Unstable cohomology of $SL_n\mathbb{Z}$ and Hopf algebras

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joint with

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Borel-Serre (1973)

$$H^k(SL_n \mathbb{Z}; \mathbb{Q} \otimes M) \cong 0 \quad \text{for } k > \binom{n}{2}$$

Church-Farb-Putman Conjecture (2014)

$$H^k(SL_n \mathbb{Z}; \mathbb{Q}) \cong 0 \quad \text{for } k > \binom{n-1}{2}$$

$$\Leftrightarrow H^{\binom{n}{2}-i}(SL_n \mathbb{Z}; \mathbb{Q}) \cong 0 \quad \text{for } i \leq n-2$$

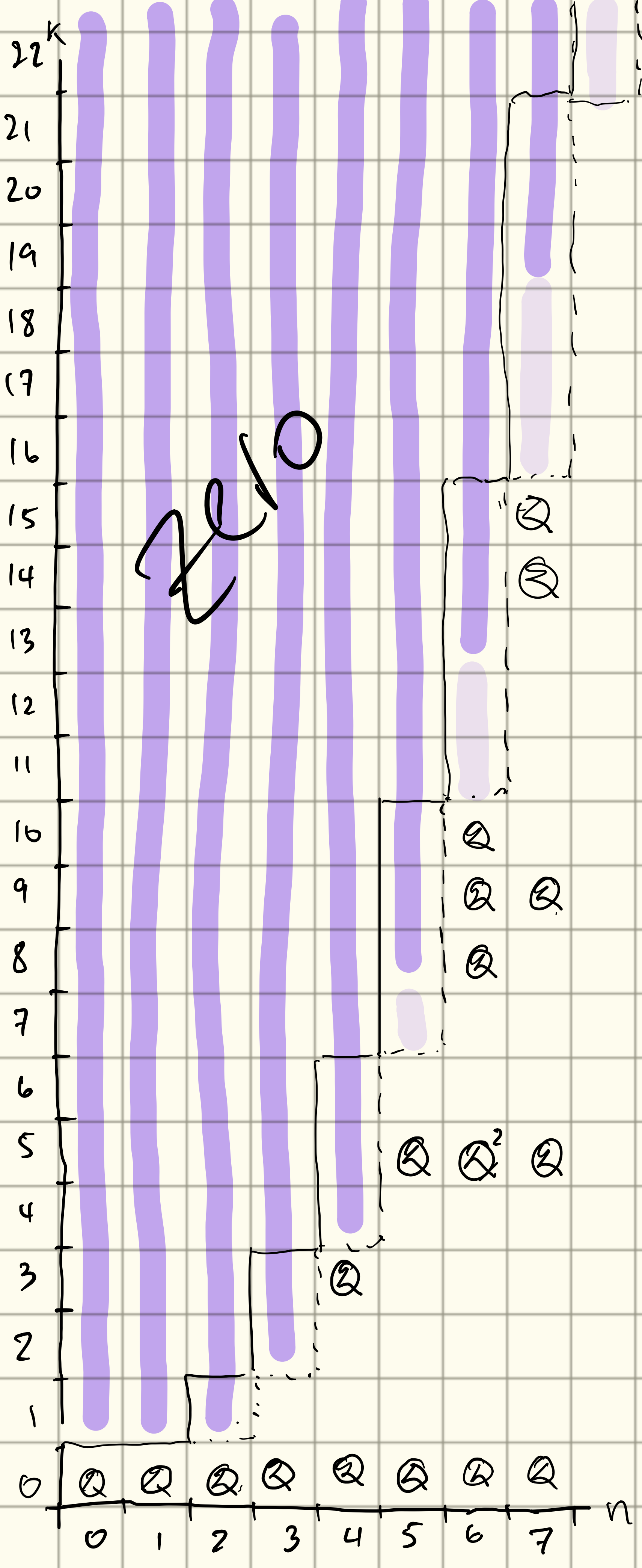
Proved for

$i=0$ Lee-Szczarba (1976)

$i=1$ Church-Putman (2015)

$i=2$ Brück-Miller-P.-Sroka-Wilson (2024)

$H^k(SL_n\mathbb{Z};\mathbb{Q})$



Borel (1974)

$$H^*(SL_n\mathbb{Z};\mathbb{Q}) \cong \Lambda^*[\sigma_5, \sigma_9, \sigma_{13}, \dots]$$

for $* \leq n-2$

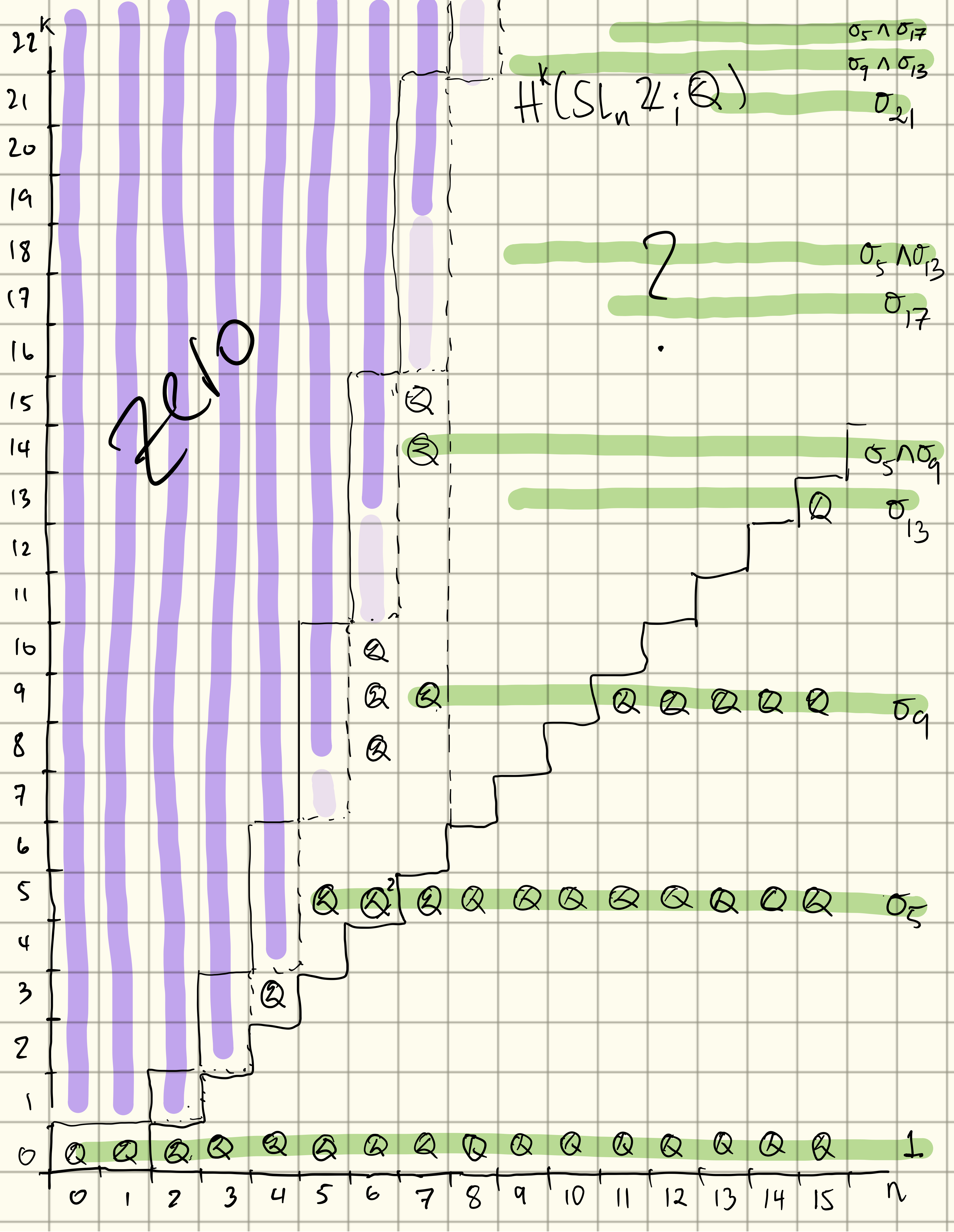
range due to Li-Sun (2019)

Franke (2009)

$$\mathrm{im}\left(\Lambda^*[\sigma_5, \sigma_9, \sigma_{13}, \dots] \longrightarrow H^*(SL_n\mathbb{Z};\mathbb{Q})\right)$$

$$= \Lambda^*[\sigma_5, \dots, \sigma_{4k+1}]$$

for $n = 2k+3$ or $2k+4$



$H^k(SL_n\mathbb{Z};\mathbb{Q})$
zero
?
$\sigma_5 \wedge \sigma_{17}$
$\sigma_9 \wedge \sigma_{13}$
σ_{21}
$\sigma_5 \wedge \sigma_{13}$
σ_{17}
$\sigma_5 \wedge \sigma_9$
σ_{13}
σ_9
σ_5
1
$\mathbb{Q}^2$
k
n

Borel-Serre Duality (1973)

$$H^{\binom{n}{2}-k}(SL_n\mathbb{Z}; \mathbb{Q}\otimes M) \cong H_k(SL_n\mathbb{Z}; \mathbb{Q}\otimes M \otimes St_n\mathbb{Q})$$

Steinberg module

Tits building

F field, $T_n(F)$ simplicial cpx

vertex : $0 \subsetneq V \subsetneq F^n$ subspace

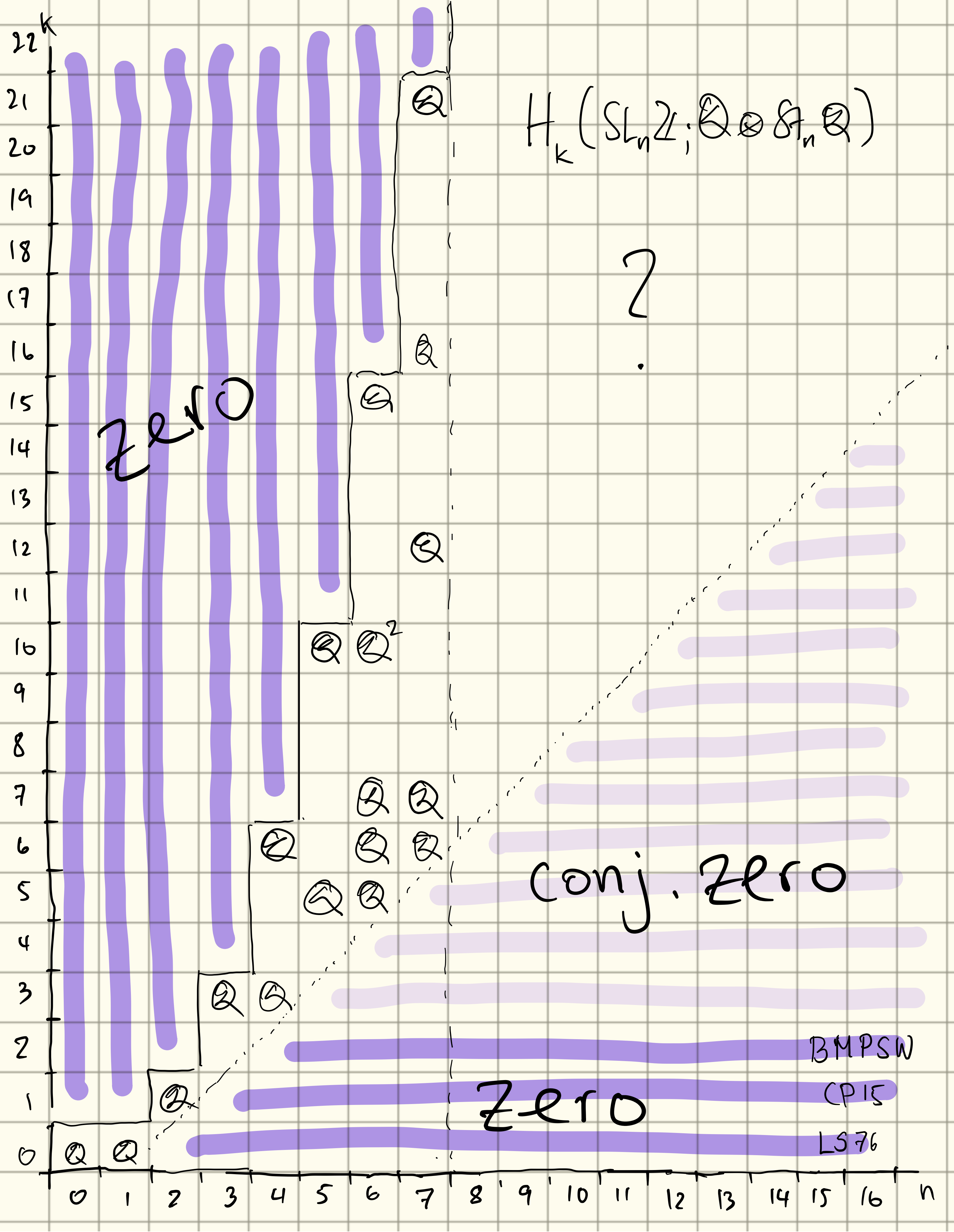
p-simplex : $V_0 \subsetneq \cdots \subsetneq V_p$

Solomon-Tits (1969)

$$T_n(F) \simeq \bigvee S^{n-2}$$

Steinberg module

$$St_n F := \tilde{H}_{n-2}(T_n(F))$$



$H_k(SL_n\mathbb{Z}; \mathbb{Q} \otimes \mathfrak{S}_n \mathbb{Q})$
?
zero
conj. zero
zero
BMPSW
CP15
LS76
k
n

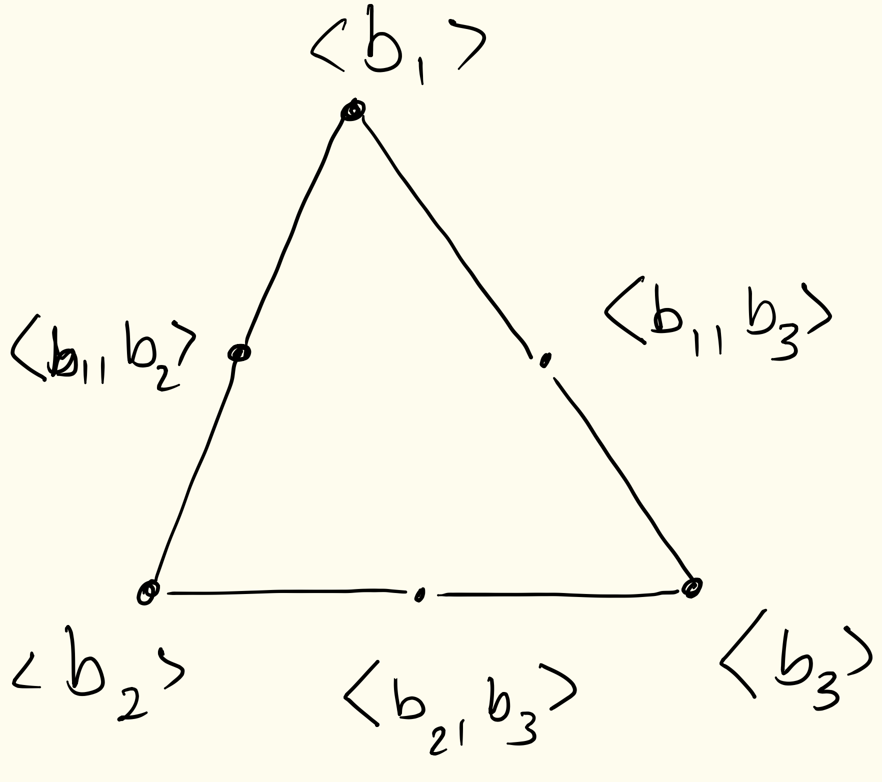
Apartment

$b_1, \dots, b_n$ basis of F^n

$[b_1, \dots, b_n] \subset T_n(F)$ full subcpx on spans of proper non empty subsets of $\{b_1, \dots, b_n\}$

$\cong S^{n-2}$

Ex $n=3$



$\rightsquigarrow [\![b_1, \dots, b_n]\!] \in \tilde{H}_{n-2}(T_n(F)) = St_n F$

Steinberg as an algebra

[Miller - Nagpal - P. 2020]

$$\mathrm{St}_n\mathbb{Q} \otimes \mathrm{St}_m\mathbb{Q} \longrightarrow \mathrm{St}_{n+m}\mathbb{Q}$$

$$[\![b_1,\dots,b_n]\!] \otimes [\![b_1',\dots,b_m']\!] \longmapsto [\![b_1,\dots,b_n,b_1',\dots,b_m']\!]$$

$GL_n\mathbb{Z} \times GL_m\mathbb{Z}$ - equivariant

$$\rightsquigarrow H_k(SL_n\mathbb{Z};\mathrm{St}_n\mathbb{Q}) \otimes H_\ell(SL_m\mathbb{Z};\mathrm{St}_m\mathbb{Q})$$

$$\longrightarrow H_{k+\ell}(SL_{n+m}\mathbb{Z};\mathrm{St}_{n+m}\mathbb{Q})$$

$$\rightsquigarrow \bigoplus_{n,k\geq 0} H_k(SL_n\mathbb{Z};\mathrm{St}_n\mathbb{Q})$$

bigraded commutative algebra

Observation

for $n \geq 1$

$$1 \longrightarrow SL_n \mathbb{Z} \longrightarrow GL_n \mathbb{Z} \longrightarrow C_2 \longrightarrow 1$$

$$\Rightarrow \quad H_k(SL_n \mathbb{Z}; \mathbb{Q} \otimes St_n \mathbb{Q})$$

$$\cong H_k(GL_n \mathbb{Z}; \mathbb{Q}C_2 \otimes St_n \mathbb{Q})$$

$$\cong H_k(GL_n \mathbb{Z}; \mathbb{Q} \otimes St_n \mathbb{Q})$$

$H^{\binom{n}{2}-k}(GL_n \mathbb{Z}; \mathbb{Q})$ if n odd $\quad H^{\binom{n}{2}-k}(GL_n \mathbb{Z}; \mathbb{Q} \otimes \det)$ if n even

$$\oplus\, H_k(GL_n \mathbb{Z}; \mathbb{Q} \otimes \det \otimes St_n \mathbb{Q})$$

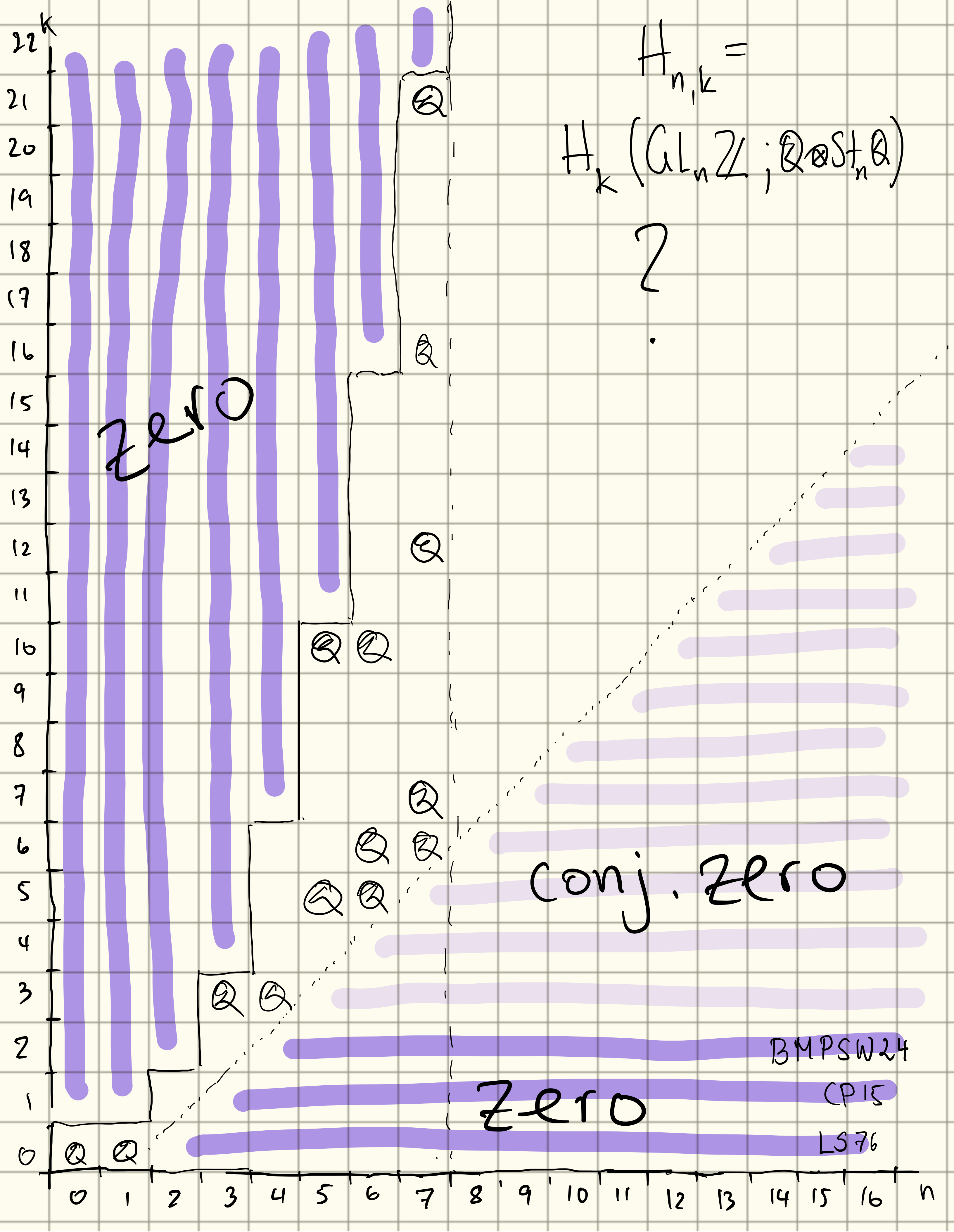
$H^{\binom{n}{2}-k}(GL_n \mathbb{Z}; \mathbb{Q} \otimes \det) = 0$ if n odd $\quad H^{\binom{n}{2}-k}(GL_n \mathbb{Z}; \mathbb{Q})$ if n even

In fact,
$$\bigoplus H_k(SL_n \mathbb{Z}; \mathbb{Q} \otimes St_n \mathbb{Q})$$

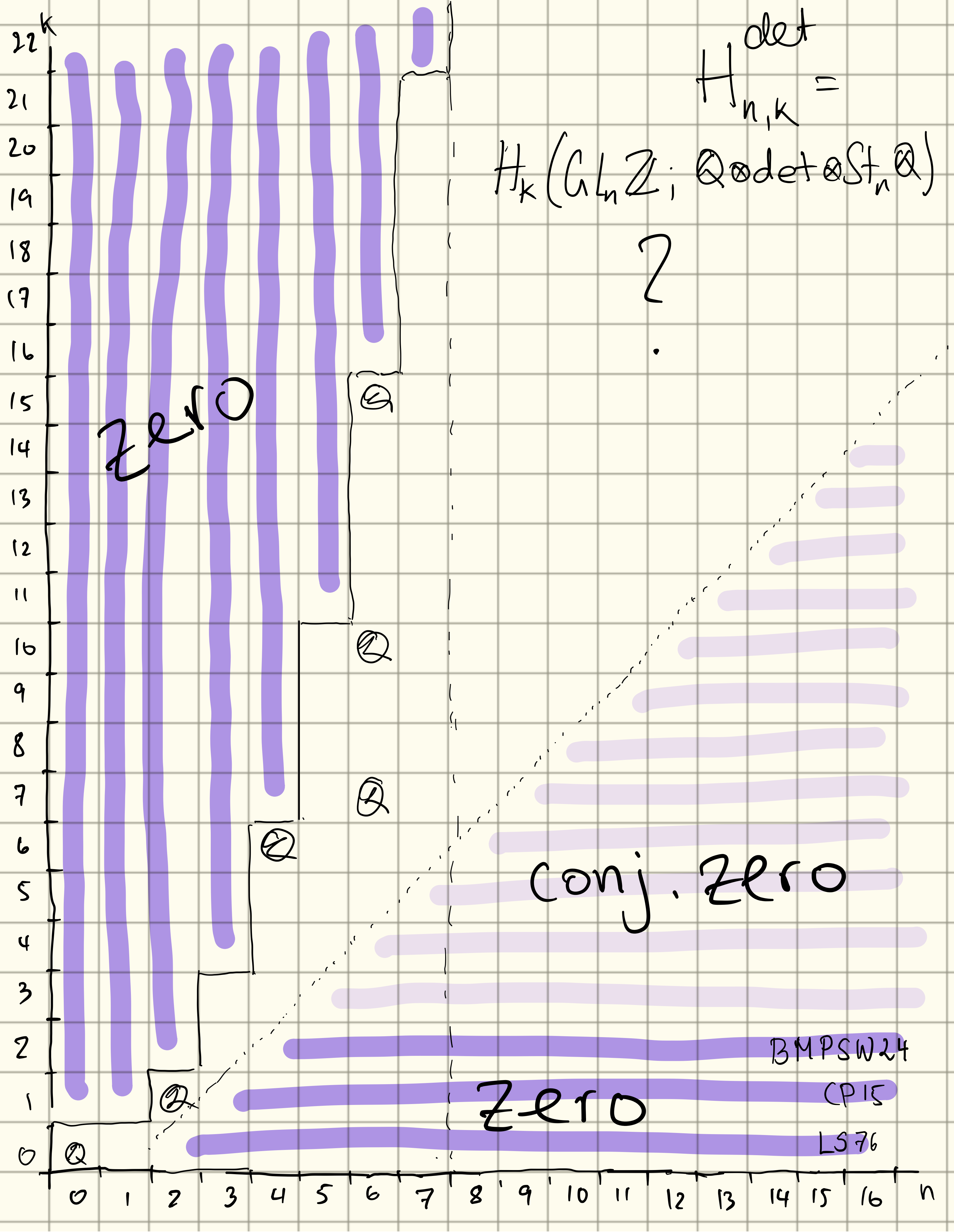
$$\cong \bigoplus H_k(GL_n \mathbb{Z}; \mathbb{Q} \otimes St_n \mathbb{Q}) =: H$$

$$\vee \bigoplus H_k(GL_n \mathbb{Z}; \mathbb{Q} \otimes \det \otimes St_n \mathbb{Q}) =: H^{\det}$$

as algebras



$H_{n,k} = H_k(GL_n\mathbb{Z}; \mathbb{Q}\otimes St_n\mathbb{Q})$?
zero
conj. zero
Zero
BMPSW24
CP15
LS76
k
n



$H^{det}_{n,k} = H_k(GL_n\mathbb{Z}; \mathbb{Q}\otimes\det\otimes St_n\mathbb{Q})$
?
zero
conj. zero
zero
BMPSW24
CP15
LS76
k
n

Thm (Ash–Miller–P., Brown–Chan–Galatius–Payne)

H is free graded commutative and

$$\bigoplus_m \Lambda^*\left[\sigma_5^{2m+3}, \ldots, \sigma_{4m+1}^{2m+3}\right] \subset H^*(GL_{2m+3}\mathbb{Z}; \mathbb{Q})$$

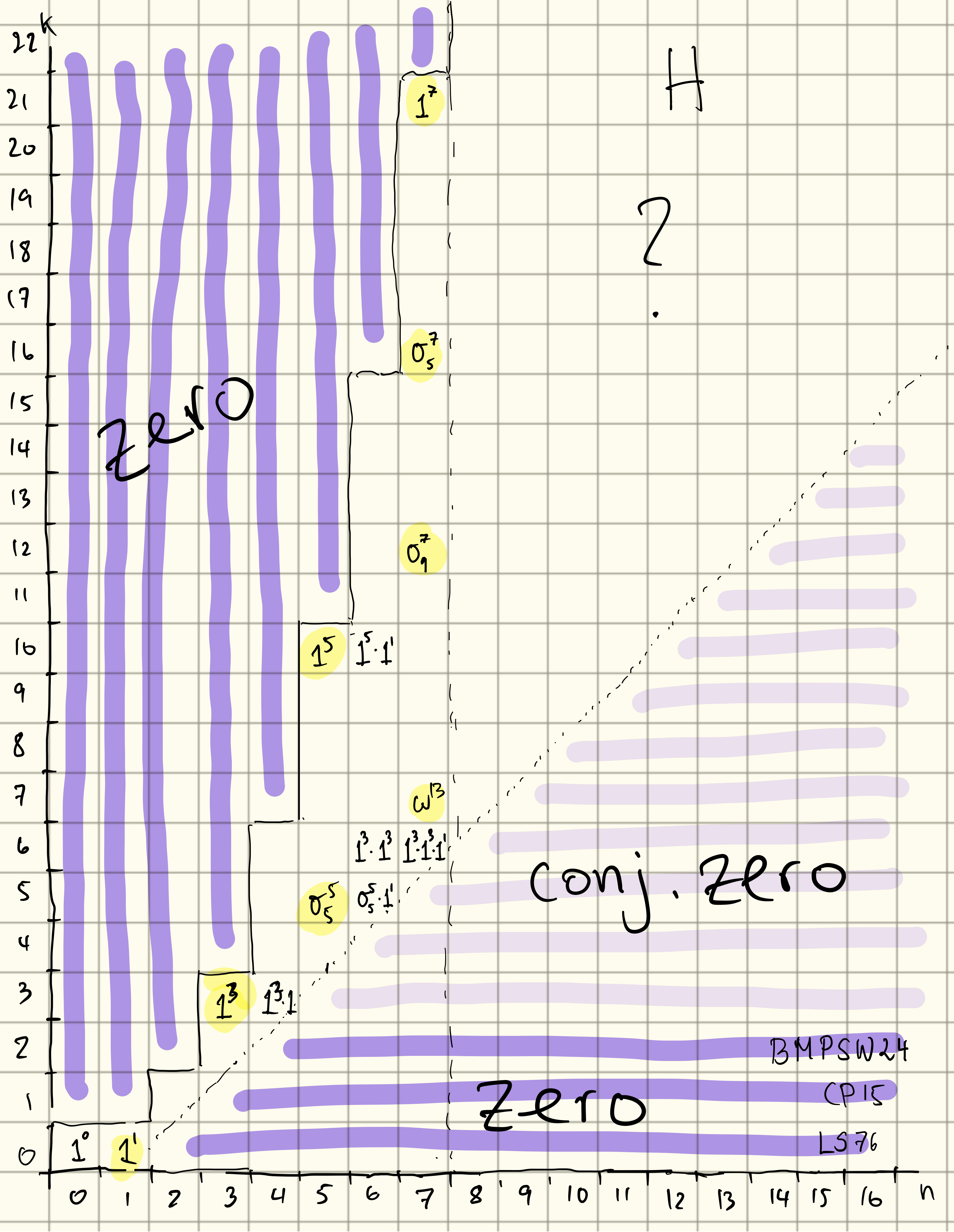
generates a free graded commutative subalgebra.

Thm (Ash–Miller–P., Brown–Hu–Panzer)

H^{det} is free graded commutative and

$$\bigoplus_m \Lambda^*\left[\sigma_5^{2m+4}, \ldots, \sigma_{4m+1}^{2m+4}\right] \subset H^*(GL_{2m+4}\mathbb{Z}; \mathbb{Q})$$

generates a free graded commutative subalgebra.



H
?
zero
conj. zero
Zero
BMPSW24
CP15
LS76
1^7
0^7_5
0^7_9
1^5
$1^5 \cdot 1^1$
ω^{13}
$1^3 \cdot 1^3$
$1^3 \cdot 1^3 \cdot 1^1$
0^5_5
$0^5_5 \cdot 1^1$
1^3
$1^3 \cdot 1$
1^0
1^1
k
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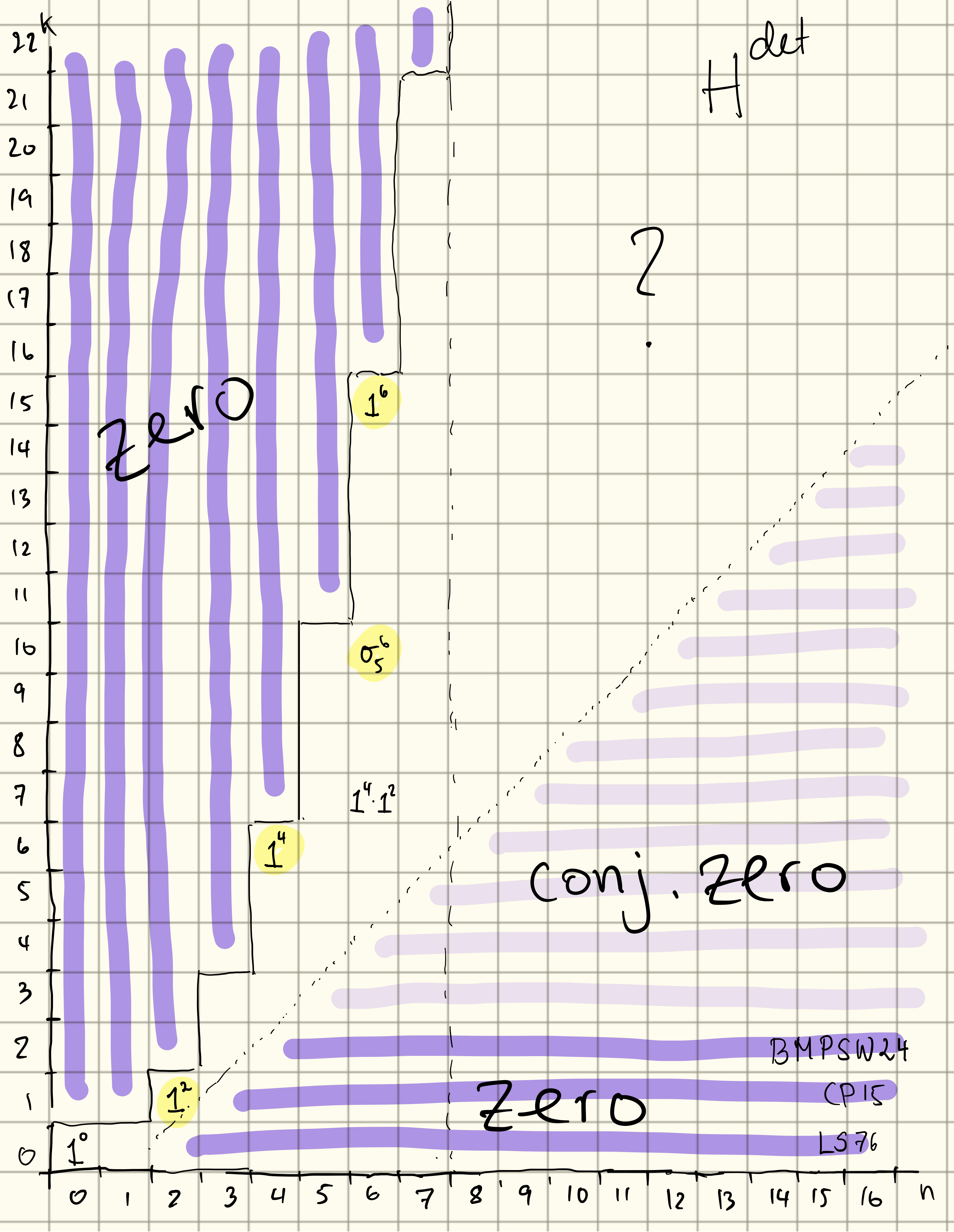
H^det
k
n
zero
?
1^0
1^2
1^4
1^4·1^2
0_5^6
1^6
conj. zero
zero
BMPSW24
CP15
LS76

Hopf algebras

Let $B = \bigoplus_{n \geq 0} B_n$ be a graded bialgebra over $\mathbb{Q}$ with $B_0 = \mathbb{Q}$.

Fact: B is a Hopf algebra.

Thm (Leray)

If B is commutative, then

$B \cong \mathrm{Sym}(Q(B))$, where

$Q(B) = B_+ / B_+^2$ are the indecomposables of B.

Steinberg as a coalgebra

[Ash–Miller–P.]

$$\mathrm{St}_n\mathbb{Q} \longrightarrow \bigoplus_{U \subset \mathbb{Q}^n} \mathrm{St}(U) \otimes \mathrm{St}(\mathbb{Q}^n/U)$$

$$[\![b_1, \ldots, b_n]\!] \longmapsto \sum_{\pi \text{ shuffles}} (-1)^{\pi} [\![b_{\pi(1)}, \ldots, b_{\pi(p)}]\!] \otimes [\![\overline{b}_{\pi(p+1)}, \ldots, \overline{b}_{\pi(n)}]\!]$$

$$U = \mathrm{span}(b_{\pi(1)}, \ldots, b_{\pi(p)})$$

$GL_n\mathbb{Z}$ - equivariant

$$\rightsquigarrow \quad H_k(GL_n\mathbb{Z}; \mathrm{St}_n\mathbb{Q})$$

$$\to \bigoplus_{\substack{p+q=n \\ a+b=k}} H_a(GL_p\mathbb{Z}; \mathrm{St}_p\mathbb{Q}) \otimes H_b(GL_q\mathbb{Z}; \mathrm{St}_q\mathbb{Q})$$

Thm (Ash-Miller-P., Brown-Chan-Galatius-Payne)

H is a graded comm. Hopf algebra.

Thm (Brown-Chan-Galatius-Payne)

$$\Lambda^*\left[\sigma_5^{2m+3}, \ldots, \sigma_{4m+1}^{2m+3}\right] \subset H$$

is indecomposable.

Thm (Ash-Miller-P.)

H^{det} is a graded comm. Hopf algebra.

Thm (Brown-Hu-Panzer)

$$\Lambda^*\left[\sigma_5^{2m+4}, \ldots, \sigma_{4m+1}^{2m+4}\right] \subset H^{det}$$

is indecomposable.

Dual Picture: $B = \bigoplus_{n \geq 0} B_n$ Hopf algebra with

$B_0 = \mathbb{Q}$ and B_n fin. dim

$B^\vee := \bigoplus_{n \geq 0} B_n^\vee$ dual Hopf algebra

product $\longleftrightarrow$ coproduct

commutative $\longleftrightarrow$ cocommutative

indecomposables $\longleftrightarrow$ primitives

$P(B) = \{ x \in B \mid \Delta(x) = 1 \otimes x + x \otimes 1 \}$

Thm (Milnor - Moore 1965)

If B is a cocommutative Hopf algebra,

then $B \cong \mathcal{U}(P(B))$.

Lie algebra

universal enveloping algebra

Thm (Brown-Chan-Galatius-Payne)

The duals of

$$W_{2k+3} = \sigma_5 \wedge \cdots \wedge \sigma_{4k+1} \qquad k = 0, \ldots, 10$$

in $\mathcal{P}(H^\vee)$ generate a free Lie algebra.

Cor $W_3^\vee, \ldots, W_{23}^\vee$ generate a free associative algebra in $H^\vee$.

Open Questions:

① What is the Lie algebra generated by

$$\left(\bigoplus_{m\geq 0}\Lambda^*\left[\sigma_5^{2m+3},\dots,\sigma_{4m+1}^{2m+3}\right]\right)^\vee \subset P(H^\vee)$$

and by

$$\left(\bigoplus_{m\geq 0}\Lambda^*\left[\sigma_5^{2m+4},\dots,\sigma_{4m+1}^{2m+4}\right]\right)^\vee \subset P((H^{\det})^\vee)\ ?$$

② How does the cup product of $H^*(GL_n\mathbb{Z})$ interact with the Hopf algebra structure?

③ Are all indecomposables of H in degree n odd?