathbb{N}) \to \mathbb{N}$ as follows. If $s = \{s_0, s_1, s_2, \dots, s_n\}$ where $s_0 < s_1 < \dots < s_n$, then

$$g(S) = 2^{s_0+1} 3^{s_1+1} \dots p_n^{s_n+1}$$

where p_i is the i^{th} prime. Also we define $g(\emptyset) = 1$. Clearly g is an injection and so $\mathrm{Fin}(\mathbb{N}) \preceq \mathbb{N}$.

By Cantor-Bernstein, $\mathbb{N} \sim \mathrm{Fin}(\mathbb{N})$. □

Exercise 6.8. If $a < b$ are reals, then $(a, b) \sim (0, 1)$.

Exercise 6.9. If $a < b$ are reals, then $[a, b] \sim (0, 1)$.

Exercise 6.10. $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$.

Exercise 6.11. If $A \sim B$ and $C \sim D$, then $A \times C \sim B \times D$.

Definition 6.12. If A and B are sets, then

$$B^A = \{f \mid f\colon A \to B\}.$$

Theorem 6.13. $\mathcal{P}(\mathbb{N}) \sim \mathbb{N}^{\mathbb{N}}$.

Proof. For each $S \subseteq \mathbb{N}$ we define the corresponding characteristic function $\chi_S\colon \mathbb{N} \to \{0, 1\}$ by

$$\chi_S(n) = 1 \text{ if } n \in S$$
$$\chi_S(n) = 0 \text{ if } n \notin S$$

Let $f\colon \mathcal{P}(\mathbb{N}) \to \mathbb{N}^{\mathbb{N}}$ be the function defined by $f(S) = \chi_S$. Clearly f is an injection and so $\mathcal{P}(\mathbb{N}) \preceq \mathbb{N}^{\mathbb{N}}$.

Now we define a function $g\colon \mathbb{N}^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ by

$$g(\pi) = \{2^{\pi(0)+1}, 3^{\pi(1)+1}, \dots, p_n^{\pi(n)+1}, \dots\} \qquad \square$$

where p_n is the n^{th} prime. Clearly g is an injection. Hence $\mathbb{N}^{\mathbb{N}} \preceq \mathcal{P}(\mathbb{N})$.

By Cantor-Bernstein, $\mathcal{P}(\mathbb{N}) \sim \mathbb{N}^{\mathbb{N}}$.

Heuristic Principle Let S be an infinite set.

1. If each $s \in S$ is determined by a *finite* amount of data, then S is countable.
2. If each $s \in S$ is determined by *infinitely many independent* pieces of data, then S is uncountable.

Definition 6.14. A function $f\colon \mathbb{N} \to \mathbb{N}$ is *eventually constant* iff there exists $a, b \in \mathbb{N}$ such that

$$f(n) = b \text{ for all } n \geq a.$$

$\mathrm{EC}(\mathbb{N}) = \{f \in \mathbb{N}^{\mathbb{N}} \mid f \text{ is eventually constant }\}$.

Theorem 6.15. $\mathbb{N} \sim \mathrm{EC}(\mathbb{N})$.

Proof. For each $n \in \mathbb{N}$, let $c_n\colon \mathbb{N} \to \mathbb{N}$ be the function defined by

$$c_n(t) = n \text{ for all } t \in \mathbb{N}.$$

Then we can define an injection $f\colon \mathbb{N} \to \mathrm{EC}(\mathbb{N})$ by $f(n) = c_n$. Hence $\mathbb{N} \preceq \mathrm{EC}(\mathbb{N})$.

Next we define a function $g\colon \mathrm{EC}(\mathbb{N}) \to \mathbb{N}$ as follows. Let $\pi \in \mathrm{EC}(\mathbb{N})$. Let $a, b \in \mathbb{N}$ be chosen so that:

1. $\pi(n) = b$ for all $n \geq a$

2. a is the least such integer.

Then

$$g(\pi) = 2^{\pi(0)+1} 3^{\pi(1)+1} \dots p_a^{\pi(a)+1}$$

where p_i is the i^{th} prime. Clearly g is an injection. Thus $\text{EC}(\mathbb{N}) \preceq \mathbb{N}$.

By Cantor-Bernstein, $\mathbb{N} \sim \text{EC}(\mathbb{N})$. □

7 The proof of Cantor-Berstein

Next we turn to the proof of the Cantor-Bernstein Theorem. We shall make use of the following result.

Definition 7.1. If $f\colon A \to B$ and $C \subseteq A$, then

$$f[C] = \{f(c) \mid c \in C\}.$$

Lemma 7.2. *If $f\colon A \to B$ is an injection and $C \subseteq A$, then*

$$f[A \smallsetminus C] = f[A] \smallsetminus f[C].$$

Proof. Suppose that $x \in f[A \smallsetminus C]$. Then there exists $a \in A \smallsetminus C$ such that $f(a) = x$. In particular $x \in f[A]$. Suppose that $x \in f[C]$. Then there exists $c \in C$ such that $f(c) = x$. But $a \neq c$ and so this contradicts the fact that f is an injection. Hence $x \notin f[C]$ and so $x \in f[A] \smallsetminus f[C]$.

Conversely suppose that $x \in f[A] \smallsetminus f[C]$. Since $x \in f[A]$, there exists $a \in A$ such that $f(a) = x$. Since $x \notin f[C]$, it follows that $a \notin C$. Thus $a \in A \smallsetminus C$ and $x = f(a) \in f[A \smallsetminus C]$. □

Theorem 7.3. *(Cantor-Bernstein) If $A \preceq B$ and $B \preceq A$, then $A \sim B$.*

Proof. Since $A \preceq B$ and $B \preceq A$, there exists injections $f\colon A \to B$ and $g\colon B \to A$. Let $C = g[B] = \{g(b) \mid b \in B\}$.

Claim 7.4. $B \sim C$.

Proof of Claim 7.4. The map $b \mapsto g(b)$ is a bijection from B to C. □

Thus it is enough to prove that $A \sim C$. For then, $A \sim C$ and $C \sim B$, which implies that $A \sim B$.

Let $h = g \circ f\colon A \to C$. Then h is an injection.

Define by induction on $n \geq 0$.

$$A_0 = A \qquad\qquad C_0 = C$$
$$A_{n+1} = h[A_n] \qquad\qquad C_{n+1} = h[C_n]$$

Define $k\colon A \to C$ by $k(x) =$

$= h(x)$ if $x \in A_n \smallsetminus C_n$ for some n

$= x$ otherwise

Claim 7.5. k is an injection.

Proof of Claim 7.5. Suppose that $x \neq x'$ are distinct elements of A. We consider three cases.

Case 1:

Suppose that $x \in A_n \smallsetminus C_n$ and $x' \in A_m \smallsetminus C_m$ for some n, m. Since h is an injection,
$$k(x) = h(x) = x \neq x' = h(x) = k(x).$$

Case 2:

Suppose that $x \notin A_n \smallsetminus C_n$ for all n and that $x' \notin A_n \smallsetminus C_n$ for all n. Then
$$k(x) = x \neq x' = k(x).$$

Case 3:

Suppose that $x \in A_n \smallsetminus C_n$ and $x' \notin A_m \smallsetminus C_m$ for all m. Then
$$k(x) = h(x) \in h[A_n \smallsetminus C_n]$$
and
$$h[A_n \smallsetminus C_n] = h[A_n] \smallsetminus h[C_n] = A_{n+1} \smallsetminus C_{n+1}.$$
Hence $k(x) = h(x) \neq x' = k(x')$. □

Claim 7.6. k is a surjection.

Proof of Claim 7.6. Let $x \in C$. There are two cases to consider.

Case 1:

Suppose that $x \notin A_n \smallsetminus C_n$ for all n. Then $k(x) = x$.

Case 2:

Suppose that $x \in A_n \smallsetminus C_n$. Since $x \in C$, we must have that $n = m + 1$ for some m. Since
$$h[A_m \smallsetminus C_m] = A_n \smallsetminus C_n,$$
there exists $y \in A_m \smallsetminus C_m$ such that $k(y) = h(y) = x$. □

This completes the proof of the Cantor-Bernstein Theorem. □

Theorem 7.7. $\mathbb{R} \sim \mathbb{R} \times \mathbb{R}$

Proof. Since $(0,1) \sim \mathbb{R}$, it follows that $(0,1) \times (0,1) \sim \mathbb{R} \times \mathbb{R}$. Hence it is enough to prove that $(0,1) \sim (0,1) \times (0,1)$.

First define $f \colon (0,1) \to (0,1) \times (0,1)$ by $f(r) = \langle r, r \rangle$. Clearly f is an injection and so $(0,1) \preceq (0,1) \times (0,1)$.

Next define $g \colon (0,1) \times (0,1) \to (0,1)$ as follows. Suppose that $r, s \in (0,1)$ have decimal expansions
$$r = 0.r_0 r_1 \ldots r_n \ldots$$

$s = 0.s_0s_1 \dots s_n \dots$

Then

$g(\langle r,s\rangle) = 0.r_0s_0r_1s_1 \dots r_ns_n \dots$

Clearly g is an injection and so $(0,1)\times(0,1) \preceq (0,1)$.

By Cantor-Bernstein, $(0,1) \sim (0,1)\times(0,1)$. □

Exercise 7.8. $\mathbb{R}\smallsetminus\mathbb{N} \sim \mathbb{R}$

Exercise 7.9. $\mathbb{R}\smallsetminus\mathbb{Q} \sim \mathbb{R}$

Exercise 7.10. Let $\mathrm{Sym}(\mathbb{N}) = \{f \mid f\colon \mathbb{N} \to \mathbb{N} \text{ is a bijection }\}$. Prove that $\mathcal{P}(\mathbb{N}) \sim \mathrm{Sym}(\mathbb{N})$.

Definition 7.11. Let A be any set. Then a *finite sequence* of elements of A is an object

$\langle a_0, a_1, \dots, a_n\rangle$, $n \geq 0$

so that each $a_i \in A$, chosen so that

$\langle a_0, a_1, \dots, a_n\rangle = \langle b_0, b_1, \dots, b_n\rangle$

iff $n = m$ and $a_i = b_i$ for $0 \leq i \leq n$.

FinSeq(A) is the set of all finite sequences of elements of A.

Theorem 7.12. *If A is a nonempty countable set, then* $\mathbb{N} \sim \mathrm{FinSeq}(A)$.

Proof. First we prove that $\mathbb{N} \preceq \mathrm{FinSeq}(A)$. Fix some $a \in A$. Then we define $f\colon \mathbb{N} \to \mathrm{FinSeq}(A)$ by

$$f(n) = \langle\underbrace{a, a, a, a, a, \dots, a}_{n+1 \text{ times}}\rangle.$$

Clearly f is an injection and so $\mathbb{N} \preceq \mathrm{FinSeq}(A)$.

Next we prove that $\mathrm{FinSeq}(A) \preceq \mathbb{N}$. Since A is countable, there exists an injection $e\colon A \to \mathbb{N}$. Define $g\colon \mathrm{FinSeq}(A) \to \mathbb{N}$ by

$g(\langle a_0, a_1, \dots, a_n\rangle) = 2^{e(a_0)+1} \dots p_n^{e(a_n)+1}$

where p_i is the n^{th} prime. Clearly g is an injection. Hence $\mathrm{FinSeq}(A) \preceq \mathbb{N}$.

By Cantor-Bernstein, $\mathbb{N} \sim \mathrm{FinSeq}(A)$. □

8 Binary relations

Definition 8.1. A *binary relation* on a set A is a subset $R \subseteq A\times A$. We usually write aRb instead of writing $\langle a,b\rangle \in R$.

Example 8.2. 1. The order relation on $\mathbb{N}$ is given by

$$\{\langle n,m\rangle \mid n,m \in \mathbb{N},\ n < m\}.$$

2. The division relation D on $\mathbb{N} \smallsetminus \{0\}$ is given by

$$D = \{\langle n, m \rangle \mid n, m \in \mathbb{N},\ n \text{ divides } m\}.$$

Observation Thus $\mathcal{P}(\mathbb{N}\times\mathbb{N})$ is the collection of all binary relations on $\mathbb{N}$. Clearly $\mathcal{P}(\mathbb{N}\times\mathbb{N}) \sim \mathcal{P}(\mathbb{N})$ and so $\mathcal{P}(\mathbb{N}\times\mathbb{N})$ is uncountable.

Definition 8.3. Let R be a binary relation on A.

1. R is *reflexive* iff xRx for all $x \in A$.
2. R is *symmetric* iff xRy implies yRx for all $x, y \in A$.
3. R is *transitive* iff xRy and yRz implies xRz for all $x, y, z \in A$.

R is an *equivalence relation* iff R is reflexive, symmetric, and transitive.

Example 8.4. Define the relation R on $\mathbb{Z}$ by

$$aRb \quad \text{iff} \quad 3|a - b.$$

Proposition 8.5. *R is an equivalence relation.*

Exercise 8.6. Let $A = \{\langle a, b \rangle \mid a, b \in \mathbb{Z},\ b \neq 0\}$. Define the relation S on A by

$$\langle a, b \rangle S \langle c, d \rangle \quad \text{iff} \quad ad - bc = 0.$$

Prove that S is an equivalence relation.

Definition 8.7. Let R be an equivalence relation on A. For each $x \in A$, the *equivalence class* of x is

$$[x] = \{y \in A \mid xRy\}.$$

Example 8.4 Cont. The distinct equivalence classes are

$$[0] = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$$
$$[1] = \{\ldots, -5, -2, 1, 4, 7, \ldots\}$$
$$[2] = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$$

Definition 8.8. Let A be a nonempty set. Then $\{B_i \mid i \in I\}$ is a *partition* of A iff the following conditions hold:

1. $\emptyset \neq B_i$ for all $i \in I$.
2. If $i \neq j \in I$, then $B_i \cap B_j = \emptyset$.
3. $A = \bigcup_{i \in I} B_i$.

Theorem 8.9. *Let R be an equivalence relation on A.*

1. *If $a \in A$ then $a \in [a]$.*
2. *If $a, b \in A$ and $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$.*

Hence the set of distinct equivalence classes forms a partition of A.

Proof.

1. Let $a \in A$. Since R is reflexive, aRa and so $a \in [a]$.
2. Suppose that $c \in [a] \cap [b]$. Then aRc and bRc. Since R is symmetric, cRb. Since R is transitive, aRb. We claim that $[b] \subseteq [a]$. To see this, suppose that $d \in [b]$. Then bRd. Since aRb and bRd, it follows that aRd. Thus $d \in [a]$. Similarly, $[a] \subseteq [b]$ and so $[a] = [b]$.

□

Theorem 8.10. *Let $\{B_i \mid i \in I\}$ be a partition of A. Define a binary relation R on A by*

$$a\,R\,b \text{ iff there exists } i \in I \text{ such that } a, b \in B_i.$$

Then R is an equivalence relation whose equivalence classes are precisely $\{B_i | i \in I\}$. □

Example 8.11. How many equivalence relations can be defined on $A = \{1, 2, 3\}$?

Sol'n This is the same as asking how many partitions of A exist.

$\{\{1, 2, 3\}\}$,
$\{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{2, 3\}, \{1\}\}$,
$\{\{1\}, \{2\}, \{3\}\}$

Hence there are 5 equivalence relations on $\{1, 2, 3\}$.

Exercise 8.12. How many equivalence relations can be defined on $A = \{1, 2, 3, 4\}$?

Challenge Let $\mathrm{EQ}(\mathbb{N})$ be the collection of equivalence relations on $\mathbb{N}$. Prove that $\mathrm{EQ}(\mathbb{N}) \sim \mathcal{P}(\mathbb{N})$.

9 Linear orders

Definition 9.1. Let R be a binary relation on A.

1. R is *irreflexive* iff $\langle a, a\rangle \notin R$ for all $a \in A$.
2. R satisfies the *trichotomy property* iff for all $a, b \in A$, exactly one of the following holds:

$$a\,R\,b, a = b, b\,R\,a.$$

$\langle A, R\rangle$ is a *linear order* iff R is irreflexive, transitive, and satisfies the trichotomy property.

Example 9.2. Each of the following are linear orders.

1. $\langle \mathbb{N}, <\rangle$
2. $\langle \mathbb{N}, >\rangle$
3. $\langle \mathbb{Z}, <\rangle$
4. $\langle \mathbb{Q}, <\rangle$
5. $\langle \mathbb{R}, <\rangle$

Definition 9.3. Let R be a binary relation on A. Then $\langle A, R\rangle$ is a *partial order* iff R is irreflexive and transitive.

Example 9.4. Each of the follow are partial orders, but *not* linear orders.

1. Let A be any nonempty set containing at least two elements. Then $\langle \mathcal{P}(A), \subset\rangle$ is a partial order.
2. Let D be the strict divisability relation on $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. Then $\langle \mathbb{N}^+, D\rangle$ is a partial order.

Definition 9.5. Let $\langle A, <\rangle$ and $\langle B, <\rangle$ be partial orders. A map $f\colon A \to B$ is an *isomorphism* iff the following conditions are satisfied.

1. f is a bijection
2. For all $x, y \in A$, $x < y$ iff $f(x) < f(y)$.

In this case, we say that $\langle A, <\rangle$ and $\langle B, <\rangle$ are isomorphic and write $\langle A, <\rangle \cong \langle B, <\rangle$.

Example 9.6. $\langle \mathbb{Z}, <\rangle \cong \langle \mathbb{Z}, >\rangle$

Proof. Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be the map defined by $f(x) = -x$. Clearly f is a bijection. Also, if $x, y \in \mathbb{Z}$, then $x < y$
iff $-x > -y$
iff $f(x) > f(y)$.
Thus f is an isomorphism. □

Example 9.7. $\langle \mathbb{N}, <\rangle \ncong \langle \mathbb{Z}, <\rangle$.

Proof. Suppose that $f\colon \mathbb{N} \to \mathbb{Z}$ is an isomorphism. Let $f(0) = z$. Since f is a bijection, there exists $n \in \mathbb{N}$ such that $f(n) = z - 1$. But then $n > 0$ and $f(n) < f(0)$, which is a contradiction. □

Exercise 9.8. Prove that $\langle \mathbb{Z}, < \rangle \not\cong \langle \mathbb{Q}, < \rangle$.

Example 9.9. $\langle \mathbb{Q}, < \rangle \not\cong \langle \mathbb{R}, < \rangle$.

Proof. Since $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable, there does not exist a bijection $f\colon \mathbb{Q} \to \mathbb{R}$. Hence there does not exist an isomorphism $f\colon \mathbb{Q} \to \mathbb{R}$. □

Example 9.10. $\langle \mathbb{R}, < \rangle \not\cong \langle \mathbb{R} \smallsetminus \{0\}, < \rangle$.

Proof. Suppose that $f\colon \mathbb{R} \smallsetminus \{0\} \to \mathbb{R}$ is an isomorphism. For each $n \geq 1$, let $r_n = f(1/n)$. Then

$$r_1 > r_2 > \ldots > r_n > \ldots > f(-1).$$

Let s be the greatest lower bound of $\{r_n \mid n \geq 1\}$. Then there exists $t \in \mathbb{R} \smallsetminus \{0\}$ such that $f(t) = s$. Clearly $t < 0$. Hence $f(t/2) > s$. But then there exists $n \geq 1$ such that $r_n < f(t/2)$. But this means that $t/2 < 1/n$ and $f(t/2) > f(1/n)$, which is a contradiction. □

Question 9.11. Is $\langle \mathbb{Q}, < \rangle \cong \langle \mathbb{Q} \smallsetminus \{0\}, < \rangle$?

Definition 9.12. For each prime p,

$$\mathbb{Z}[1/p] = \{a/p^n \mid a \in \mathbb{Z},\ n \in \mathbb{N}\}.$$

Question 9.13. Is $\langle \mathbb{Z}[1/2], < \rangle \cong \langle \mathbb{Z}[1/3], < \rangle$?

Definition 9.14. A linear order $\langle D, < \rangle$ is a *dense linear order without endpoints* or DLO iff the following conditions hold.

1. For all $a, b \in D$, if $a < b$, then there exists $c \in D$ such that $a < c < b$.
2. For all $a \in D$, there exists $b \in D$ such that $a < b$.
3. For all $a \in D$, there exists $b \in D$ such that $b < a$.

Example 9.15. The following are DLOs.

1. $\langle \mathbb{Q}, < \rangle$
2. $\langle \mathbb{R}, < \rangle$
3. $\langle \mathbb{Q} \smallsetminus \{0\}, < \rangle$
4. $\langle \mathbb{R} \smallsetminus \{0\}, < \rangle$

Theorem 9.16. *For each prime p, $\langle \mathbb{Z}[1/p], < \rangle$ is a DLO.*

Proof. Clearly $\langle \mathbb{Z}[1/p], < \rangle$ linear order without endpoints. Hence it is enough to show that $\mathbb{Z}[1/p]$ is dense. Suppose that $a, b \in \mathbb{Z}[1/p]$. Then there exists $c, d \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $a = c/p^n$ and $b = d/p^n$. Clearly $a < a + (1/p^n) \leq b$. Consider

$$r = \frac{c}{p^n} + \frac{1}{p^n} = \frac{cp+1}{p^{n+1}} \in \mathbb{Z}[1/p].$$

Then $a < r < b$. ☐

Theorem 9.17. *If $\langle A, < \rangle$ and $\langle B, < \rangle$ are countable DLOs then $\langle A, < \rangle \cong \langle B, < \rangle$.*

Corollary 9.18. $\langle \mathbb{Q}, < \rangle \cong \langle \mathbb{Q} \setminus \{0\}, < \rangle$. ☐

Corollary 9.19. $\langle \mathbb{Z}[1/2], < \rangle \cong \langle \mathbb{Z}[1/3], < \rangle$. ☐

Corollary 9.20. *If p is any prime, then $\langle \mathbb{Z}[1/p], < \rangle \cong \langle \mathbb{Q}, < \rangle$.* ☐

Proof of Theorem 9.17. Let $A = \{a_n \mid n \in \mathbb{N}\}$ and $B = \{b_n \mid n \in \mathbb{N}\}$. First define $A_0 = \{a_0\}$ and $B_0 = \{b_0\}$ and let $f_0 \colon A_0 \to B_0$ be the map defined by $f_0(a_0) = b_0$.

Now suppose inductively that we have defined a function $f_n \colon A_n \to B_n$ such that the following conditions are satisfied.

1. $\{a_0, \dots, a_n\} \subseteq A_n \subseteq A$.
2. $\{b_0, \dots, b_n\} \subseteq B_n \subseteq B$.
3. $f_n \colon A_n \to B_n$ is an order preserving bijection.

We now extend f_n to a suitable function f_{n+1}.

Step 1 If $a_{n+1} \in A_n$, then let $A'_n = A_n$, $B'_n = B_n$, and $f'_n = f_n$. Otherwise, suppose for example that

$$c_0 < c_1 < \ldots < c_i < a_{n+1} < c_{i+1} < \ldots < c_m$$

where $A_n = \{c_0, \dots, c_m\}$. Choose some element $b \in B$ such that $f_n(c_i) < b < f_n(c_{i+1})$ and define

$$\begin{aligned} A'_n &= A_n \cup \{a_{n+1}\} \\ B'_n &= B_n \cup \{b\} \\ f'_n &= f_n \cup \{\langle a_{n+1}, b \rangle\} \end{aligned}$$

Step 2 If $b_{n+1} \in B'_n$, then let $A_{n+1} = A'_n$, $B_{n+1} = B'_n$, and $f_{n+1} = f'_n$. Otherwise, suppose for example that

$$d_0 < d_1 < \ldots < d_j < b_{n+1} < d_{j+1} < \ldots < d_t$$

where $B'_n = \{d_0, \dots, d_t\}$. Choose some element $a \in A$ such that $(f'_n)^{-1}(d_j) < a < (f'_n)^{-1}(d_{j+1})$ and define

$$\begin{aligned} A_{n+1} &= A'_n \cup \{a\} \\ B_{n+1} &= B'_n \cup \{b_{n+1}\} \\ f_{n+1} &= f'_n \cup \{\langle a, b_{n+1}\rangle\}. \end{aligned}$$

Finally, let $f = \bigcup_{n\geq 0} f_n$. Then $f\colon A \to B$ is an isomorphism. □

10 Extensions

Definition 10.1. Suppose that R, S are binary relations on A. Then S *extends* R iff $R \subseteq R$.

Example 10.2. Consider the binary relations R, S on $\mathbb{N}$ defined by

$$\begin{aligned} R &= \{\langle n, m\rangle \mid n < m\} \\ S &= \{\langle n, m\rangle \mid n \leq m\} \end{aligned}$$

Then S extends R.

Example 10.3. Consider the partial order $\prec$ on $\{a, b, c, d, e\}$ which is

$$\{\langle d, b\rangle, \langle d, a\rangle, \langle d, e\rangle, \langle d, c\rangle, \langle a, b\rangle, \langle e, b\rangle, \langle c, b\rangle\}.$$

Then we can extend $\prec$ to the linear order $<$ defined by the transitive closure of

$$d < e < c < a < b.$$

Exercise 10.4. If $\langle A, \prec\rangle$ is a finite partial order, then we can extend $\prec$ to a linear ordering $<$ of A.

Question 10.5. Does the analogous result hold if $\langle A, \prec\rangle$ is a infinite partial order?

Definition 10.6. If A is a set and $n \geq 1$, then

$$A^n = \{\langle a_1, \dots, a_n\rangle \mid a_1, \dots, a_n \in A\}.$$

An *n-ary relation* on A is a subset $R \subseteq A^n$.
An *n-ary operation* on A is a function $f\colon A^n \to A$.

11 Propositional logic

"The study of how the truth value of compound statements depends on those of simple statements."

A reminder of truth-tables.

and $\wedge$

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

or $\vee$

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

not $\neg$

A	$\neg A$
T	F
F	T

material implication $\rightarrow$

A	B	$A \rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

iff $\leftrightarrow$

A	B	$A \leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

Now our study actually begins... First we introduce our *formal language*.

Definition 11.1. The *alphabet* consists of the following symbols:

1. the sentence connectives
$$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$$
2. the punctuation symbols
$$(,)$$
3. the sentence symbols
$$A_1, A_2, \ldots, A_n, \ldots,\ n \geq 1$$

Remark 11.2. Clearly the alphabet is countable.

Definition 11.3. An *expression* is a finite sequence of symbols from the alphabet.

Example 11.4. The following are expressions:

$$(A_1 \wedge A_2), \quad ((\neg \rightarrow ())A_3$$

Remark 11.5. Clearly the set of expressions is countable.

Definition 11.6. The set of *well-formed formulas* (wffs) is defined recursively as follows:

1. Every sentence symbol A_n is a wff.
2. If α and β are wffs, then so are
$$(\neg\alpha),\ (\alpha \wedge \beta),\ (\alpha \vee \beta),\ (\alpha \rightarrow \beta),\ (\alpha \leftrightarrow \beta)$$
3. No expression is a wff unless it is compelled to be so by repeated applications of (1) and (2).

Remark 11.7.

1. From now on we omit clause (3) in any further recursive definitions.
2. Clearly the set of wffs is countably infinite.
3. Because the definition of a wff is recursive, most of the properties of wffs are proved by induction on the length of a wff.

Example 11.8.

1. $(A_1 \rightarrow (\neg A_2))$ is a wff.
2. $((A_1 \wedge A_2)$ is not a wff. How can we prove this?

Proposition 11.9. *If α is a wff, then α has the same number of left and right parentheses.*

Proof. We argue by induction on the length $n \geq 1$ of the wff α. First suppose that $n = 1$. Then α must be a sentence symbol, say A_n. Clearly the result holds in this case.

Now suppose that $n > 1$ and that the result holds for all wffs of length less than n. Then α must have one of the following forms:

$$(\neg\beta), (\beta \wedge \gamma), (\beta \vee \gamma), (\beta \rightarrow \gamma), (\beta \leftrightarrow \gamma)$$

for some wffs β, γ of length less than n. By induction hypothesis the result holds for both β and γ. Hence the result also holds for α. □

Definition 11.10. $\mathcal{L}$ is the set of sentence symbols. $\bar{\mathcal{L}}$ is the set of wffs. $\{T, F\}$ is the set of truth values.

Definition 11.11. A *truth assignment* is a function $v\colon \mathcal{L} \to \{T, F\}$.

Definition 11.12. Let v be a truth assignment. Then we define the extension $\bar{v}\colon \bar{\mathcal{L}} \to \{T, F\}$ recursively as follows.

0. If $A_n \in \mathcal{L}$ then $\bar{v}(A_n) = v(A_n)$.

For any $\alpha, \beta \in \bar{\mathcal{L}}$

1. $\bar{v}((\neg\alpha)) =$
   $= T$ if $\bar{v}(\alpha) = F$
   $= F$ otherwise
2. $\bar{v}((\alpha \wedge \beta)) =$
   $= T$ if $\bar{v}(\alpha) = \bar{v}(\beta) = T$
   $= F$ otherwise
3. $\bar{v}((\alpha \vee \beta)) =$
   $= F$ if $\bar{v}(\alpha) = \bar{v}(\beta) = F$
   $= T$ otherwise
4. $\bar{v}((\alpha \to \beta)) =$
   $= F$ if $\bar{v}(\alpha) = T$ and $\bar{v}(\beta) = F$
   $= T$ otherwise
5. $\bar{v}((\alpha \leftrightarrow \beta)) =$
   $= T$ if $\bar{v}(\alpha) = \bar{v}(\beta)$
   $= F$ otherwise

Possible problem. Suppose there exists a wff α such that α has both the forms $(\beta \to \gamma)$ and $(\sigma \wedge \varphi)$ for some wffs $\beta, \gamma, \sigma, \varphi$. Then there will be two (possibly conflicting) clauses which define $\bar{v}(\alpha)$.

Fortunately no such α exists...

Theorem 11.13 (Unique readability). *If α is a wff of length greater than 1, then there exists eactly one way of expressing α in the form:*

$$(\neg\beta), (\beta \wedge \gamma), (\beta \vee \gamma), (\beta \to \gamma), \quad \textit{or} \quad (\beta \leftrightarrow \gamma)$$

for some shorter wffs β, γ.

We shall make use of the following result.

Lemma 11.14. *Any proper initial segment of a wff contains more left parentheses than right parentheses. Thus no proper initial segment of a wff is a wff.*

Proof. We argue by induction on the length $n \geq 1$ of the wff α. First suppose that $n = 1$. Then α is a sentence symbol, say A_n. Since A_n has no proper initial segments, the result holds vacuously.

Now suppose that $n > 1$ and that the result holds of all wffs of length less than n. Then α must have the form

$$(\neg\beta), (\beta \wedge \gamma), (\beta \vee \gamma), (\beta \rightarrow \gamma), \quad \text{or} \quad (\beta \leftrightarrow \gamma)$$

for some shorter wffs β and γ. By induction hypothesis, the result holds for both β and γ. We just consider the case when α is $(\beta \wedge \gamma)$. (The other cases are similar.) The proper initial segments of α are:

1. $($
2. $(\beta_0$ where β_0 is an initial segment of β
3. $(\beta\wedge$
4. $(\beta \wedge \gamma_0$ where γ_0 is an initial segment of γ.

Using the induction hypothesis and the previous proposition (Proposition 11.9), we see that the result also holds for α. □

Proof of Theorem 11.13. Suppose, for example, that

$$\alpha = (\beta \wedge \gamma) = (\sigma \wedge \varphi).$$

Deleting the first (we obtain that

$$\beta \wedge \gamma) = \sigma \wedge \varphi).$$

Suppose that $\beta \neq \sigma$. Then wlog β is a proper initial segment of σ. But then β isn't a wff, which is a contradiction. Hence $\beta = \sigma$. Deleting β and σ, we obtain that

$$\wedge\gamma) = \wedge\varphi).$$

and so $\gamma = \varphi$.

Next suppose that

$$\alpha = (\beta \wedge \gamma) = (\sigma \rightarrow \varphi).$$

Arguing as above, we find that $\beta = \sigma$ and so

$$\wedge\gamma) =\rightarrow \varphi)$$

which is a contradiction.

The other cases are similar. □

Definition 11.15. Let $\upsilon\colon \mathcal{L} \to \{T, F\}$ be a truth assignment.

1. If φ is a wff, then υ *satisfies* φ iff $\bar{\upsilon}(\varphi) = T$.
2. If Σ is a set of wffs, then υ *satisfies* Σ iff $\bar{\upsilon}(\sigma) = T$ for all $\sigma \in \Sigma$.
3. Σ is *satisfiable* iff there exists a truth assignment υ which satisfies Σ.

Example 11.16.

1. Suppose that $\upsilon\colon \mathcal{L} \to \{T, F\}$ is a truth assignment and that $\upsilon(A_1) = F$ and $\upsilon(A_2) = T$. Then υ satisfies $(A_1 \to A_2)$.
2. $\Sigma = \{A_1, (\neg A_2), (A_1 \to A_2)\}$ is *not* satisfiable.

Exercise 11.17. Suppose that φ is a wff and υ_1, υ_2 are truth assignments which agree on all sentence symbols appearing in φ. Then $\bar{\upsilon_1}(\varphi) = \bar{\upsilon_2}(\varphi)$. (*Hint:* argue by induction on the length of φ.)

Definition 11.18. Let Σ be a set of wffs and let φ be a wff. Then Σ *tautologically implies* φ, written $\Sigma \models \varphi$, iff every truth assignment which satisfies Σ also satisfies φ.

Important Observation. Thus $\Sigma \models \varphi$ iff $\Sigma \cup \{\neg\varphi\}$ is not satisfiable.

Example 11.19. $\{A_1, (A_1 \to A_2)\} \models A_2$.

Definition 11.20. The wffs φ, ψ are *tautologically equivalent* iff both $\{\varphi\} \models \psi$ and $\{\psi\} \models \varphi$.

Example 11.21. $(A_1 \to A_2)$ and $((\neg A_2) \to (\neg A_1)$ are tautologicaly equivalent.

Exercise 11.22. Let σ, τ be wffs. Then the following statements are equivalent.

1. σ and τ are tautologically equivalent.
2. $(\sigma \leftrightarrow \tau)$ is a tautology.

(Hint: do *not* argue by induction on the lengths of the wffs.)

12 The compactness theorem

Question 12.1. Suppose that Σ is an infinite set of wffs and that $\Sigma \models \tau$. Does there necessarily exists a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \tau$?

A positive answer follows from the following result...

Theorem 12.2 (The Compactness Theorem). *Let Σ be a set of wffs. If every finite subset $\Sigma_0 \subseteq \Sigma$ is satisfiable, then Σ is satisfiable.*

Definition 12.3. A set Σ of wffs is *finitely satisfiable* iff every finite subset $\Sigma_0 \subset \Sigma$ is satisfiable.

Theorem 12.4 (The Compactness Theorem). *If Σ is a finitely satisfiable set of wffs, then Σ is satisfiable.*

Before proving the compactness theorem, we present a number of its applications.

Corollary 12.5. *If $\Sigma \models \tau$, then there exists a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \tau$.*

Proof. Suppose not. Then for every finite subset $\Sigma_0 \subseteq \Sigma$, we have that $\Sigma_0 \not\models \tau$ and hence $\Sigma_0 \cup \{(\neg\tau)\}$ is satisfiable. Thus $\Sigma \cup \{(\neg\tau)\}$ is finitely satisfiable. By the Compactness Theorem, $\Sigma \cup \{(\neg\tau)\}$ is satisfiable. But this means that $\Sigma \not\models \tau$, which is a contradiction. $\square$

13 A graph-theoretic application

Definition 13.1. Let E be a binary relation on the set V. Then $\Gamma = \langle V, E\rangle$ is a *graph* iff:

1. E is irreflexive; and
2. E is symmetric.

Example 13.2. Let $V = \{0, 1, 2, 3, 4\}$ and let $E = \{\langle i, j\rangle \mid j = i + 1 \mod 5\}$. This is called the *cycle* of length five.

Definition 13.3. Let $k \geq 1$. Then the graph $\Gamma = \langle V, E\rangle$ is *k-colorable* iff there exists a function $\chi \colon V \to \{1, 2, \dots k\}$. such that for all $a, b \in V$,

$$\text{if} \quad aEb, \quad \text{then} \quad \chi(a) \neq \chi(b).$$

Example 13.4. Any cycle of even length is two-colorable. Any cycle of odd length is three-colorable but not two-colorable.

Theorem 13.5 (Erdös). *A countable graph $\Gamma = \langle V, E\rangle$ is k-colorable iff every finite subgraph $\gamma_0 \subset \Gamma$ is k-colorable.*

Proof. $\Rightarrow$ Suppose that Γ is k-colorable and let $\chi \colon V \to \{1, 2, \dots, k\}$ is any k-coloring. Let $\Gamma_0 = \langle V_0, E_0\rangle$ be any finite subgraph of Γ. Then $\chi_0 = \chi | V_0$ is a k-coloring of Γ_0.

$\Leftarrow$ In this direction we use the Compactness Theorem.

Step 1 We choose a suitable propositional language. The idea is to have a sentence symbol for every decision we must make. So our language has sentence symbols:

$$C_{v,i} \quad \text{for each} \quad v \in V, \quad 1 \leq i \leq k.$$

(The intended meaning of $C_{v,i}$ is: "color vertex v with color i.")

Step 2 We write down a suitable set of wffs which imposes a suitable set of constraints on our truth assignments. Let Σ be the set of wffs of the following forms:

(a) $C_{v,1} \vee C_{v,2} \vee \ldots \vee C_{v,k}$ for each $v \in V$.

(b) $\neg(C_{v,i} \wedge C_{v,j})$ for each $v \in V$ and $1 \leq i \neq j \leq k$.

(c) $\neg(C_{v,i} \wedge C_{w,i})$ for each pair $v, w \in V$ of adjacent vertices and each $1 \leq i \leq k$.

Step 3 We check that we have chosen a suitable set of wffs.

Claim 13.6. Suppose that υ is a truth assignment which satisfies Σ. Then we can define a k-coloring $\chi\colon \Gamma \to \{1, \ldots, k\}$ by

$$\chi(v) = i \quad \text{iff} \quad \upsilon(C_{v,i}) = T.$$

Proof. By (a) and by (b), for each $v \in V$, there exists a unique $1 \leq i \leq k$ such that $\upsilon(C_{v,i}) = T$. Thus $\chi\colon V \to \{1, \ldots\}$ is a function. By (c), if $v, w \in V$ are adjacent, then $\chi(v) \neq \chi(w)$. Hence χ is a k-coloring. □

Step 4 We next prove that Σ is finitely satisfiable. So let $\Sigma_0 \subseteq \Sigma$ be any finite subset. Let $V_0 \subseteq V$ be the finite set of vertices that are mentioned in Σ_0. Then the finite subgraph $\Gamma_0 = \langle V_0, E_0 \rangle$ is k-colorable. Let

$$\chi\colon V_0 \to \{1, \ldots, k\}$$

be a k-coloring of Γ_0. Let υ_0 be a truth assignment such that if $v \in V_0$ and $1 \leq i \leq k$, then

$$\upsilon(C_{v,i}) = T \ \text{ iff } \ \chi_0(v) = i.$$

Clearly υ_0 satisfies Σ_0.

By the Compactness Theorem, Σ is satisfiable. Hence Γ is k-colorable. □

14 Extending partial orders

Theorem 14.1. *Let $\langle A, \prec \rangle$ be a countable partial order. Then there exists a linear ordering $<$ of A which extends $\prec$.*

Proof. We work with the propositional language which has sentence symbols

$$L_{a,b} \quad \text{for} \quad a \neq b \in A$$

Let Σ be the following set of wffs:

(a) $L_{a,b} \vee L_{b,a}$ for $a \neq b \in A$

(b) $\neg(L_{a,b} \wedge L_{b,a})$ for $a \neq b \in A$

(c) $((L_{a,b} \wedge L_{b,c}) \to L_{a,c})$ for distinct $a, b, c \in A$

(d) $L_{a,b}$ for distinct $a, b \in A$ with $a \prec b$.

Claim 14.2. Suppose that υ is a truth assignment which satisfies Σ. Define the binary relation $<$ on A by
$$a < b \quad \text{iff} \quad \upsilon(L_{a,b}) = T.$$
Then $<$ is a linear ordering of A which extends $\prec$.

Proof. Clearly $<$ is irreflexive. By (a) and (b), $<$ has the trichotomy property. By (c), $<$ is transitive. Finally, by (d), $<$ extends $\prec$. □

Next we prove that Σ is finitely satisfiable. So let $\Sigma_0 \subseteq \Sigma$ be any finite subset. Let $A_0 \subseteq A$ be the finite set of elements that are mentioned in Σ_0 and consider the partial order $\langle A_0, \prec_0 \rangle$. Then there exists a partial ordering $<_0$ of A_0 extending $\prec_0$. Let υ_0 be the truth assignment such that if $a \neq b \in A_0$, then
$$\upsilon_0(L_{a,b}) = T \quad \text{iff} \quad a \leq_0 b.$$
Clearly υ_0 satisfies Σ_0.

By the compactness theorem, Σ is satisfiable. Hence there exists a linear ordering $<$ of A which extends $\prec$. □

15 Hall's Theorem

Definition 15.1. Suppose that S is a set and that $\langle S_i \mid i \in I \rangle$ is an indexed collection of (not necessarily distinct) subsets of S. A system of *distinct representatives* is a choice of elements $x_i \in S_i$ for $i \in I$ such that if $i \neq j \in I$, then $x_i \neq x_j$.

Example 15.2. Let $S = \mathbb{N}$ and let $\langle S_n \mid n \in \mathbb{N} \rangle$ be defined by
$$S_n = \{n, n+1\}$$
Thus $S_0 = \{0, 1\}$, $S_1 = \{1, 2\}$, ... Then we can take $x_i = i \in S_i$.

Theorem 15.3 (Hall's Matching Theorem (1935)). *Let S be any set and let $n \in \mathbb{N}^+$. Let $\langle S_1, S_2, \ldots, S_n \rangle$ be an indexed collection of subsets of S. Then a necessary and sufficent condition for the existance of a system of distinct representatives is:*

(H) For every $1 \leq k \leq n$ and choice of k distinct indices $1 \leq i_1, \ldots, i_k \leq n$, we have $|S_{i_1} \cup \ldots \cup S_{i_k}| \geq k$.

Challange: Prove this!

Problem 15.4. State and prove an infinite analogue of Hall's Matching Theorem.

First Attempt Let S be any set and let $\langle S_n \mid n \in \mathbb{N}^+\rangle$ be an indexed collection of subsets of S. Then a necessary and sufficient condition for the existence of a system of distinct representatives is:

(H^*) For every $k \in \mathbb{N}^+$ and choice of k distinct indices $i_1, \dots, i_k \in \mathbb{N}$, we have $|S_{i_1} \cup \ldots \cup S_{i_k}| \geq k$.

Counterexample Take $S_1 = \mathbb{N}$, $S_2 = \{0\}$, $S_3 = \{1\}$, ..., $S_n = \{n-2\}$, ... Clearly (H^*) is satisfied and yet there is *no* system of distinct representatives.

Question 15.5. Where does the compactness argument break down?

Theorem 15.6 (Infinite Hall's Theorem). *Let S be any set and let $\langle S_n \mid n \in \mathbb{N}^+\rangle$ be an indexed collection of* finite *subsets of S. Then a necessary and sufficient condition for the existence of a system of distinct representatives is:*

(H^*) For every $k \in \mathbb{N}^+$ and choice of k distinct indices $i_1, \dots, i_k \in \mathbb{N}$, we have $|S_{i_1} \cup \ldots \cup S_{i_k}| \geq k$.

Proof. We work with the propositional language with sentence symbols

$$C_{n,x}. \quad n \in \mathbb{N}^+, \quad x \in S_n.$$

Let Σ be the following set of wffs:

(a) $\neg(C_{n,x} \wedge C_{m,x})$ for $n \neq m \in \mathbb{N}^+$, $x \in S_n \cap S_m$.

(b) $\neg(C_{n,x} \wedge C_{n,y})$ for $n \in \mathbb{N}^+$, $x \neq y \in S_n \cap S_m$.

(c) $(C_{n,x_1} \vee \ldots \vee C_{n,x_k})$ for $n \in \mathbb{N}^+$, where $S_n = \{x_1, \dots, x_k\}$.

Claim 15.7. Suppose that v is a truth assignment which satisfies Σ. Then we can define a system of distinct representatives by

$$x \in S_n \text{ iff } v(C_{n,x}) = T.$$

Proof. By (b) and (c), each S_n gets assigned a unique representative. By (a), distinct sets $S_m \neq S_m$ get assigned distinct representatives. $\square$

Next we prove that Σ is finitely satisfiable. So let $\Sigma_0 \subseteq \Sigma$ be any finite subset. Let $\{i_1, \dots, i_l\}$ be the indices that are mentioned in Σ_0. Then $\{S_{i_1}, \dots S_{i_l}\}$ satisfies condition (H). By Hall's Theorem, there exists a set of distinct representatives for $\{S_{i_1}, \dots S_{i_l}\}$; say, $x_r \in S_{i_r}$. Let v_0 be the truth assignment such that for $1 \leq r \leq l$ and $x \in S_{i_r}$,

$$v(C_{i_r,x}) = T \text{ iff } x = x_r.$$

Clearly v_0 satisfies Σ_0.

By the Compactness Theorem, Σ is satisfiable. Hence there exists a system of distinct representatives. $\square$

16 Proof of compactness

Theorem 16.1 (The Compactness Theorem). *If Σ is a finitely satisfiable set of wffs, then Σ is satisfiable.*

Basic idea Imagine that for each sentence symbol A_n, either $A_n \in \Sigma$ or $\neg A_n \in \Sigma$. Then there is only one possibility for a truth assignment v which satisfies Σ: namely,

$$v(A_n) = T \text{ iff } A_n \in \Sigma.$$

Presumably this v works...

In the general case, we extend Σ to a finitely satisfiable set Δ as above. For technical reasons, we construct Δ so that for *every* wff α, either $\alpha \in \Delta$ or $\neg\alpha \in \Delta$.

Lemma 16.2. *Suppose that Σ is a finitely satisfiable set of wffs. If α is any wff, then either $\Sigma \cup \{\alpha\}$ is finitely satisfiable or $\Sigma \cup \{\neg\alpha\}$ is finitely satisfiable.*

Proof. Suppose that $\Sigma \cup \{\alpha\}$ isn't finitely satisfiable. Then there exists a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \cup \{\alpha\}$ isn't satisfiable. Thus $\Sigma \models \neg\alpha$. We claim that $\Sigma \cup \neg\alpha$ is fintely satisfiable. Let $\Delta \subseteq \Sigma \cup \{\neg\alpha\}$ be any finite subset. If $\Delta \subseteq \Sigma$ then Δ is satisfiable. Hence we can suppose that $\Delta = \Delta_0 \cup \{\neg\alpha\}$ for some finite subset $\Delta_0 \subseteq \Sigma$. Since Σ is finitely satisfiable, ther exists a truth assignment v which satisfies $\Sigma_0 \cap \Delta_0$. Since $\Sigma_0 \models \neg\alpha$, it follows that $\bar{v}(\neg\alpha) = T$. Hence v satisfies $\Delta_0 \cup \{\neg\alpha\}$. □

Proof of the Compactness Theorem. Let Σ be a finitely satisfiable set of wffs. Let

$$\alpha_1, \alpha_2, \dots, \alpha_n, \dots \quad n \geq 1$$

be an enumeration of all the wffs $\alpha \in \bar{\mathcal{L}}$. We shall inductively define an increasing sequence of finitely satisfiable sets of wffs

$$\Delta_0 \subseteq \Delta_1 \subseteq \dots \subseteq \Delta_n \subseteq \dots$$

First let $\Delta_0 = \Sigma$. Suppose inductively that Δ_n has been defined. Then

$$\begin{aligned} \Delta_{n+1} &= \Delta_n \cup \{\alpha_{n+1}\}, \text{ if this is finitely satisfiable} \\ &= \Delta_n \cup \{(\neg\alpha_{n+1})\}, \text{ otherwise.} \end{aligned}$$

By the lemma, Δ_{n+1} is also finitely satisfiable. Finally define

$$\Delta = \bigcup_n \Delta_n.$$

Claim 16.3. Δ is finitely satisfiable.

Proof. Suppose that $\Phi \subseteq \Delta$ is a finite subset. Then there exists an n such that $\Phi \subseteq \Delta_n$. Since Δ_n is finitely satisfiable, Φ is satisfiable. $\square$

Claim 16.4. If α is any wff, then either $\alpha \in \Delta$ or $(\neg\alpha) \in \Delta$.

Proof. There exists an $n \geq 1$ such that $\alpha = \alpha_n$. By construction, either $\alpha_n \in \Delta_{n+1}$ or $(\neg\alpha_n) \in \Delta_{n+1}$; and $\Delta_{n+1} \subseteq \Delta$. $\square$

Define a truth assignment $v \colon \mathcal{L} \to \{T, F\}$ by

$$v(A_l) = T \quad \text{iff} \quad A_l \in \Delta.$$

Claim 16.5. For every wff α, $\bar{v}(\alpha) = T$ iff $\alpha \in \Delta$.

Proof. We argue by induction on the length m of the wff α. First suppose that $m = 1$. Then α is a sentence symbol; say, $\alpha = A_l$. By definition

$$\bar{v}(A_l) = v(A_l) = T \quad \text{iff} \quad A_l \in \Delta.$$

Now suppose that $m > 1$. Then α has the form

$$(\neg\beta), (\beta \wedge \gamma), (\beta \vee \gamma), (\beta \to \gamma), (\beta \leftrightarrow \gamma)$$

for some shorter wffs β, γ.

Case 1 Suppose that $\alpha = (\neg\beta)$. Then

$$\begin{aligned} \bar{v}(\alpha) = T \quad &\text{iff} \quad \bar{v}(\beta) = F \\ &\text{iff} \quad \beta \notin \Delta \ \text{ by induction hypothesis} \\ &\text{iff} \quad (\neg\beta) \in \Delta \ \text{ by Claim 16.4} \\ &\text{iff} \quad \alpha \in \Delta \end{aligned}$$

Case 2 Suppose that α is $(\beta \vee \gamma)$. First suppose that $\bar{v}(\alpha) = T$. Then $\bar{v}(\beta) = T$ or $\bar{v}(\gamma) = T$. By induction hypothesis, $\beta \in \Delta$ or $\gamma \in \Delta$. Since Δ is finitely satisfiable, $\{\beta, (\neg(\beta \vee \gamma))\} \nsubseteq \Delta$ and $\{\gamma, (\neg(\beta \vee \gamma))\} \nsubseteq \Delta$. Hence $(\neg(\beta \vee \gamma)) \notin \Delta$ and so $(\beta \vee \gamma) \in \Delta$.

Conversely suppose that $(\beta \vee \gamma) \in \Delta$. Since Δ is finitely satisfiable, $\{(\neg\beta), (\neg\gamma), (\beta \vee \Gamma)\} \nsubseteq \Delta$. Hence $(\neg\beta) \notin \Delta$ or $(\neg\gamma) \notin \Delta$; and so $\beta \in \Delta$ or $\gamma \in \Delta$. By induction hypothesis, $\bar{v}(\beta) = T$ or $\bar{v}(\gamma) = T$. Hence $\bar{v}(\beta \vee \gamma) = T$.

Exercise 16.6. Write out the details for the other cases.

$\square$

Thus v satisfies Δ. Since $\Sigma \subseteq \Delta$, it follows that v satisfies Σ. $\square$

17 Trees and Konig's Lemma

Definition 17.1. A partial order $\langle T, \prec \rangle$ is a *tree* iff the following conditions are satisfied.

1. There exists a unique minimal element $t_0 \in T$ called the *root*.
2. For each $t \in T$, the set
$$\mathrm{Pr}_T(t) = \{s \in T \mid s \prec t\}$$
is a finite set which is linearly ordered by $\prec$.

Example 17.2. The complete binary tree is defined to be

$$T_2 = \{f \mid f \colon n \to \{0, 1\}\}$$

ordered by
$$f \prec g \ \text{ iff } \ f \subset g.$$

Definition 17.3. Let $\langle T, \prec \rangle$ be a tree.

1. If $t \in T$, then the *height* of t is defined to be
$$\mathrm{ht}_T(t) = |\mathrm{Pr}_T(t)|.$$
2. For each $n \geq 0$, the n^{th} *level* of T is
$$\mathrm{Lev}_T(n) = \{t \in T \mid \mathrm{ht}_T(t) = n\}.$$
3. For each $t \in T$, the set of *immediate successors* of t is
$$\mathrm{succ}_T(t) = \{s \in T \mid t \prec s \text{ and } \mathrm{ht}_T(s) = \mathrm{ht}_T(t) + 1\}.$$
4. T is *finitely branching* iff each $t \in T$ has a finite (possibly empty) set of immediate successors.
5. A *branch* $\mathcal{B}$ of T is a maximal linearly ordered subset of T.

Example 17.4. Consider the complete binary tree T_2. If $\varphi \colon \mathbb{N} \to \{0, 1\}$, then we can define a corresponding branch of T_2 by

$$\mathcal{B}_\varphi = \{\varphi | n \mid n \in \mathbb{N}\}.$$

Conversely, let $\mathcal{B}$ be an arbitrary branch of T_2. Let $\varphi = \bigcup \mathcal{B}$. Then $\varphi \colon \mathbb{N} \to \{0, 1\}$ and $\mathcal{B} = \mathcal{B}_\varphi$.

Exercise 17.5. Let $\langle T, \prec \rangle$ be a tree. Then the following are equivalent:

1. T is finitely branching.
2. $\mathrm{Lev}_T(n)$ is finite for all $n \geq 0$.

Lemma 17.6 (König). *Suppose that T is an infinite finitely branching tree. Then there exists an infinite branch $\mathcal{B}$ through T.*

Remark 17.7. Note that such a branch $\mathcal{B}$ necessarily satisfies:

$$|\mathcal{B} \cap \mathrm{Lev}_T(n)| = 1 \quad \text{for all } n \geq 0.$$

First we shall give a proof of König's Lemma, using the Compactness Theorem.

Proof of König's Lemma. Let $\langle T, \prec \rangle$ be an infinite finitely branching tree. Then each level $\mathrm{Lev}_T(n)$ is finite and so T is countably infinite. We shall work with the propositional language with sentence symbols $\{B_t \mid t \in T\}$. Let Σ be the following set of wffs:

(a) $B_{t_1} \vee \ldots \vee B_{t_l}$ where $\mathrm{Lev}_T(n) = \{t_1, \ldots, t_l\}$ and $n \geq 0$.

(b) $\neg(B_{t_i} \wedge B_{t_j})$ where $\mathrm{Lev}_T(n) = \{t_1, \ldots, t_l\}$, $n \geq 0$, and $1 \leq i < j \leq l$.

(c) $(B_s \rightarrow B_t)$ for $s, t \in T$ with $s \prec t$.

Claim 17.8. Suppose that v is a truth assignment which satisfies Σ. Then

$$\mathcal{B} = \{t \in T \mid v(B_t) = T\}$$

is an infinite branch through T.

Proof. By (a) and (b), $\mathcal{B}$ intersects every level in a unique point. Suppose that $s \neq t \in \mathcal{B}$. Then wlog we have that $\mathrm{ht}_T(s) < \mathrm{ht}_T(t)$. Let $n = \mathrm{ht}_T(s)$. By (c), $\mathcal{B}$ must contain the predecessor of t in $\mathrm{Lev}_T(n)$, which must be equal to s. Thus $s \prec t$. It follows that $\mathcal{B}$ is linearly ordered. □

We claim that Σ is finitely satisfiable. Let $\Sigma_0 \subseteq \Sigma$ be a finite subset. Then there exists $n \geq 0$ such that if $t \in T$ is mentioned in Σ_0, then $\mathrm{ht}_T(t) < n$. Choose $t_0 \in \mathrm{Lev}_T(n)$ and let v_0 be the truth assignment such that for all $t \in T$ with $\mathrm{ht}_T(t) < n$,

$$v_0(B_t) = T \quad \text{iff} \quad t < t_0.$$

Clearly v_0 satisfies Σ_0. By Compactness, Σ is satisfiable and hence T has an infinite branch. □

Next we shall give a direct proof of König's Lemma.

Proof of König's Lemma. Let T be an infinite finitely branching tree. We shall define a sequence of elements $t_n \in T$ inductively so that the following conditions are satisfied:

(a) $t_n \in \mathrm{Lev}_T(n)$

(b) If $m < n$ then $t_m \prec t_n$.

(c) $\{s \in T \mid t_n \prec s\}$ is infinite.

First let $t_0 \in \mathrm{Lev}_T(0)$ be the root. Clearly the above conditions are satisfied.. Assume inductively that t_n has been defined. Then t_n has a finite set of immediate successors; say $\{a_1, \dots, a_l\}$. If $t_n \prec s$ and $\mathrm{ht}_T(s) > n+1$, then there exists $1 \leq i \leq l$ such that $a_i \prec s$. By the pigeon hole principle, there exists $1 \leq i \leq l$ such that a_i satisfies (c). Then we define $t_{n+1} = a_i$. Clearly $\mathcal{B} = \{t_n \mid n \geq 0\}$ is an infinite branch through T. $\square$

Next we present an application of König's Lemma.

Theorem 17.9 (Erdös). *A countably infinite graph Γ is k-colorable iff every finite subgraph of Γ is k-colorable.*

Proof. ($\Rightarrow$) Trival!

($\Leftarrow$) Suppose that every finite subgraph of Γ is k-colorable. Let $\Gamma = \{v_1, v_2, \dots, v_n, \dots\}$; and for each $n \geq 1$, let $\Gamma_n = \{v_1, \dots, v_n\}$ and let C_n be the set of k-colorings of Γ_n. Let T be the tree with levels defined by

$$\begin{aligned} \mathrm{Lev}_T(0) &= \{\emptyset\} \\ \mathrm{Lev}_T(n) &= C_n \quad \text{for } n \geq 0 \end{aligned}$$

partially ordered as follows. Suppose that $\chi \in \mathrm{Lev}_T(n)$ and $\theta \in \mathrm{Lev}_T(m)$ where $1 \leq n < m$. Then

$$\chi \prec \theta \ \text{ iff } \ \chi = \theta | \{v_1, \dots, v_n\}.$$

Clearly T is an infinite finitely branching tree. By König's Lemma, there exists an infinite branch $\mathcal{B} = \{\chi_n \mid n \in \mathbb{N}\}$ through T, where $\chi_n \in \mathrm{Lev}_T(n)$. We claim that $\chi = \bigcup_n \chi_n$ is a k-coloring of Γ. It is clear that $\chi\colon \Gamma \to \{1, \dots, k\}$. Next suppose that $a \neq b \in \Gamma$ are adjacent vertices. Then there exists $n \geq 1$ such that $a, b \in \Gamma_n$. By definition, we have that $\chi(a) = \chi_n(a)$ and $\chi(b) = \chi_n(b)$. Since χ_n is a k-coloring of Γ_n, it follows that $\chi_n(a) \neq \chi_n(b)$. Thus $\chi(a) \neq \chi(b)$. $\square$

Finally we use König's Lemma to give a proof of the Compactness Theorem.

Proof of Compactness Theorem. Suppose that Σ is a finitely satisfiable set of wffs in the propositional language with sentence symbols $\{A_1, A_2, \dots, A_n, \dots\}$. We define a tree T as follows.

- $\mathrm{Lev}_T(0) = \{\emptyset\}$

- If $n \geq 1$, then $\mathrm{Lev}_T(n)$ is the set of all partial truth assignments $\upsilon\colon \{A_1, \dots, A_n\} \to \{T, F\}$ which satisfy every $\sigma \in \Sigma$ which only mention $A_1, \dots, A_n$.

We partially order T as follows. Suppose that $\upsilon \in \mathrm{Lev}_T(n)$ and $\upsilon' \in \mathrm{Lev}_T(m)$, where $1 \leq n < m$. Then

$$\upsilon \prec \upsilon' \text{ iff } \upsilon = \upsilon' | \{A_1, \dots, A_n\}.$$

Clearly $|\mathrm{Lev}_T(n)| \leq 2^n$ and so each level $\mathrm{Lev}_T(n)$ is finite.

Claim 17.10. For each $n \geq 0$, $\mathrm{Lev}_T(n) \neq \emptyset$.

Proof. Clearly we can suppose that $n \geq 1$. Let Σ_n be the set of wffs in Σ which only involve $A_1, \dots, A_n$. If Σ_n is finite, the result holds by the finite satisfiability of Σ. Hence we can suppose that Σ_n is infinite; say $\Sigma_n = \{\sigma_1, \sigma_2, \dots, \sigma_t, \dots\}$. For each $t \geq 1$, let $\Delta_t = \{\sigma_1, \dots, \sigma_t\}$. Then there exists a partial truth assignment $\omega_t\colon \{A_1, \dots, A_n\} \to \{T, F\}$ which satisfies Δ_t. By the pigeon hole principle, there exists a *fixed* $\omega\colon \{A_1, \dots, A_n\} \to \{T, F\}$ such that $\omega_t = \omega$ for infinitely many $t \geq 1$. Clearly $\omega \in \mathrm{Lev}_T(n)$. □

Thus T is an infinite, finitely branching tree. By König's Lemma, there exists an infinite branch $\mathcal{B} = \{\upsilon_n \mid n \in \mathbb{N}\}$ through T, where $\upsilon_n \in \mathrm{Lev}_T(n)$. It follows that $\upsilon = \bigcup_n \upsilon_n$ is a truth assignment which satisfies Σ. □

18 First Order Logic

Definition 18.1. The *alphabet* of a first order language $\mathcal{L}$ consists of:

A. Symbols common to all languages (Logical Symbols)

   (a) Parentheses (,)

   (b) Connectives $\to, \neg$

   (c) Variables $v_1, v_2, \dots, v_n, \dots$ $n \geq 0$

   (d) Quantifier $\forall$

   (e) Equality symbol $=$

B. Symbols particular to the language (Non-logical Symbols)

   (a) For each $n \geq 1$, a (possibly empty) countable set of n-place predicate symbols.

   (b) A (possibly empty) countable set of constant symbols.

   (c) For each $n \geq 1$, a (possibly empty) countable set of n-place function symbols.

Remark 18.2. It is easily checked that the alphabet is countable.

Definition 18.3. An *expression* is a finite sequence of symbols from the alphabet.

Remark 18.4. The set of expressions is countable.

Definition 18.5. The set of terms is defined inductively as follows:

1. Each variable and each constant symbol is a term.
2. If f is an n-place function symbol and $t_1, \ldots, t_n$ are terms, then $ft_1 \ldots t_n$ is a term.

Definition 18.6. An *atomic formula* is an expression of the form

$$Pt_1 \ldots t_n$$

where P is an n-place predicate symbol and $t_1, \ldots, t_n$ are terms.

Remark 18.7. The equality symbol $=$ is a two-place predicate symbol. Hence every language has atomic formulas.

Definition 18.8. The set of *well-formed formulas* (wffs) is defined inductively as follows:

1. Every atomic formula is a wff.
2. If α and β are wffs and v is a variable, then

$$(\neg\alpha), (\alpha \to \beta), \quad \text{and} \quad \forall v\alpha$$

are wffs.

Some abbreviations We usually write

$$\begin{array}{rcl} (\alpha \vee \beta) & \text{instead of} & ((\neg\alpha) \to \beta) \\ (\alpha \wedge \beta) & \text{"} & (\neg(\alpha \to (\neg\beta))) \\ \exists v\alpha & \text{"} & (\neg\forall v(\neg\alpha)) \\ u = t & \text{"} & = ut \\ u \neq t & \text{"} & (\neg = ut) \end{array}$$

We also use common sense in our use of parentheses.

Definition 18.9. Let x be a variable.

1. If α is atomic, then x *occurs free* in α iff x occurs in α.
2. x occurs free in $(\neg\alpha)$ iff x occurs free in α.
3. x occurs free in $(\alpha \to \beta)$ iff x occurs free in α or x occurs free in β.
4. x occurs free in $\forall v\alpha$ iff x occurs free in α and $x \neq v$.

Definition 18.10. The wff σ is a *sentence* iff σ has no free variables.

19 Truth and Structures

Definition 19.1. A *structure* $\mathcal{A}$ for the first order language $\mathcal{L}$ consists of:

1. a non-empty set A, the *universe* of $\mathcal{A}$.
2. for each n-place predicate symbol P, an n-ary relation $P^{\mathcal{A}} \subseteq A^n$.
3. for each constant symbol c, an element $c^{\mathcal{A}} \in A$.
4. for each function symbol f, an n-ary operation $f^{\mathcal{A}}\colon A^n \to A$.

Example 19.2. Suppose that $\mathcal{L}$ has the following non-logical symbols:

1. a 1-place predicate symbol S
2. a 2-place predicate symbol R
3. a constant symbol c
4. a 1-place function symbol f.

Then we can define a structure

$$\mathcal{A} = \langle A; S^{\mathcal{A}}, R^{\mathcal{A}}, c^{\mathcal{A}}, f^{\mathcal{A}} \rangle$$

for $\mathcal{L}$ as follows:

1. $A = \{1, 2, 3, 4\}$
2. $S^{\mathcal{A}} = \{\langle 2 \rangle, \langle 3 \rangle\}$
3. $R^{\mathcal{A}} = \{\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 4 \rangle, \langle 4, 1 \rangle\}$
4. $c^{\mathcal{A}} = 1$
5. $f^{\mathcal{A}}\colon A \to A$ where $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 4$, and $4 \mapsto 1$.

Target Let $\mathcal{L}$ be any first order language. For each sentence σ and each structure $\mathcal{A}$ for $\mathcal{L}$, we want to define

$$\mathcal{A} \models \sigma$$

"$\mathcal{A}$ satisfies σ" or "σ is true in $\mathcal{A}$".

Example 19.3 (Example Cont.). Let σ be the sentence

$$\forall x \forall y (fx = y \to Rxy)$$

Clearly

$$\mathcal{A} \models \sigma.$$

First we need to define a more involved notion. Let

- φ be a wff
- $\mathcal{A}$ be a structure for $\mathcal{L}$
- $s\colon V \to A$ be a function, where v is the set of variables.

Then we will define

$$\mathcal{A} \models \varphi[s]$$

"φ is true in $\mathcal{A}$ if each free occurence of x in φ is interpreted as $s(x)$ in A."

Step 1
Let T be the set of terms. We first define an extension $\bar{s}\colon T \to A$ as follows:

1. For each variable $v \in V$, $\bar{s}(v) = s(v)$.
2. For each constant symbol c, $\bar{s}(c) = c^{\mathcal{A}}$.
3. If f is an n-place function symbol and $t_1, \ldots, t_n$ are terms, then

$$\bar{s}(ft_1 \ldots t_n) = f^{\mathcal{A}}(\bar{s}(t_1), \ldots, \bar{s}(t_n)).$$

Step 2 Atomic formulas.

(a). $\mathcal{A} \models\ = t_1 t_2[s] \quad \text{iff} \quad \bar{s}(t_1) = \bar{s}(t_2)$.

(b). If P is an n-place predicate symbol different from $=$ and $t_1, \ldots, t_n$ are terms, then

$$\mathcal{A} \models Pt_1 \ldots t_n[s] \;\text{ iff }\; \langle \bar{s}(t_1), \ldots, \bar{s}(t_n) \rangle \in P^{\mathcal{A}}.$$

Step 3 Other wffs.

(c). $\mathcal{A} \models (\neg\alpha)[s]$ iff $\mathcal{A} \not\models \alpha[s]$.

(d). $\mathcal{A} \models (\alpha \to \beta)[s]$ iff $\mathcal{A} \not\models \alpha[s]$ or $\mathcal{A} \models \beta[s]$.

(e). $\mathcal{A} \models \forall x \alpha[s]$ iff for all $a \in A$, $\mathcal{A} \models \alpha[s(x|a)]$ where $s(x|a)$ is defined by

$$\begin{aligned} s(x|a)(y) &= s(y), \quad y \neq x \\ &= a, \quad y = x \end{aligned}$$

Theorem 19.4. *Assume that $s_1, s_2 \colon V \to A$ agree on all free variables (if any) of the wff φ. Then*

$$\mathcal{A} \models \varphi[s_1] \quad \textit{iff} \quad \mathcal{A} \models \varphi[s_2].$$

Proof slightly delayed.

Corollary 19.5. *If σ is a sentence, then either*

1. $\mathcal{A} \models \sigma[s]$ *for all* $s \colon V \to A$ *or*
2. $\mathcal{A} \not\models \sigma[s]$ *for all* $s \colon V \to A$.

Definition 19.6. Let σ be a sentence. Then $\mathcal{A} \models \sigma$ iff $\mathcal{A} \models \sigma[s]$ for all $s \colon V \to A$.

Exercise 19.7. Let $\mathcal{A}$ be a structure and let t be a term. If $s_1, s_2 \colon V \to A$ agree on all variables (if any) in t, then $\bar{s_1}(t) = \bar{s_2}(t)$.

Proof of Theorem 19.4. We argue by induction on the complexity of φ.

Case 1 Suppose that φ is an atomic formula. First suppose that φ is $= t_1 t_2$. By the Exercise, $\bar{s_1}(t_1) = \bar{s_2}(t_1)$ and $\bar{s_1}(t_2) = \bar{s_2}(t_2)$. Hence

$$\begin{aligned} \mathcal{A} \models \ = t_1 t_2[s_1] \quad & \text{iff} \quad \bar{s_1}(t_1) = \bar{s_1}(t_2) \\ & \text{iff} \quad \bar{s_2}(t_1) = \bar{s_2}(t_2) \\ & \text{iff} \quad \mathcal{A} \models \ = t_1 t_2[s_2]. \end{aligned}$$

Next suppose that φ is $P t_1 \dots t_n$. Again by the Exercise, $\bar{s_1}(t_i) = \bar{s_2}(t_i)$ for $1 \le i \le n$. Hence

$$\begin{aligned} \mathcal{A} \models P t_1 \dots t_n[s_1] \quad & \text{iff} \quad \langle \bar{s_1}(t_1), \dots, \bar{s_1}(t_n) \rangle \in P^{\mathcal{A}} \\ & \text{iff} \quad \langle \bar{s_2}(t_1), \dots, \bar{s_2}(t_n) \rangle \in P^{\mathcal{A}} \\ & \text{iff} \quad \mathcal{A} \models P t_1 \dots t_n[s_2]. \end{aligned}$$

Case 2 Suppose that φ is $(\neg\psi)$. Then s_1, s_2 agree on the free variables of ψ. Hence

$$\begin{aligned} \mathcal{A} \models (\neg\psi)[s_1] \quad & \text{iff} \quad \mathcal{A} \not\models \psi[s_1] \\ & \text{iff} \quad \mathcal{A} \not\models \psi[s_2] \text{ by ind. hyp.} \\ & \text{iff} \quad \mathcal{A} \models (\neg\psi)[s_2]. \end{aligned}$$

Case 3 A similar argument deals with the case when φ is $(\psi \to \theta)$.

Case 4 Suppose that φ is $\forall x\psi$. Then s_1, s_2 agree on all free variables of ψ except possibly x. Hence for all $a \in A$, $s_1(x|a)$ and $s_2(x|a)$ agree on all free variables of ψ. Thus

$$\begin{aligned} \mathcal{A} \models \forall x\psi[s_1] \quad & \text{iff} \quad \text{for all } a \in A,\ \mathcal{A} \models \psi[s_1(x|a)] \\ & \text{iff} \quad \text{for all } a \in A,\ \mathcal{A} \models \psi[s_2(x|a)] \\ & \text{iff} \quad \mathcal{A} \models \forall x\psi[s_2]. \end{aligned}$$

□

20 Compactness in first order logic

Definition 20.1. Let Σ be a set of wffs.

(a) $\mathcal{A}$ *satisfies* Σ with s iff $\mathcal{A} \models \sigma[s]$ for all $\sigma \in \Sigma$.

(b) Σ is *satisfiable* iff there exists a structure $\mathcal{A}$ and a function $s \colon V \to A$ such that $\mathcal{A}$ satisfies Σ with s.

(c) Σ is *finitely satisfiable* iff every finite subset of Σ is satisfiable.

One of the deepest results of the course:

Theorem 20.2 (Compactness). *Let Σ be a set of wffs in the first order language $\mathcal{L}$. If Σ is finitely satisfiable, then Σ is satisfiable.*

Application of the Compactness Theorem Let $\mathcal{L}$ be the language of arithmetic; ie $\mathcal{L}$ has non-logical symbols $\{+, \times, <, 0, 1\}$. Let

$$\mathrm{Th}\mathbb{N} = \{\sigma \mid \sigma \text{ is a sentence satisfied by } \langle \mathbb{N}; +, \times, <, 0, 1\rangle\}.$$

Consider the following set Σ of wffs:

$$\mathrm{Th}\mathbb{N} \cup \{x > \underbrace{1 + \ldots + 1}_{n \text{ times}} | n \geq 1\}.$$

We claim that Σ is finitely satisfiable. To see this, suppose that $\Sigma_0 \subseteq \Sigma$ is any finite subset; say, $\Sigma_0 = T \cup \{x > \underbrace{1 + \ldots + 1}_{n_1}, \ldots, x > \underbrace{1 + \ldots + 1}_{n_t}\}$, where $T \subseteq \mathrm{Th}\mathbb{N}$. Let $m = \max\{n_1, \ldots, n_t\}$ and let $s \colon V \to \mathbb{N}$ with $s(x) = m + 1$. Then $\mathbb{N}$ satisfies Σ_0

with s. By the Compactness Theorem, there exists a structure $\mathcal{A}$ for $\mathcal{L}$ and a function $s\colon V \to A$ such that $\mathcal{A}$ satisfies Σ with s. Thus $\mathcal{A}$ is a "model of artihmetic" containing the "infinite natural number" $s(x) \in A$.

Discussion of the order relation in $\mathcal{A}$....

Now we return to the systematic development of first order logic.

Definition 20.3. Let $\mathcal{A}, \mathcal{B}$ be structures for the language $\mathcal{L}$. A function $f\colon A \to B$ is an *isomorphism* iff the following conditions are satisfied.

1. f is a bijection.

2. For each n-ary predicate symbol P and any n-tuple $a_1, \dots, a_n \in A$,
$$\langle a_1, \dots, a_n \rangle \in P^{\mathcal{A}} \ \text{ iff } \ \langle f(a_1), \dots, f(a_n) \rangle \in P^{\mathcal{B}}.$$

3. For each constant symbol c, $f(c^{\mathcal{A}}) = c^{\mathcal{B}}$.

4. For each n-ary function symbol h and n-tuple $a_1, \dots, a_n \in A$,
$$f(h^{\mathcal{A}}(a_1, \dots, a_n)) = h^{\mathcal{B}}(f(a_1), \dots, f(a_n)).$$

We write $\mathcal{A} \cong \mathcal{B}$ iff $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

Theorem 20.4. *Suppose that $\varphi\colon A \to B$ is an isomorphism. If σ is any sentence, then $\mathcal{A} \models \sigma$ iff $\mathcal{B} \models \sigma$.*

In order to prove the above theorem, we must prove the following more general statement.

Theorem 20.5. *Suppose that $\varphi\colon A \to B$ is an isomorphism and $s\colon V \to A$. Then for any wff α*
$$\mathcal{A} \models \alpha[s] \ \ \textit{iff} \ \ \mathcal{B} \models \alpha[\varphi \circ s].$$

We shall make use of the following result.

Lemma 20.6. *With the above hypotheses, for each term t,*
$$\varphi(\bar{s}(t)) = (\overline{\varphi \circ s})(t).$$

Proof. Exercise. □

Proof of Theorem 20.5. We argue by induction of the complexity of α. First suppose that α is atomic, say $Pt_1 \dots t_n$. Then
$$\begin{aligned}
\mathcal{A} \models Pt_1 \dots t_n[s] \quad & \text{iff} \quad \langle \bar{s}(t_1), \dots, \bar{s}(t_n) \rangle \in P^{\mathcal{A}} \\
& \text{iff} \quad \langle \varphi(\bar{s}(t_1)), \dots, \varphi(\bar{s}(t_n)) \rangle \in P^{\mathcal{B}} \\
& \text{iff} \quad \langle (\overline{\varphi \circ s})(t_1)), \dots, (\overline{\varphi \circ s})(t_n) \rangle \in P^{\mathcal{B}} \\
& \text{iff} \quad \mathcal{B} \models Pt_1 \dots t_n[\varphi \circ s]
\end{aligned}$$

Next suppose that α is $\neg\beta$. Then

$$\begin{aligned} \mathcal{A} \models \neg\beta[s] \quad &\text{iff} \quad \mathcal{A} \not\models \beta[s] \\ &\text{iff} \quad \mathcal{B} \not\models \beta[\varphi \circ s] \\ &\text{iff} \quad \mathcal{B} \models \neg\beta[\varphi \circ s] \end{aligned}$$

A similar argument deals with the case when α is $(\beta \implies \gamma)$.

Finally suppose that α is $\forall v\beta$. Then

$$\begin{aligned} \mathcal{A} \models \forall v\beta[s] \quad &\text{iff} \quad \mathcal{A} \models \beta[s(v|a)], \text{ for all } a \in A \\ &\text{iff} \quad \mathcal{B} \models \beta[\varphi \circ s(v|a)], \text{ for all } a \in A \\ &\text{iff} \quad \mathcal{B} \models \beta[(\varphi \circ s)(v|\varphi(a))], \text{ for all } a \in A \\ &\text{iff} \quad \mathcal{B} \models \beta[(\varphi \circ s)(v|b)], \text{ for all } b \in B \\ &\text{iff} \quad \mathcal{B} \models \forall v\beta[\varphi \circ s] \end{aligned}$$

□

Example 20.7. $\langle \mathbb{N}, < \rangle \ncong \langle \mathbb{Z}, < \rangle$.

Proof. Consider the sentence σ given by

$$(\exists x)(\forall y)(y = x \vee x < y).$$

Then $\langle \mathbb{N}, < \rangle \models \sigma$ and $\langle \mathbb{Z}, < \rangle \not\models \sigma$. Thus $\langle \mathbb{N}, < \rangle \ncong \langle \mathbb{Z}, < \rangle$. □

Example 20.8. $\langle \mathbb{Z}, < \rangle \ncong \langle \mathbb{Q}, < \rangle$.

Proof. Consider the sentence σ given by

$$(\forall x)(\forall y)(x < y \rightarrow (\exists z)(x < z \wedge z < y)).$$

□

Definition 20.9. Let T be a set of sentences.

1. $\mathcal{A}$ is a *model* for T iff $\mathcal{A} \models \sigma$ for every $\sigma \in T$.

2. $\mathrm{Mod}(T)$ is the class of all models of T.

Abbreviation If E is a binary predicate symbol, then we usually write xEy instead of Exy.

Example 20.10. Let T be the following set of sentences:

$$\neg(\exists x)(xEx)$$

$$(\forall x)(\forall y)(xEy \rightarrow yEx).$$

Then $\mathrm{Mod}(T)$ is the class of graphs.

Example 20.11. Let T be the following set of sentences:

$$\neg(\exists x)(xEx)$$

$$(\forall x)(\forall y)(\forall z)((xEy \wedge yEz) \rightarrow xEz)$$
$$(\forall x)(\forall y)(x = y \vee xEy \vee yEx)$$

Then $\mathrm{Mod}(T)$ is the class of linear orders.

Definition 20.12. A class $\mathcal{C}$ of structures is *axiomatizable* iff there is a set T of sentences such that $\mathcal{C} = \mathrm{Mod}(T)$. If there exists a finite set T of sentences such that $\mathcal{C} = \mathrm{Mod}(T)$, then $\mathcal{C}$ is *finitely axiomatizable*.

Example 20.13. The class of graphs is finitely axiomatizable.

Example 20.14. The class of infinite graphs is axiomatizable.

Proof. For each $n \geq 1$ let $\mathcal{O}_n$ be the sentence

"There exist at least n elements."

For example $\mathcal{O}_3$ is the sentence

$$(\exists x)(\exists y)(\exists z)(x \neq y \wedge y \neq z \wedge z \neq x).$$

Then $\mathcal{C} = \mathrm{Mod}(T)$, where T is the following set of sentences:

$$\neg(\exists x)(xEx)$$

$$(\forall x)(\forall y)(xEy \rightarrow yEx)$$
$$\mathcal{O}_n, \quad n \geq 1.$$

□

Question 20.15. Is the class of infinite graphs finitely axiomatizable?

Question 20.16. Is the class of finite graphs axiomatizable?

Another application of the Compactness Theorem...

Theorem 20.17. *Let T be a set of sentences in a first order language $\mathcal{L}$. If T has arbitrarily large finite models, then T has an infinite model.*

Proof. For each $n \geq 1$, let $\mathcal{O}_n$ be the sentence which says:

"There exist at least n elements."

Let Σ be the set of sentences $T \cup \{\mathcal{O}_n \mid n \geq 1\}$. We claim that Σ is finitely satisfiable. Suppose $\Sigma_0 \subseteq \Sigma$ is any finite subset. Then wlog

$$\Sigma_0 = T \cup \{\mathcal{O}_{n_1}, \dots, \mathcal{O}_{n_t}\}.$$

Let $m = \max\{n_1, \dots, n_t\}$. Then there exists a finite model $\mathcal{A}_0$ of T such that $\mathcal{A}_0$ has at least m elements. Clearly $\mathcal{A}_0$ satisfies Σ_0. By the Compactness Theorem, there exists a model $\mathcal{A}$ of Σ. Clearly $\mathcal{A}$ is an infinite model of T. □

Corollary 20.18. *The class $\mathcal{F}$ of finite graphs is* not *axiomatizable.*

Proof. Suppose T is a set of sentences such that $\mathcal{F} = \mathrm{Mod}(T)$. Clearly there are arbitrarily large finite graphs and hence T has arbitrarily large finite models. But this means that T has an infinite model, which is a contradiction. □

Corollary 20.19. *The class $\mathcal{C}$ of infinite graphs is not finitely axiomatizable.*

Proof. Suppose that there exists a finite set $T = \{\varphi_1, \dots, \varphi_n\}$ of sentences such that $\mathcal{C} = \mathrm{Mod}(T)$. Consider the following set T' of sentences.

$$\neg(\exists x)(xEx)$$

$$(\forall x)(\forall y)(xEy \to yEx)$$

$$\neg(\varphi_1 \wedge \dots \wedge \varphi_n).$$

Then clearly $\mathrm{Mod}(T')$ is the class of finite graphs, which is a contradiction. □

21 Valid sentences

Definition 21.1. Let Σ be a set of wffs and let φ be a wff. Then Σ *logically implies/semantically implies* φ iff for every structure $\mathcal{A}$ and for every function $s\colon V \to A$, if $\mathcal{A}$ satisfies Σ with s, then $\mathcal{A}$ satisfies φ with s. In this case we write $\Sigma \models \varphi$.

Definition 21.2. The wff φ is *valid* iff $\emptyset \models \varphi$; *ie*, for all structures $\mathcal{A}$ and functions $s\colon V \to A$, $\mathcal{A} \models \varphi[s]$.

Example 21.3. $\{\forall x Px\} \models Pc$.

Question 21.4. Suppose that Σ is an infinite set of wffs and that $\Sigma \models \varphi$. Does there exist a finite set $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \varphi$?

Answer Yes. We shall show that $\Sigma \models \varphi$ iff there exists a proof of φ from Σ. Such a proof will only use a finite subset $\Sigma_0 \subseteq \Sigma$.

We now return to the syntactic aspect of first order languages. We will next define rigorously the notion of a *deduction* or proof.

Notation Λ will denote the set of *logical axioms*. These will be defined explicitly a little later.

$$\text{eg} \;\; (\forall x(\alpha \to \beta) \to (\forall x\alpha \to \forall x\beta)).$$

Each logical axiom will be valid.

Definition 21.5. Let Γ be a set of wffs and φ a wff. A *deduction* of φ from Γ is a finite sequence of wffs

$$\langle \alpha_1, \dots, \alpha_n \rangle$$

such that $\alpha_n = \varphi$ and for each $1 \le i \le n$, either:

(a) $\alpha_i \in \Lambda \cup \Gamma$; or

(b) there exist $j, k < i$ such that α_k is $(\alpha_j \to \alpha_i)$.

Remark 21.6. In case (b), we have

$$\langle \alpha_1, \dots, \alpha_j, \dots, (\alpha_j \to \alpha_i), \dots, \alpha_i, \dots, \alpha_n \rangle$$

We say that α_i follows from α_j and $(\alpha_j \to \alpha_i)$ by modus ponens (MP).

Definition 21.7. φ is a *theorem* of Γ, written $\Gamma \vdash \varphi$, iff there exists a deduction of φ from Γ.

The two main results of this course...

Theorem 21.8 (Soundness). *If* $\Gamma \vdash \varphi$*, then* $\Gamma \models \varphi$.

Theorem 21.9 (Completeness (Godel)). *If* $\Gamma \models \varphi$*, then* $\Gamma \vdash \varphi$.

22 The Logical Axioms Λ

φ is a *generalization* of ψ iff for some $n \ge 0$ and variables $x_1, \dots, x_n$, we have that φ is

$$\forall x_1 \dots \forall x_n \psi.$$

The logical axioms are all generalizations of all wffs of the following forms:

1. Tautologies.
2. $(\forall x\alpha \to \alpha^x_t)$, where t is a term which is substitutable for x in α.

3. $(\forall x(\alpha \to \beta) \to (\forall x\alpha \to \forall x\beta))$.
4. $(\alpha \to \forall x\alpha)$, where x doesn't appear free in α.
5. $x = x$.
6. $(x = y \to (\alpha \to \alpha'))$, where α is atomic and α' is obtained from α be replacing some (possibly none) of the occurrences of x by y.

Explanation 1. A tautology is a wff that can be obtained from a propostional tautology by substituting wffs for sentence symbols.

$$\textit{eg} \quad (P \to \neg Q) \to (Q \to \neg P)$$

is a propositional tautology.

$$(\forall x\alpha \to \neg\beta) \to (\beta \to \neg\forall x\alpha)$$

is a first order tautology.

Explanation 2. α^x_t is the result of replacing each free occurrence of x by t. We say that t is *substitutable* for x in α iff no variable of t gets bound by a quantifier in α^x_t.

eg Let α be $\neg\forall y(x = y)$. Then y is *not* substitutable for x in α. Note that in this case

$$\forall x\alpha \to \alpha^x_t$$

becomes

$$\forall x\neg\forall y(x = y) \to \neg\forall y(y = y)$$

which is *not* valid. So we need the above restriction.

Explanation 4. A typical example is

$$Pyz \to \forall xPyz.$$

Here "$\forall x$" is a "dummy quantifier" which does nothing. Note that

$$x = 0 \to \forall x(x = 0)$$

is *not* valid. So we need the above restriction.

23 Some examples of deductions

Example 23.1. $\vdash (Px \to \exists y Py)$

Proof. Note that $(Px \to \exists y Py)$ is an abbreviation of $(Px \to \neg\forall y \neg Py)$. The following is a deduction from $\emptyset$.

1. $(\forall y \neg Py \to \neg Px) \to (Px \to \neg\forall y \neg Py)$ [Axiom 1]
2. $(\forall y \neg Py \to \neg Px)$ [Axiom 2]
3. $(Px \to \neg\forall y \neg Py)$ [MP, 1, 2]

$\square$

Example 23.2. $\vdash \forall x(Px \to \neg\forall y \neg Py)$

Proof. The following is a deduction from $\emptyset$.

1. $\forall x((\forall y \neg Py \to \neg Px) \to (Px \to \neg\forall y \neg Py))$ [Axiom 1]
2. $\forall x(\forall y \neg Py \to \neg Px)$ [Axiom 2]
3. $\forall x((\forall y \neg Py \to \neg Px) \to (Px \to \neg\forall y \neg Py)) \to (\forall x(\forall y \neg Py \to \neg Px) \to \forall x(Px \to \neg\forall y \neg Py))$ [Axiom 3]
4. $\forall x(\forall y \neg Py \to \neg Px) \to \forall x(Px \to \neg\forall y \neg Py)$ [MP, 1, 3]
5. $\forall x(Px \to \neg\forall y \neg Py))$ [MP, 2, 4]

$\square$

24 Soundness Theorem

Theorem 24.1 (Soundness). *If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.*

We shall make use of the following result.

Lemma 24.2. *Every logical axiom $\varphi \in \Lambda$ is valid*

Proof. We just consider the case where φ has the form

$$(\alpha \to \forall x \alpha)$$

where x isn't free in α. Let $\mathcal{A}$ be any structure and $s \colon V \to A$. If $\mathcal{A} \not\models \alpha[s]$, then $\mathcal{A} \models (\alpha \to \forall x \alpha)[s]$. So suppose that $\mathcal{A} \models \alpha[s]$. Let $a \in A$ be any element. Then s and $s(x|a)$ agree on the free variables of α. Hence $\mathcal{A} \models \alpha[s(x|a)]$ and so $\mathcal{A} \models (\alpha \to \forall x \alpha)[s]$. $\square$

Remark 24.3. The other cases are equally easy, except for the case of

$$(\forall x\alpha \to \alpha^x_t)$$

which is harder. We will give a detailed proof of this case later.

Exercise 24.4. Show that

$$(\forall x(\alpha \to \beta) \to (\forall x\alpha \to \beta))$$

is valid.

Proof of the Soundness Theorem. We argue by induction on the minimal length $n \geq 1$ of a deduction that if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.

First suppose that $n = 1$. Then $\varphi \in \Gamma \cup \Lambda$. If $\varphi \in \Gamma$ then clearly $\Gamma \models \varphi$. If $\varphi \in \Lambda$, then the lemma (24.2) says that φ is valid. Thus $\emptyset \models \varphi$ and so $\Gamma \models \varphi$.

Now suppose that $n > 1$. Let

$$\langle \alpha_1, \dots, \alpha_n = \varphi \rangle$$

be a deduction of φ from Γ. Then φ must follow from MP from two earlier wffs θ and $(\theta \to \varphi)$. Note that proper initial segments of deductions from Γ are also deductions from Γ. Thus $\Gamma \vdash \theta$ and $\Gamma \vdash (\theta \to \varphi)$ via deductions of length less than n. By induction hypothesis, $\Gamma \models \theta$ and $\Gamma \models (\theta \to \varphi)$. Let $\mathcal{A}$ be any structure and $s\colon V \to A$. Suppose that $\mathcal{A}$ satisfies Γ with s. Then $\mathcal{A} \models \theta[s]$ and $\mathcal{A} \models (\theta \to \varphi)[s]$. Hence $\mathcal{A} \models \varphi[s]$. Thus $\Gamma \models \varphi$. $\square$

Definition 24.5. A set Γ of wffs is *inconsistent* iff there exists a wff β such that $\Gamma \vdash \beta$ and $\Gamma \vdash \neg\beta$. Otherwise, Γ is *consistent*.

Corollary 24.6. *If Γ is satisfiable, then Γ is consistent.*

Proof. Suppose that Γ is satisfiable. Let $\mathcal{A}$ satisfy Γ with $s\colon V \to A$. Now suppose that Γ is inconsistent; say $\Gamma \vdash \beta$ and $\Gamma \vdash \neg\beta$. By Soundness $\Gamma \models \beta$ and $\Gamma \models \neg\beta$. But this means that $\mathcal{A} \models \beta[s]$ and $\mathcal{A} \models \neg\beta[s]$, which is a contradiction. $\square$

25 Meta-theorems

Now we turn to the proof of the Completeness Theorem. First we need to prove a number of "Meta-Theorems".

Theorem 25.1 (Generalization). *If $\Gamma \vdash \varphi$ and x doesn't occur free in any wff of Γ, then $\Gamma \vdash \forall x\varphi$.*

Remark 25.2. Note that if c is a constant symbol, then

$$\{x = c\} \vdash x = c.$$

However,

$$\{x = c\} \nvdash \forall x(x = c).$$

How do we know this? By the Soundness Theorem, it is enough to show that

$$\{x = c\} \not\models \forall x(x = c).$$

Proof of Generalization Theorem. We argue by induction on the minimal length n of a deduction of φ from Γ that $\Gamma \vdash \forall x\varphi$.

First suppose that $n = 1$. Then $\varphi \in \Gamma \cup \Lambda$.

Case 1 Suppose that $\varphi \in \Lambda$. Then $\forall x\varphi \in \Lambda$ and so $\Gamma \vdash \forall x\varphi$.

Case 2 Suppose that $\varphi \in \Gamma$. Then x doesn't occur free in φ and so $(\varphi \to \forall x\varphi) \in \Lambda$. Hence the following is a deduction of $\forall x\varphi$ from Γ.

1. φ [in Γ]
2. $\varphi \to \forall x\varphi$ [Ax 4]
3. $\forall x\varphi$ [MP, 1, 2]

Now suppose that $n > 1$. Then in a deduction of minimal length, φ follows from earlier wffs θ and $(\theta \to \varphi)$ by MP. By induction hypothesis, $\Gamma \vdash \forall x\theta$ and $\Gamma \vdash \forall x(\theta \to \varphi)$. Hence the following is a deduction of $\forall x\varphi$ from Γ.

1. … deduction of $\forall x\theta$ from Γ.

n. $\forall x\theta$

n+1. … deduction of $\forall x(\theta \to \varphi)$ from Γ.

n+m. $\forall x(\theta \to \varphi)$

n+m+1. $\forall x(\theta \to \varphi) \to (\forall x\theta \to \forall x\varphi)$ [Ax 3]

n+m+2. $\forall x\theta \to \forall x\varphi$ [MP, $n + m$, $n + m + 1$]

n+m+3. $\forall x\varphi$ [MP, n, $n + m + 2$]

□

Definition 25.3. $\{\alpha_1, \ldots, \alpha_n\}$ *tautologically implies* β iff

$$(\alpha_1 \to (\alpha_2 \to \ldots (\alpha_n \to \beta) \ldots))$$

is a tautology.

Theorem 25.4 (Rule T). *If* $\Gamma \vdash \alpha_1, \ldots, \Gamma \vdash \alpha_n$ *and* $\{\alpha_1, \ldots, \alpha_n\}$ *tautologically implies* β*, the* $\Gamma \vdash \beta$*.*

Proof. Obvious, via repeated applications of MP. □

Theorem 25.5 (Deduction). *If* $\Gamma \cup \{\gamma\} \vdash \varphi$*, then* $\Gamma \vdash (\gamma \to \varphi)$*.*

Proof. We argue by induction on the minimal length n of a deduction of φ from $\Gamma \cup \{\gamma\}$.

First suppose that $n = 1$.

Case 1 Suppose that $\varphi \in \Gamma \cup \Lambda$. Then the following is a deduction from Γ.

1. φ [in $\Gamma \cup \Lambda$]
2. $(\varphi \to (\gamma \to \varphi))$ [Ax 1]
3. $(\gamma \to \varphi)$ [MP, 1, 2]

Case 2 Suppose that $\varphi = \gamma$. In this case $(\gamma \to \varphi)$ is a tautology and so $\Gamma \vdash (\gamma \to \varphi)$.

Now suppose that $n > 1$. Then in a deduction of minimal length φ follows from earlier wffs θ and $(\theta \to \varphi)$ by MP. By induction hypothesis, $\Gamma \vdash (\gamma \to \theta)$ and $\Gamma \vdash (\gamma \to (\theta \to \varphi))$. Clearly $\{(\gamma \to \theta), (\gamma \to (\theta \to \varphi))\}$ tautologically implies $(\gamma \to \varphi)$. By Rule T, $\Gamma \vdash (\gamma \to \varphi)$. □

Theorem 25.6 (Contraposition). $\Gamma \cup \{\varphi\} \vdash \neg\psi$ *iff* $\Gamma \cup \{\psi\} \vdash \neg\varphi$*.*

Proof. Suppose that $\Gamma \cup \{\varphi\} \vdash \neg\psi$. By the deduction theorem $\Gamma \vdash (\varphi \to \neg\psi)$. By Rule T, $\Gamma \vdash (\psi \to \neg\varphi)$. Hence $\Gamma \cup \{\psi\} \vdash \psi$ amd $\Gamma \cup \{\psi\} \vdash (\psi \to \neg\varphi)$. By Rule T, $\Gamma \cup \{\psi\} \vdash \neg\varphi$.

The other direction is similar. □

Theorem 25.7 (Reductio Ad Absurdum). *If* $\Gamma \cup \{\varphi\}$ *is inconsistent, then* $\Gamma \vdash \neg\varphi$*.*

Proof. Suppose that $\Gamma \cup \{\varphi\} \vdash \beta$ and $\Gamma \cup \{\varphi\} \vdash \neg\beta$. By the Deduction Theorem , $\Gamma \vdash (\varphi \to \beta)$ and $\Gamma \vdash (\varphi \to \neg\beta)$. Since $\{(\varphi \to \beta), (\varphi \to \neg\beta)\}$ tautologically implies $\neg\varphi$, Rule T gives $\Gamma \vdash \neg\varphi$. □

Remark 25.8. If Γ is inconsistent, then $\Gamma \vdash \alpha$ for ever wff α.

Proof. Suppose that $\Gamma \vdash \beta$ and $\Gamma \vdash \neg\beta$. Clearly

$$(\beta \to (\neg\beta \to \alpha))$$

is a tautology. By Rule T, $\Gamma \vdash \alpha$. □

26 Applications: some theorems about equality

Eq 1.
$\vdash \forall x(x = x)$

Proof. This is a logical axiom. □

Eq 2.
$\vdash \forall x \forall y(x = y \to y = x)$

Proof.

1. $\vdash x = y \to (x = x \to y = x)$ [Ax 6]
2. $\vdash x = x$ [Ax 5]
3. $\vdash x = y \to y = x$ [Rule T, 1, 2]
4. $\vdash \forall y(x = y \to y = x)$ [Gen, 3]
5. $\vdash \forall x \forall y(x = y \to y = x)$ [Gen, 4]

□

Eq 3.
$\vdash \forall x \forall y \forall z(x = y \to (y = z \to x = z))$

Proof.

1. $\vdash y = x \to (y = z \to x = z)$ [Ax 6]
2. $\vdash x = y \to y = x$ [Shown in proof of Eq 2]
3. $\vdash x = y \to (y = z \to x = z)$ [Rule T, 1, 2]
4. $\vdash \forall x \forall y \forall z(x = y \to (y = z \to x = z))$ [Gen cubed, 3]

□

27 Generalization on constants

Theorem 27.1 (Generalization on constants). *Assume that $\Gamma \vdash \varphi$ and that c is a constant symbol which doesn't occur in Γ. Then there exists a variable y (which doesn't occur in φ) such that $\Gamma \vdash \forall y \varphi^c_y$.*

Furthermore, there exists a deduction of $\forall y \varphi^c_y$ from Γ in which c doesn't occur.

Remark 27.2. Intuitively, suppose that Γ says nothing about c and that $\Gamma \vdash \varphi(c)$. Then $\Gamma \vdash \forall y \varphi(y)$. In other words, to prove $\forall y \varphi(y)$, let c be arbitrary and prove $\varphi(c)$.

Remark 27.3. Suppose that Γ is a consistent set of wffs in the language $\mathcal{L}$. Let $\mathcal{L}^+$ be the language obtained by adding a new constant symbol c. Then Γ is still consistent in $\mathcal{L}^+$.

Why? Suppose not. Then there exists a wff β in $\mathcal{L}^+$ such that $\Gamma \vdash \beta \wedge \neg\beta$ in $\mathcal{L}^+$. By the above theorem, for some variable y which doesn't occur in β,

$$\Gamma \vdash \forall y(\beta^c_y \wedge \neg\beta^c_y)$$

via a deduction that doesn't involve c. Since

$$\forall y(\beta^c_y \wedge \neg\beta^c_y) \to (\beta^c_y \wedge \neg\beta^c_y)$$

is a logical axiom,

$$\Gamma \vdash \beta^c_y \wedge \neg\beta^c_y$$

in $\mathcal{L}$. This implies that Γ is inconsistent in $\mathcal{L}$, which is a contradiction.

Proof of Generalization on Constants. Suppose that

$$(*) \quad \langle \alpha_1, \ldots, \alpha_n \rangle$$

is a deduction of φ from Γ. Let y be a variable which doesn't occur in any of the α_i. We claim that

$$(**) \quad \langle (\alpha_1)^c_y, \ldots, (\alpha_n)^c_y \rangle$$

is a deduction of φ^c_y from Γ. We shall prove that, for all $i \leq n$, either $(\alpha_i)^c_y \in \Gamma \cup \Lambda$ or $(\alpha_i)^c_y$ follows from earlier wffs in $(**)$ via MP.

Case 1 Suppose that $\alpha_i \in \Gamma$. Since c doesn't occur in Γ, it follows that $(\alpha_i)^c_y = \alpha_i \in \Gamma$.

Case 2 Suppose that $\alpha_i \in \Lambda$. Then it is easily checked that $(\alpha_i)^c_y \in \Lambda$.

Case 3 Suppose there exist $j, k < i$ such that α_k is $(\alpha_j \to \alpha_i)$. Then $(\alpha_k)^c_y$ is $((\alpha_j)^c_y \to (\alpha_i)^c_y)$. Hence $(\alpha_i)^c_y$ follows from $(\alpha_k)^c_y$ and $(\alpha_j)^c_y$ by MP.

Let Φ be the finite subset of Γ which occurs in $(**)$. Then $\Phi \vdash \varphi^c_y$ via a deduction in which c doesn't occur. By the Generalization Theorem, since y doesn't occur free in Φ, it follows that $\Phi \vdash \forall y \varphi^c_y$ via a deduction in which c doesn't occur. It follows that $\Gamma \vdash \forall y \varphi^c_y$ via a deduction in which c doesn't occur. $\square$

Exercise 27.4.
1. Show by induction on φ that if y doesn't occur in φ, then x is substitutable for y in φ^x_y and $(\varphi^x_y)^y_x = \varphi$.
2. Find a wff φ such that $(\varphi^x_y)^y_x \neq \varphi$.

Corollary 27.5. *Suppose that $\Gamma \vdash \varphi^x_c$, where c is a constant symbol that doesn't occur in Γ or φ. Then $\Gamma \vdash \forall x \varphi$, via a deduction in which c doesn't occur.*

Proof. By the above theorem, $\Gamma \vdash \forall y (\varphi^x_c)^c_y$ for some variable y which doesn't occur in φ^x_c. Since c doesn't occur in φ, $(\varphi^x_c)^c_y = \varphi^x_y$. Thus $\Gamma \vdash \forall y \varphi^x_y$. By the exercise, the following is a logical axiom: $\forall y \varphi^x_y \to \varphi$. Thus $\forall y \varphi^x_y \vdash \varphi$. Since x doesn't occur free in $\forall y \varphi^x_y$, Generalization gives that $\forall y \varphi^x_y \vdash \forall x \varphi$. Hence Deduction yields that $\vdash \forall y \varphi^x_y \to \forall x \varphi$. Since $\Gamma \vdash \forall y \varphi^x_y$, Rule T gives $\Gamma \vdash \forall x \phi$. □

Theorem 27.6 (Existence of Alphabetic Variants). *Let φ be a wff, t a term and x a variable. Then there exists a wff φ' (which differs from φ only in the choice of quantified variables) such that:*

(a) $\varphi \vdash \varphi'$ and $\varphi' \vdash \varphi$.

(b) t is substitutable for x in φ'.

Proof Omitted

28 Completeness

Now we are ready to begin the proof of:

Theorem 28.1 (Completeness). *If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.*

We shall base our strategy on the following observation.

Proposition 28.2. *The following statements are equivalent:*

(a) The Completeness Theorem: ie if $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.

(b) If Γ is a consistent set of wffs, then Γ is satisfiable.

Proof. (a) $\Rightarrow$ (b)

Suppose that Γ is consistent. Then there exists a wff φ such that $\Gamma \nvdash \varphi$. By Completeness, $\Gamma \not\models \varphi$. Hence there exists a structure $\mathcal{A}$ and a function $s\colon V \to A$ such that $\mathcal{A}$ satisfies Γ with s and $\mathcal{A} \not\models \varphi[s]$. In particular, Γ is satisfiable.

(b) $\Rightarrow$ (a)

Suppose that $\Gamma \nvdash \varphi$. Applying Reductio ad Absurdum, $\Gamma \cup \{\neg\varphi\}$ is consistent. It follows that $\Gamma \cup \{\neg\varphi\}$ is satisfiable and hence $\Gamma \not\models \varphi$. □

Now we prove:

Theorem 28.3 (Completeness'). *If Γ is a consistent set of wffs in a countable language $\mathcal{L}$, then there exists a countable structure $\mathcal{A}$ and $s\colon V \to A$ such that $\mathcal{A}$ satisfies Γ with s.*

Proof. **Step 1** Expand $\mathcal{L}$ to a larger language $\mathcal{L}^+$ by adding a countably infinte set of new constant symbols. Then Γ remains consistent as a set of wffs in $\mathcal{L}^+$.

Proof of Step 1. Suppose not. Then there exists a wff β of $\mathcal{L}^+$ such that $\Gamma \vdash \beta \wedge \neg\beta$ in $\mathcal{L}^+$. Suppose that $c_1, \dots, c_n$ includes the new constants (if any) which appear in β. By Generalization on Constants, there are variables $y_1, \dots, y_n$ such that:

(a) $\Gamma \vdash \forall y_1 \dots \forall y_n(\beta' \wedge \neg\beta')$, where β' is the result of replacing each c_i by y_i; and

(b) the deduction doesn't involve any new constants.

Since y_i is substitutable for y_i in β', we obtain that $\Gamma \vdash \beta' \wedge \neg\beta'$. But this means that Γ is inconsistent in the original language $\mathcal{L}$, which is a contradiction. □

Step 2 (We add witnesses to existential wffs.) Let

$$\langle \varphi_1, x_1 \rangle, \langle \varphi_2, x_2 \rangle, \dots, \langle \varphi_n, x_n \rangle, \dots$$

enumerate all pairs $\langle \varphi, x \rangle$, where φ is a wff of $\mathcal{L}^+$ and x is a variable. Let θ_1 be the wff

$$\neg\forall x_1 \varphi_1 \to (\neg\varphi_1)^{x_1}_{c_1},$$

where c_1 is the first new constant which doesn't occur in φ_1. If $n > 1$, then θ_n is the wff

$$\neg\forall x_n \varphi_n \to (\neg\varphi_n)^{x_n}_{c_n},$$

where c_n is the first new constant which doesn't occur in $\{\varphi_1, \dots, \varphi_n\} \cup \{\theta_1, \dots, \theta_{n-1}\}$. Let

$$\Theta = \Gamma \cup \{\theta_n \mid n \geq 1\}.$$

Claim 28.4. Θ is consistent.

Proof. Suppose not. Let $n \geq 0$ be the least integer such that $\Gamma \cup \{\theta_1, \dots, \theta_{n+1}\}$ is inconsistent. By Reductio ad Absurdum,

$$\Gamma \cup \{\theta_1, \dots, \theta_n\} \vdash \neg\theta_{n+1}.$$

Recall that θ_{n+1} has the form

$$\neg\forall x \varphi \to \neg\varphi^x_c.$$

By Rule T,

$$\Gamma \cup \{\theta_1, \dots, \theta_n\} \vdash \neg\forall x \varphi.$$

and

$$\Gamma \cup \{\theta_1, \ldots, \theta_n\} \vdash \varphi^x_c.$$

Since c doesn't occur in $\Gamma \cup \{\theta_1, \ldots, \theta_n\} \cup \{\varphi\}$, we have that

$$\Gamma \cup \{\theta_1, \ldots, \theta_n\} \vdash \forall x \varphi.$$

But this contradicts the minimality of n, or the consistency of Γ if $n = 0$. $\square$

Step 3 We extend Θ to a consistent set of wffs Δ such that for every wff φ of $\mathcal{L}^+$, either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$.

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ enumerate all the wffs of $\mathcal{L}^+$. We define inductively an increasing sequence of consistent sets of wffs

$$\Delta_0 \subseteq \Delta_1 \subseteq \ldots \subseteq \Delta_n \subseteq \ldots$$

as follows

- $\Delta_0 = \Theta$
- Suppose that Δ_n has been defined. If $\Delta_n \cup \{\alpha_{n+1}\}$ is consistent, then we set $\Delta_{n+1} = \Delta \cup \{\alpha_{n+1}\}$.

  Otherwise, if $\Delta_n \cup \{\alpha_{n+1}\}$ is inconsistent, then $\Delta \vdash \neg\alpha_{n+1}$ so we can set $\Delta_{n+1} = \Delta \cup \{\neg\alpha_{n+1}\}$.

Finally let $\Delta = \cup_{n \geq 0} \Delta_n$. Clearly Δ satisfies our requirements. $\square$

Notice that Δ is deductively closed; *ie* if $\Delta \vdash \varphi$, then $\varphi \in \Delta$. Otherwise, $\neg\varphi \in \Delta$ and so $\Delta \vdash \varphi$ and $\Delta \vdash \neg\varphi$, which contradicts the consistency of Δ.

Step 4 For each of the following wffs φ, $\Delta \vdash \varphi$ and so $\varphi \in \Delta$.

Eq 1 $\forall x(x = x)$.

Eq 2 $\forall x \forall y(x = y \to y = x)$.

Eq 3 $\forall x \forall y \forall z((x = y \wedge y = z) \to x = z)$.

Eq 4 For each n-ary predicate symbol P

$$\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n (x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \to (P x_1 \ldots x_n \leftrightarrow P y_1 \ldots y_n)$$

Eq 5 For each n-ary function symbol f

$$\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n (x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \to (f x_1 \ldots x_n = f y_1 \ldots y_n)$$

Similarly, since Δ is deductively closed and $\forall x \forall y (x = y \to y = x) \in \Delta$, if t_1, t_2 are any terms, then $(t_1 = t_2 \to t_2 = t_1) \in \Delta$ *etc.*

Step 5 We construct a structure $\mathcal{A}$ for $\mathcal{L}^+$ as follows.

Let T be the set of terms in $\mathcal{L}^+$. Define a relation E on T by

$$t_1 E t_2 \quad \text{iff} \quad (t_1 = t_2) \in \Delta.$$

Claim 28.5. E is an equivalence relation.

Proof. Suppose that $t \in T$. Then $(t = t) \in \Delta$ and so tEt. Thus E is reflexive.

Next suppose that $t_1 E t_2$. Then $(t_1 = t_2) \in \Delta$. Since $(t_1 = t_2 \to t_2 = t_1) \in \Delta$, it follows that $(t_2 = t_1) \in \Delta$. Thus $t_2 E t_1$ and so E is symmetric.

Similarly E is transitive. □

Definition 28.6. For each $t \in T$, let

$$[t] = \{s \in T \mid tEs\}.$$

Then we define

$$A = \{[t] \mid t \in T\}.$$

Definition 28.7. For each n-ary predicate symbol P, we define an n-ary relation $P^{\mathcal{A}}$ on A by

$$\langle [t_1], \ldots, [t_n] \rangle \in P^{\mathcal{A}} \ \text{ iff } \ P t_1 \ldots t_n \in \Delta.$$

Claim 28.8. $P^{\mathcal{A}}$ is well-defined.

Proof. Suppose that $[s_1] = [t_1], \ldots, [s_n] = [t_n]$. We must show that

$$P s_1 \ldots s_n \in \Delta \ \text{ iff } \ P t_1 \ldots t_n \in \Delta.$$

By assumption, $(s_1 = t_1) \in \Delta, \ldots, (s_n = t_n) \in \Delta$. Since

$$[(s_1 = t_1 \wedge \ldots \wedge s_n = t_n) \to (P s_1 \ldots s_n \leftrightarrow P t_1 \ldots t_n)] \in \Delta,$$

the result follows. □

Definition 28.9. For each constant symbol c, $c^{\mathcal{A}} = [c]$.

Definition 28.10. For each n-ary function symbol f, we define an n-ary operation $f^{\mathcal{A}} \colon A^n \to A$ by

$$f^{\mathcal{A}}([t_1], \ldots, [t_n]) = [f t_1 \ldots t_n].$$

Claim 28.11. $f^{\mathcal{A}}$ is well-defined.

Proof. Similar. $\square$

Finally we define $s\colon V \to A$ by $s(x) = [x]$.

Claim 28.12 (Target). For every wff φ of $\mathcal{L}^+$,

$$\mathcal{A} \models \varphi[s] \text{ iff } \varphi \in \Delta.$$

We shall make use of the following result.

Claim 28.13. For each term $t \in T$, $\bar{s}(t) = [t]$.

Proof. By definition, the result holds when t is a variable or a constant symbol. Suppose that t is $ft_1 \dots t_n$. Then by induction hypothesis, $\bar{s}(t_1) = [t_1], \dots, \bar{s}(t_n) = [t_n]$. Hence

$$\begin{aligned} \bar{s}(ft_1 \dots t_n) &= f^{\mathcal{A}}(\bar{s}(t_1), \dots, \bar{s}(t_1)) \\ &= f^{\mathcal{A}}([t_1], \dots, [t_1]) \\ &= [ft_1, \dots, t_1] \end{aligned}$$

$\square$

Proof of Target Claim. We argue by induction on the complexity of φ. First suppose that φ is atomic.

Case 1 Suppose that φ is $t_1 = t_2$. Then

$$\begin{aligned} \mathcal{A} \models (t_1 = t_2)[s] \quad &\text{iff} \quad \bar{s}(t_1) = \bar{s}(t_2) \\ &\text{iff} \quad [t_1] = [t_2] \\ &\text{iff} \quad (t_1 = t_2) \in \Delta \end{aligned}$$

Case 2 Suppose that φ is $Pt_1 \dots t_n$. Then

$$\begin{aligned} \mathcal{A} \models Pt_1 \dots t_n[s] \quad &\text{iff} \quad \langle \bar{s}(t_1), \dots, \bar{s}(t_n) \rangle \in P^{\mathcal{A}} \\ &\text{iff} \quad \langle [t_1], \dots, [t_n] \rangle \in P^{\mathcal{A}} \\ &\text{iff} \quad Pt_1 \dots t_n \in \Delta \end{aligned}$$

Next we consider the case when φ isn't atomic.

Case 3 Suppose that φ is $\neg\psi$. Then

$$\begin{aligned} \mathcal{A} \models \neg\psi[s] \quad &\text{iff} \quad \mathcal{A} \not\models \psi[s] \\ &\text{iff} \quad \psi \notin \Delta \\ &\text{iff} \quad \neg\psi \in \Delta \end{aligned}$$

Case 4 The case where φ is $(\theta \to \psi)$ is similar.

Case 5 Finally suppose that φ is $\forall x\psi$. We shall make use of the following result.

Lemma 28.14 (Substitution). *If the term t is substitutable for x in ψ, then*

$$\mathcal{A} \models \psi^x_t[s] \quad \textit{iff} \quad \mathcal{A} \models \psi[s(x|\bar{s}(t))].$$

Proof. Omitted. □

Recall that φ is $\forall x\psi$. By construction, for some constant c,

$$(\neg\forall\psi \to \neg\psi^x_c) \in \Delta \quad (*)$$

First suppose that $\mathcal{A} \models \forall x\psi[s]$. Then, in particular, $\mathcal{A} \models \psi[s(x|[c])]$ and so $\mathcal{A} \models \psi[s(x|\bar{s}(c))]$. By the Substitution Lemma, $\mathcal{A} \models \psi^x_c[s]$. Hence by induction hypothesis, $\psi^x_c \in \Delta$ and so $\neg\psi^x_c \notin \Delta$. By (*), $\neg\forall x\psi \notin \Delta$ and so $\forall x\psi \in \Delta$.

Conversely, suppose that $\mathcal{A} \not\models \forall x\psi[s]$. Then there exists a term $t \in T$ such that $\mathcal{A} \not\models \psi[s(x|[t])]$. Thus $\mathcal{A} \not\models \psi[s(x|\bar{s}(t))]$. Let ψ' be an alphabetic variant of ψ such that t is substitutable for x in ψ'. Then $\mathcal{A} \not\models \psi'[s(x|\bar{s}(t))]$. By the Substitution Lemma, $\mathcal{A} \not\models (\psi')^x_t[s]$. By induction hypothesis, $(\psi')^x_t \notin \Delta$. Since t is substitutable for x in ψ' is follows that $(\forall x\psi' \to (\psi')^x_t) \in \Delta$. Hence $\forall x\psi' \notin \Delta$ and so $\forall x\psi \notin \Delta$. □

Finally let $\mathcal{A}_0$ be the structure for $\mathcal{L}$ obtained from $\mathcal{A}$ by forgetting the interpretations of the new constant symbols. Then $\mathcal{A}_0$ satisfies Γ with s.

This completes the proof of the Completeness Theorem. □

Corollary 28.15. $\Gamma \models \varphi$ *iff* $\Gamma \vdash \varphi$. □

Theorem 28.16. *Let Γ be a set of wffs in a countable first order language. If Γ is finitely satisfiable, then Γ is satisfiable in some countable structure.*

Proof. Suppose that every finite subset $\Gamma_0 \subseteq \Gamma$ is satisfiable. By Soundness, every finite subset $\Gamma_0 \subseteq \Gamma$ is consistent. Hence Γ is consistent. By Completeness', Γ is satisfiable in some countable structure. □

Theorem 28.17. *Let T be a set of sentences in a first order language $\mathcal{L}$. If the class $\mathcal{C} = \mathrm{Mod}(T)$ is finitely axiomatizable, then there exists a finite subset $T_0 \subseteq T$ such that $\mathcal{C} = \mathrm{Mod}(T_0)$.*

Proof. Suppose that $\mathcal{C} = \mathrm{Mod}(T)$ is finitely axiomatizable. Then there exists a sentence σ such that $\mathcal{C} = \mathrm{Mod}(\sigma)$. Since $\mathrm{Mod}(T) = \mathrm{Mod}(\sigma)$, it follows that $T \models \sigma$. By the Completeness Theorem, $T \vdash \sigma$ and hence there exists a finite subset $T_0 \subseteq T$ such that $T_0 \vdash \sigma$. By Soundness, $T_0 \models \sigma$. Hence

$$\mathcal{C} = \mathrm{Mod}(T) \subseteq \mathrm{Mod}(T_0) \subseteq \mathrm{Mod}(\sigma) = \mathcal{C}$$

and so $\mathcal{C} = \mathrm{Mod}(T_0)$. □

Definition 28.18. Let $\mathcal{A}$, $\mathcal{B}$ be structures for the first-order language $\mathcal{L}$. Then $\mathcal{A}$ and $\mathcal{B}$ are *elementarily equivalent*, written $\mathcal{A} \equiv \mathcal{B}$, iff for every sentence σ of $\mathcal{L}$,

$$\mathcal{A} \models \sigma \text{ iff } \mathcal{B} \models \sigma.$$

Remark 28.19. If $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{A} \equiv \mathcal{B}$. Howevery, the converse does not hold, *eg* consider a nonstandard model of arithmetic.

Definition 28.20. A consistent set of sentences T is said to be *complete* iff for every sentence σ, either $T \vdash \sigma$ or $T \vdash \neg\sigma$.

Example 28.21. Let $\mathcal{A}$ be any structure and let

$$\mathrm{Th}(\mathcal{A}) = \{\sigma \mid \sigma \text{ is a sentence such that } \mathcal{A} \models \sigma\}.$$

Then $\mathrm{Th}(\mathcal{A})$ is a complete theory.

Theorem 28.22. *If T is a complete theory in the first-order language $\mathcal{L}$ and $\mathcal{A}$, $\mathcal{B}$ are models of T, then $\mathcal{A} \equiv \mathcal{B}$.*

Proof. Let σ be any sentence. Then either $T \vdash \sigma$ or $T \vdash \neg\sigma$. Suppose that $T \vdash \sigma$. By Soundness, $T \models \sigma$. Hence $\mathcal{A} \models \sigma$ and $\mathcal{B} \models \sigma$. Similarly if $T \vdash \neg\sigma$, then $\mathcal{A} \models \neg\sigma$ and $\mathcal{B} \models \neg\sigma$. □

Theorem 28.23 (Los-Vaught). *Let T be a consistent theory in a countable language $\mathcal{L}$. Suppose that*

(a) T has no finite models.

(b) If $\mathcal{A}$, $\mathcal{B}$ are countably infinite models of T, then $\mathcal{A} \cong \mathcal{B}$.

Then T is complete.

Proof. Suppose not. Then there exists a sentence σ such that $T \nvdash \sigma$ and $T \nvdash \neg\sigma$. Hence $T \cup \{\neg\sigma\}$ and $T \cup \{\sigma\}$ are both consistent. By Completeness, there exists countable structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \models T \cup \{\neg\sigma\}$ and $\mathcal{B} \models T \cup \{\sigma\}$. By (a), $\mathcal{A}$ and $\mathcal{B}$ must be countably infinite. Hence, by (b), $\mathcal{A} \cong \mathcal{B}$. But this contradicts the fact that $\mathcal{A} \models \neg\sigma$ and $\mathcal{B} \models \sigma$. □

Corollary 28.24. *Let T_{DLO} be the theory of dense linear orders without endpoints. Then T_{DLO} is complete.*

Proof. Clearly T_{DLO} has no finite models. Also, if $\mathcal{A}$, $\mathcal{B}$ are countable dense linear orders without endpoints, then $\mathcal{A} \cong \mathcal{B}$. Hence T_{DLO} is complete. □

Corollary 28.25. $\langle \mathbb{Q}, < \rangle \equiv \langle \mathbb{R}, < \rangle$.

Proof. $\langle \mathbb{Q}, < \rangle$ and $\langle \mathbb{R}, < \rangle$ are both models of the complete theory T_{DLO}. □

The rationals $\langle \mathbb{Q}, < \rangle$ are a countable linear order in which "every possible finite configuration is realized."