Canonical decomposition of rational maps

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University of Amsterdam

Geometry and Topology Seminar
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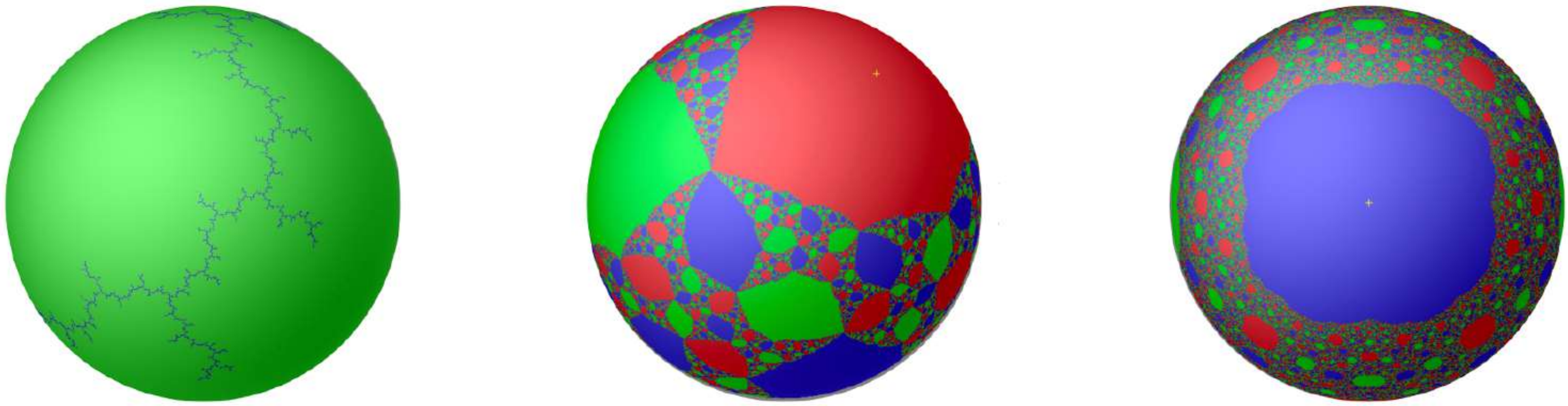
Dynamics of rational maps

Complex dynamics studies properties of holomorphic maps under iteration.

Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map, and $f^n = f \circ \cdots \circ f$ be the n-th iterate of f.

- The Julia set $\mathcal{J}_f$ is the closure of the set of repelling fixed points of f^n, $n \geq 1$.
- The Fatou set $\mathcal{F}_f := \widehat{\mathbb{C}} \setminus \mathcal{J}_f$.
  A Fatou component is a connected component of $\mathcal{F}_f$.

Intuition: f behaves "regularly" on $\mathcal{F}_f$ and "chaotically" on $\mathcal{J}_f$.



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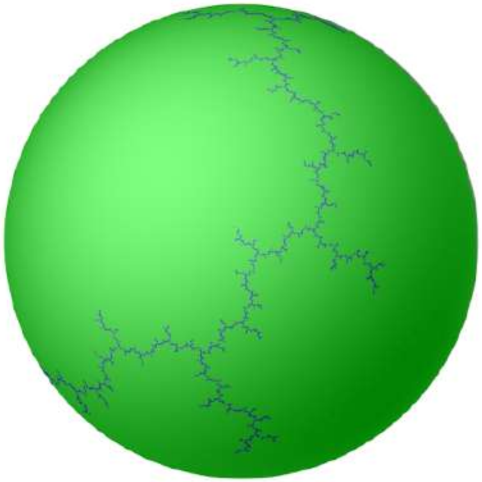
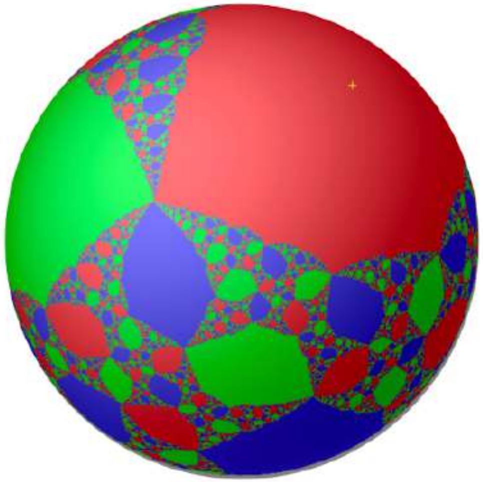
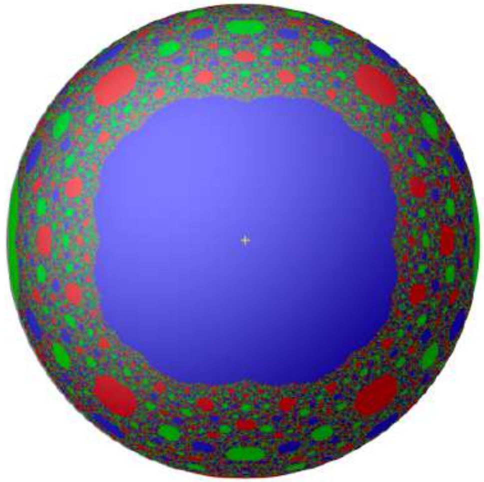

Question

What is the structure of the Julia set and how does it change when we change f?

After Lunch Menu

1. Main Course: discuss decomposition results in rational dynamics.

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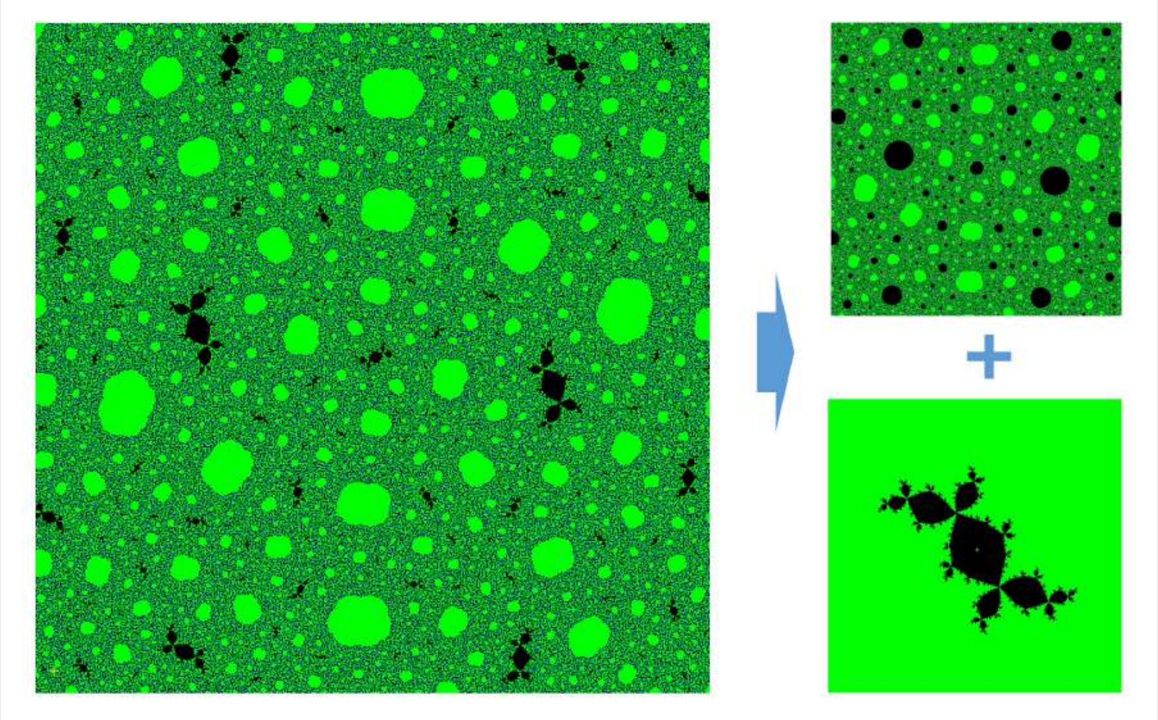
Decomposition Theorem [Dudko-H.-Schleicher]

Every postcritically-finite rational map with non-empty Fatou set can be canonically decomposed into

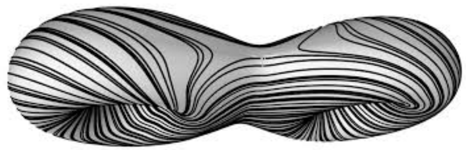


- Sierpiński carpet maps:
  Fatou components are disjoint Jordan discs
- crochet maps:
  any two Fatou components may be linked via a countable chain of "touching" Fatou components

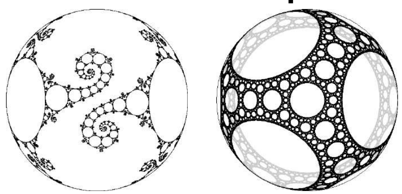
After Lunch Menu

2. Dessert: discuss some connections to other fields.

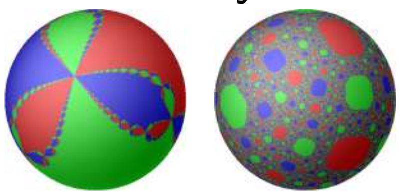
Surface Automorphisms



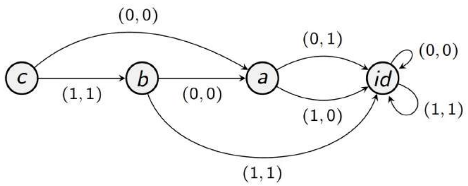
Geometric Group Theory



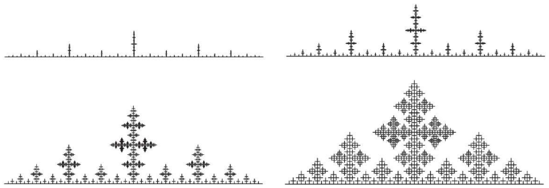
Rational Dynamics



Self-similar Groups



Fractal Geometry



Pictures courtesy of C. Bishop, D. Calegary, and C. McMullen

Starter: Geometrization of surface automorphisms

Thurston's "topology implies geometry" quest:

- geometry of 3-manifolds;
- theory of surface automorphisms;
- dynamics of rational maps.

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Let S be a closed oriented surface with a finite set P of marked points.

Every homeomorphism $f\colon (S, P) \to (S, P)$ can be canonically decomposed into

- **periodic homeomorphisms**
- and **pseudo-Anosov homeomorphisms**.

Decomposition pieces – periodic and pseudo-Anosov maps

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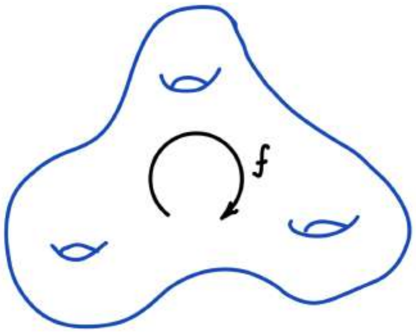

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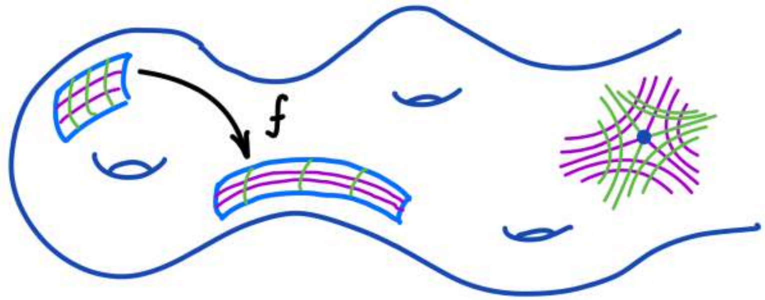
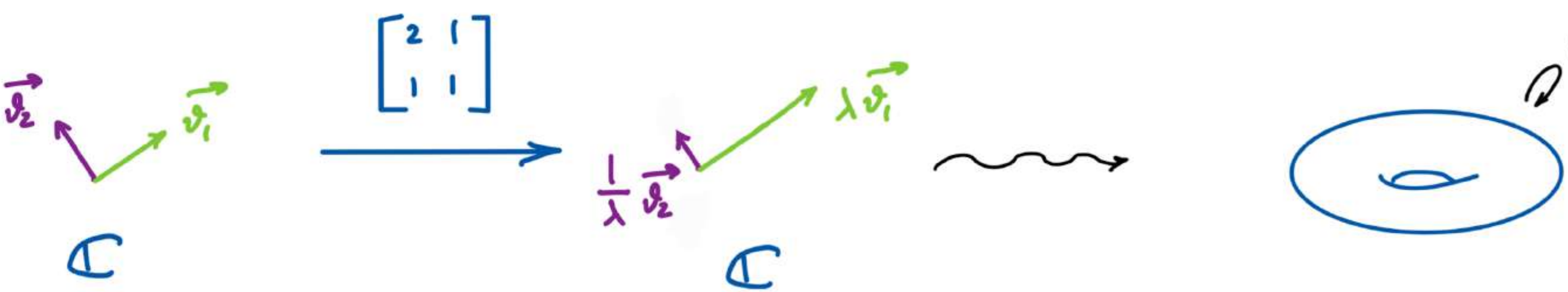


2. f admits a pseudo-Anosov structure if there are two transverse foliations of (S, P) by lines (with singularities at finitely many points) such that:
   - f is a λ-stretch (with $\lambda > 1$) along the first foliation, and
   - f is a λ-contraction along the second one.



Decomposition scissors – invariant multicurves

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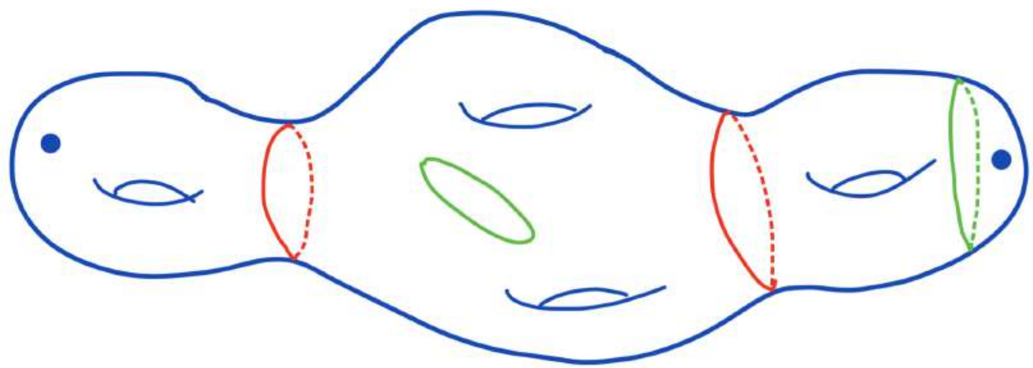
- A simple closed curve $\gamma \in S \setminus P$ is non-essential if γ bounds a disc in S with at most one marked point, and is essential otherwise.



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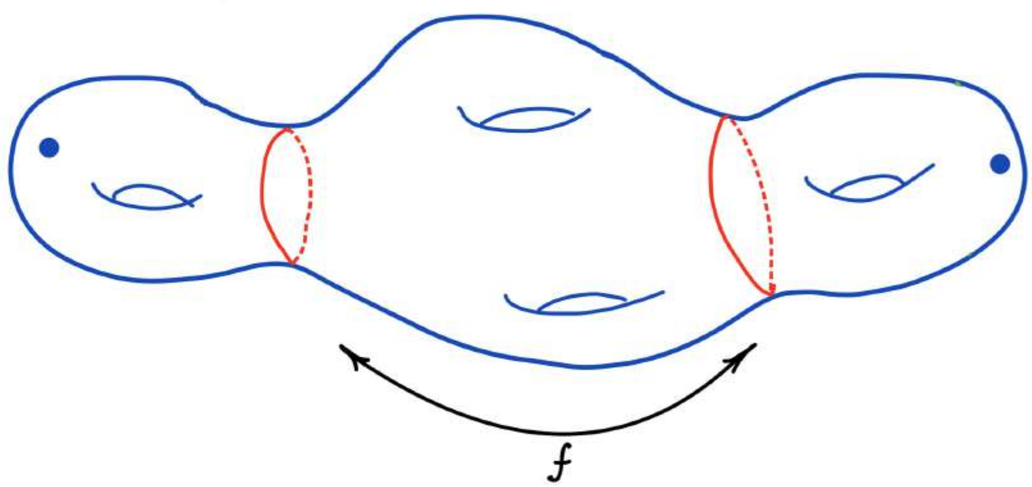

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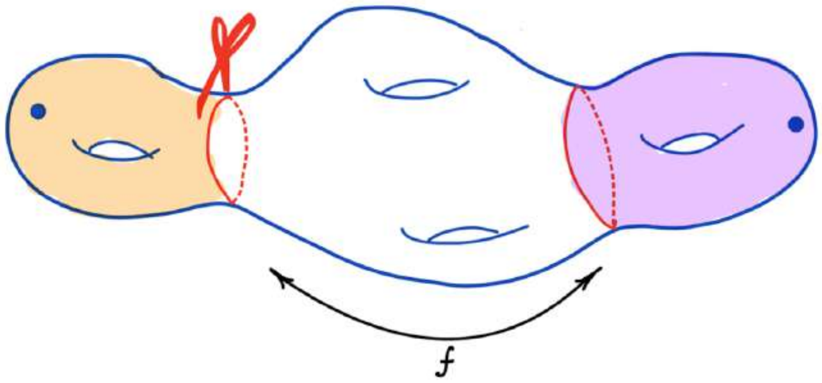
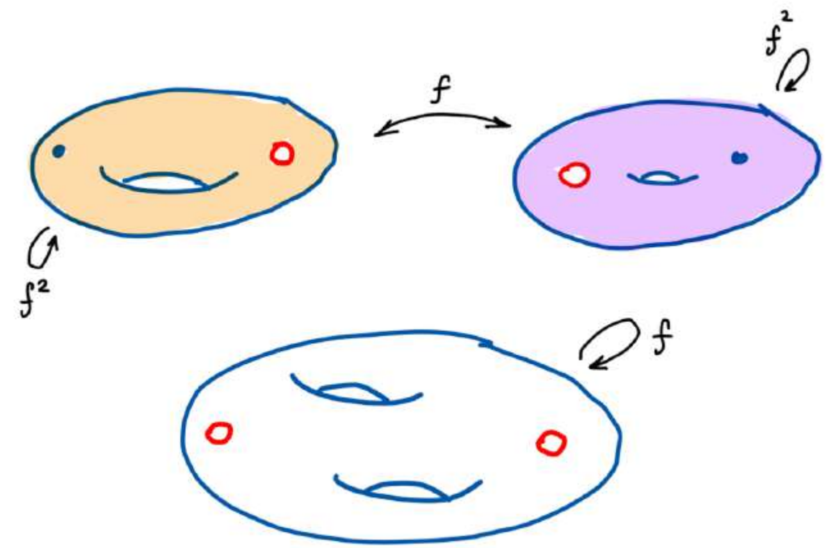
- A multicurve Γ is invariant if each $f(\gamma)$, $\gamma \in \Gamma$, is isotopic rel. P to some curve $\gamma' \in \Gamma$.

Geometrization of surface homeomorphisms





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Every homeomorphism $f\colon (S,P) \to (S,P)$ **can be canonically decomposed** along an invariant multicurve **into**

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Thurston theory of rational maps

Each rational map $f: \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a branched covering map, that is, f is

- continuous;
- surjective;
- locally $z \mapsto z^k$, $k \in \mathbb{N}$, after homeomorphic coordinate changes.

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When a branched cover $g\colon\mathbb{S}^2\to\mathbb{S}^2$ is realized by a rational map $f\colon\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$?
(i.e., f and g are conjugate up to isotopy)

Answer: Thuston's characterization of rational maps ('80s).

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Setting: postcritically finite (pcf) branched covers $g\colon \mathbb{S}^2 \to \mathbb{S}^2$.

C_g – the set of critical points of g, i.e., points where g is not locally injective.

$P_g := \bigcup_{n=1}^{\infty} g^n(C_g)$ – the postcritical set of g.

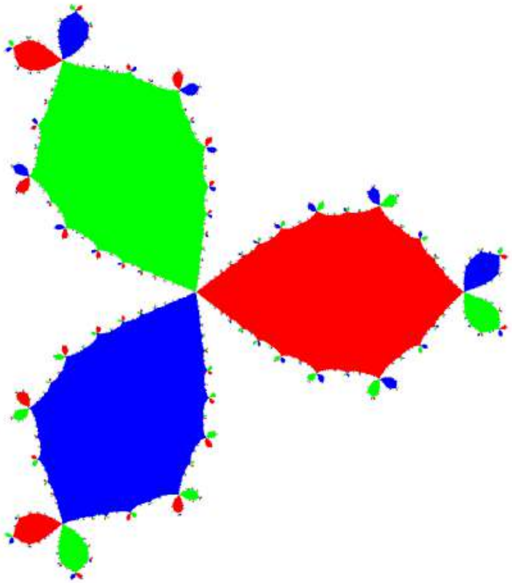
The map g is pcf if $\#P_g < \infty$, i.e., each critical point has finite orbit.

Example:

$f(z) = -\frac{1}{3}(z^4 - 4z)$

$C_f = \{1, e^{2\pi i/3}, e^{4\pi i/3}, \infty\}$

f is critically fixed, i.e., $f(c) = c \; \forall c \in C_f$.



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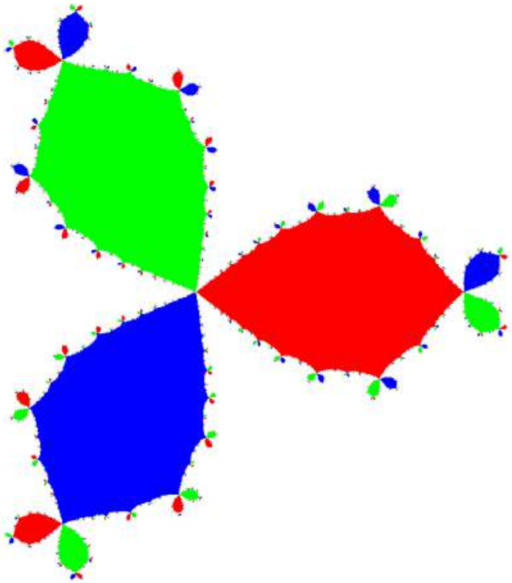

For a pcf rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$:

(i) $\mathcal{J}_f$ is a compact, connected, locally connected set in $\widehat{\mathbb{C}}$.

(ii) $\mathcal{F}_f = \big\{z \in \widehat{\mathbb{C}} : \{f^n(z)\}_{n\in\mathbb{N}}$ converges to a periodic critical cycle$\big\}$.

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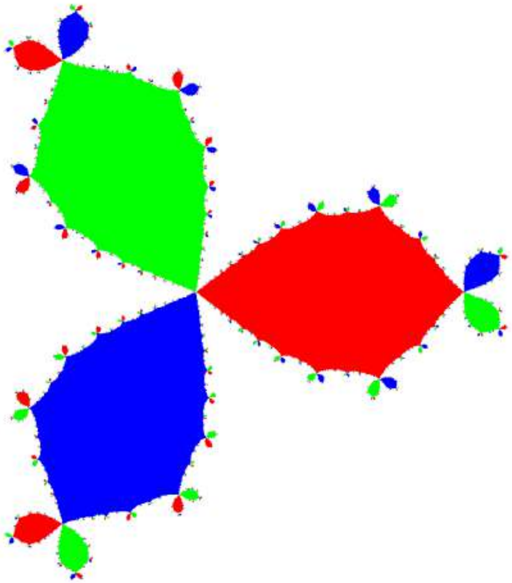

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- pcf rational maps are rather special (it is a countable family);
- BUT! they are structurally very important.

  E.g., the combinatorial structure of the Mandelbrot set  may be described using pcf maps.

Thurston's characterization — decomposition version

Theorem [Thurston'80s, Pilgrim'03, Selinger'12]

Let $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ be a **pcf branched covering map**. Then there is a canonical invariant multicurve Γ_{Th} (possibly empty) such that f **decomposes into**

- **homeomorphisms** (elliptic type);
- **quotients of torus endomorphisms** (parabolic type);
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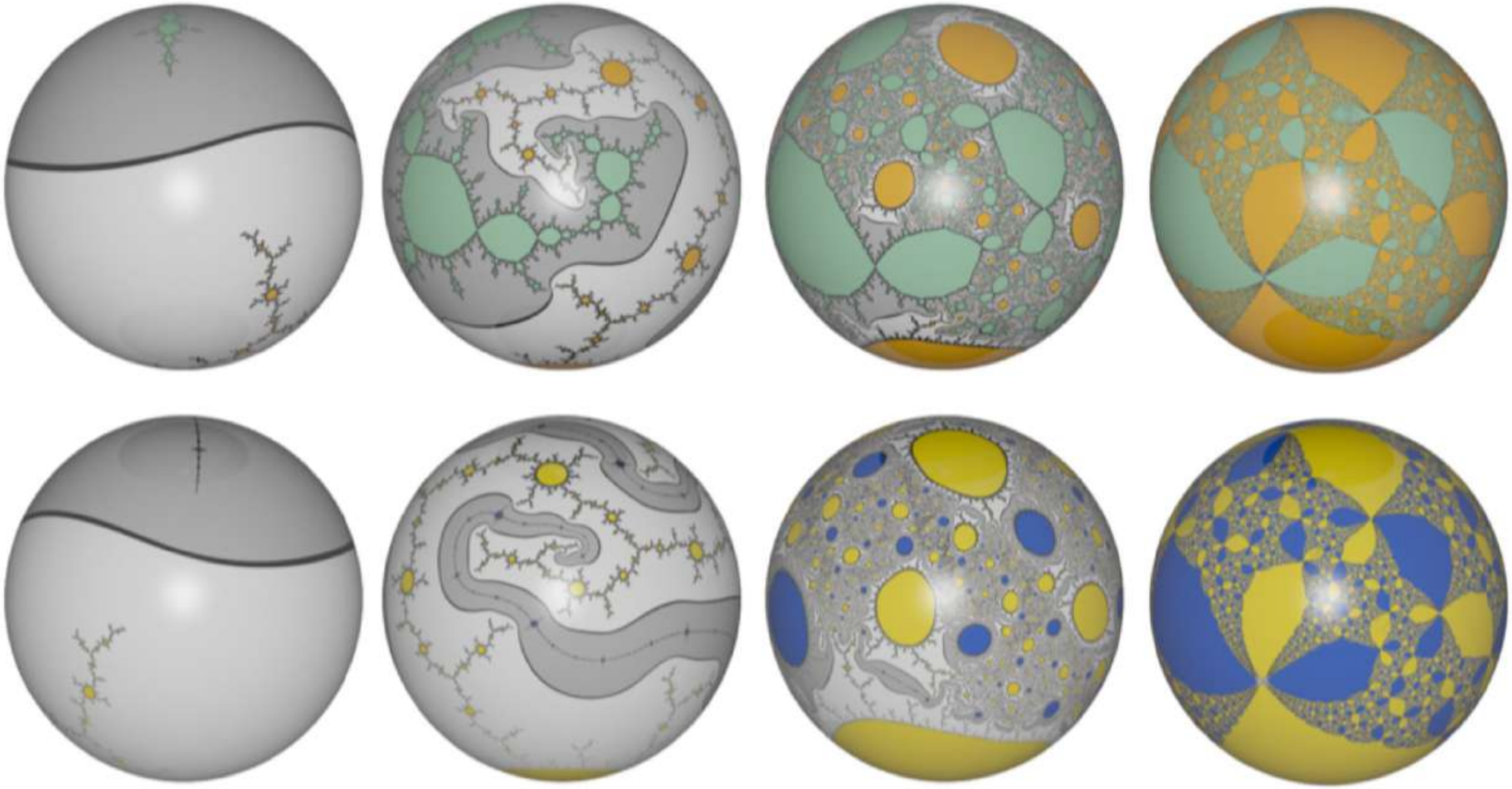
Question

Is there a natural way to decompose pcf rational maps?

Matings of polynomials / Unmating of rational maps

Mating of polynomials (Douady-Hubbard'80s):
an operation that combines two polynomials into a branched cover of $\mathbb{S}^2$.



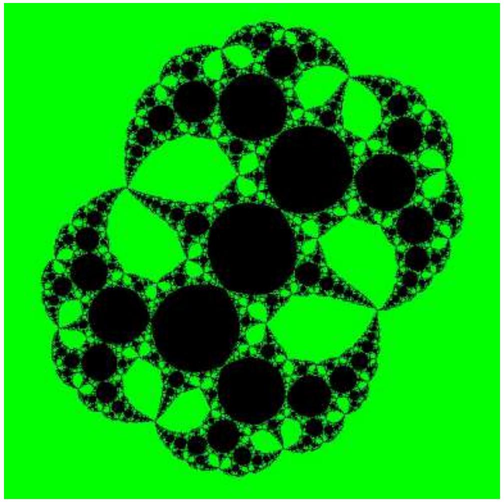
Unmating is the reverse procedure.

- not always applicable;
- non-canonical.

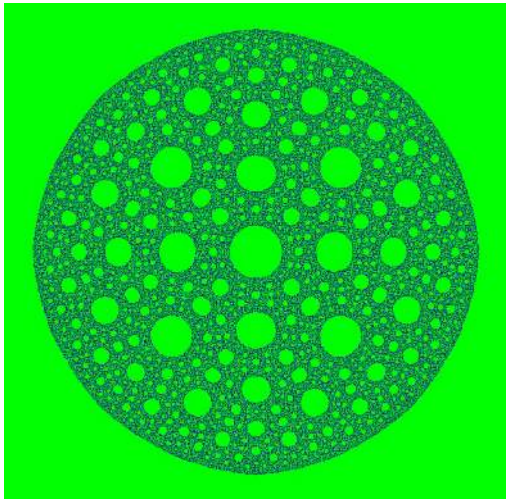
Picture from Buff, Xavier, et al. "Questions about polynomial matings."
Annales de la Faculté des sciences de Toulouse: Mathématiques . Vol. 21. No. S5. 2012.

Decomposition wrt the topology of the Julia set

Idea: Use the structure of the Julia set! Namely, touching Fatou components

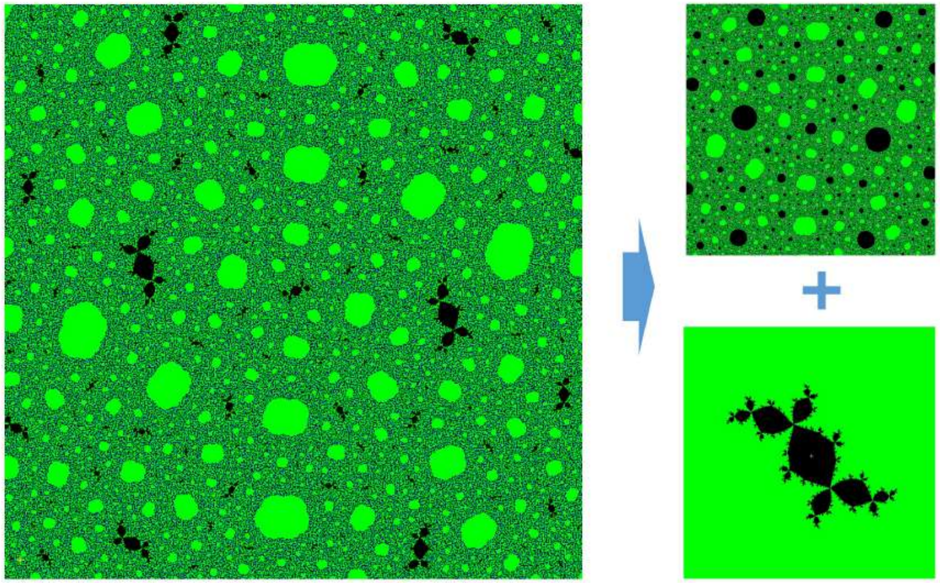


"many" touching Fatou components



no touching Fatou components

Extract maximal clusters of touching Fatou components



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- f is called a Sierpiński carpet map if $\mathcal{J}_f$ is homeomorphic to the standard Sierpiński carpet.
- f is called a crochet map if every two points in P_f may be connected by a path α such that $\alpha \cap \mathcal{J}_f$ is countable.

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Main Theorem

Every pcf rational map with $\mathcal{F}_f \neq \varnothing$ can be canonically decomposed along an invariant multicurve Γ_{cro} **into**

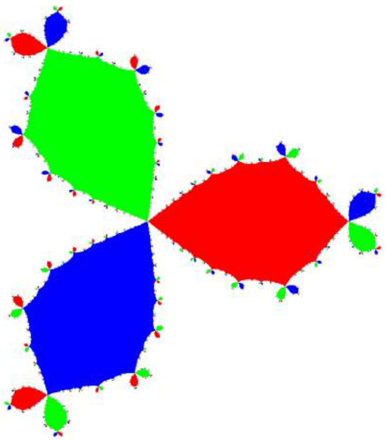
- **crochet maps**
- and **Sierpiński carpet maps**.

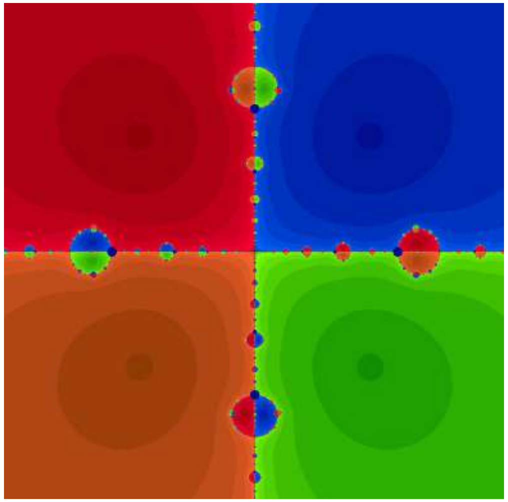
Remark: True in a more general setup of Böttcher expanding maps

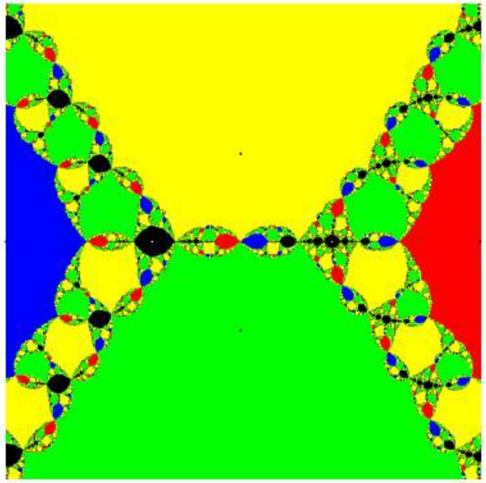
Proof ingredients I – Graphs of internal rays

Observation (Pilgrim)

Touching Fatou components contain (pre)periodic internal rays landing at a common (pre)periodic point.







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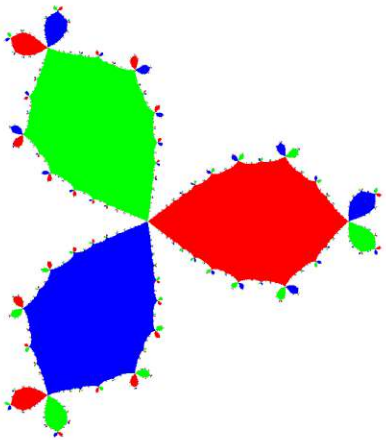

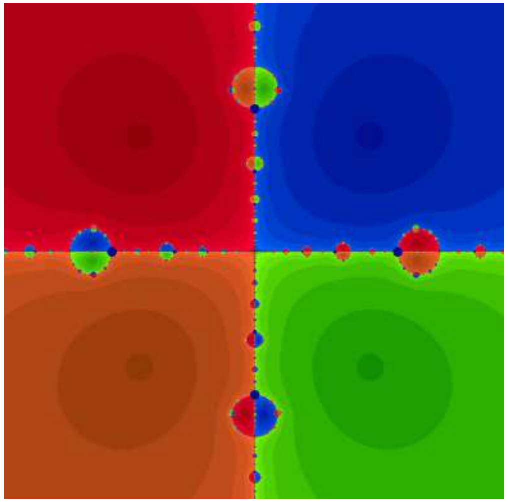

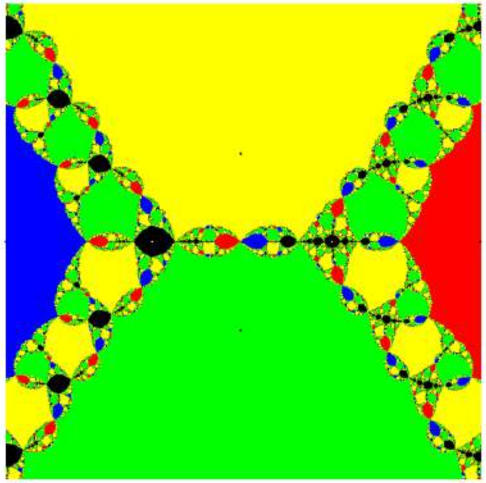

Idea: Connect postcritical points by a graph using (pre)periodic internal rays.

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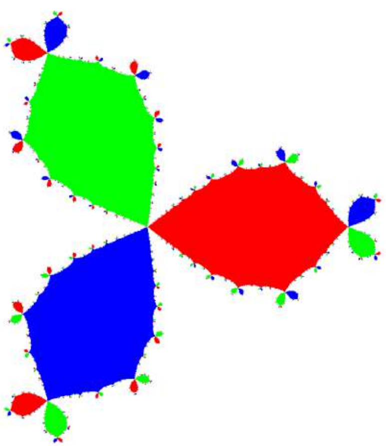

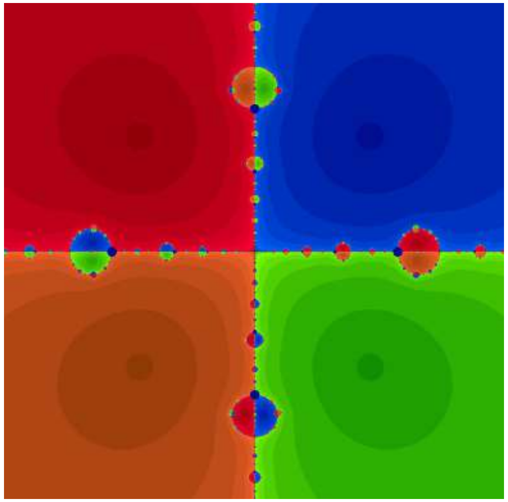

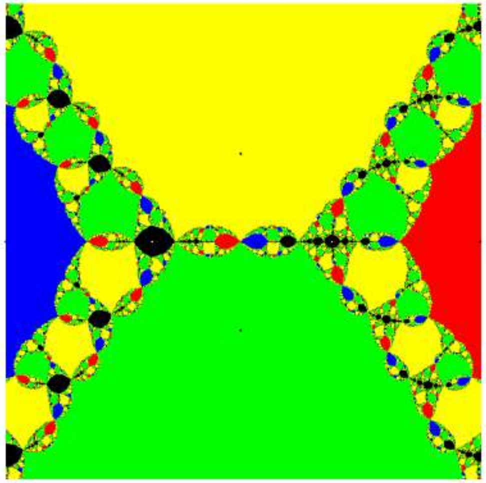

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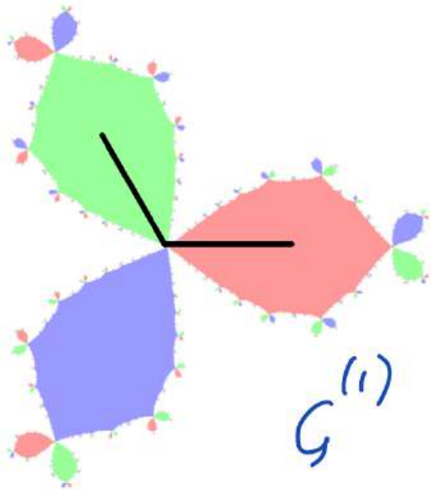
Examples of crochet maps:

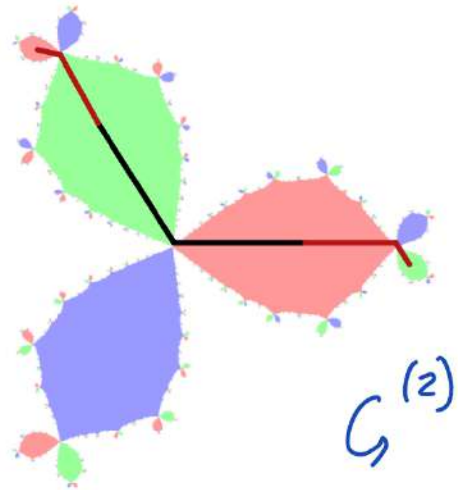
- pcf polynomials – "connect everything to ∞";
- critically fixed rational maps – "Tischler graphs" of fixed rays [H., Pilgrim et.al.];
- pcf Newton maps – "extended Newton graphs" [Lodge-Mikulich-Schleicher-Drach].

Proof ingredients II – Clusters of Fatou components

Let $\mathcal{G}$ be a finite f-invariant 0-entropy connected graph.

Define $\mathcal{G}^{(n)}$ to be the component of $f^{-n}(\mathcal{G})$ containing $\mathcal{G}$, for each $n \geq 0$.





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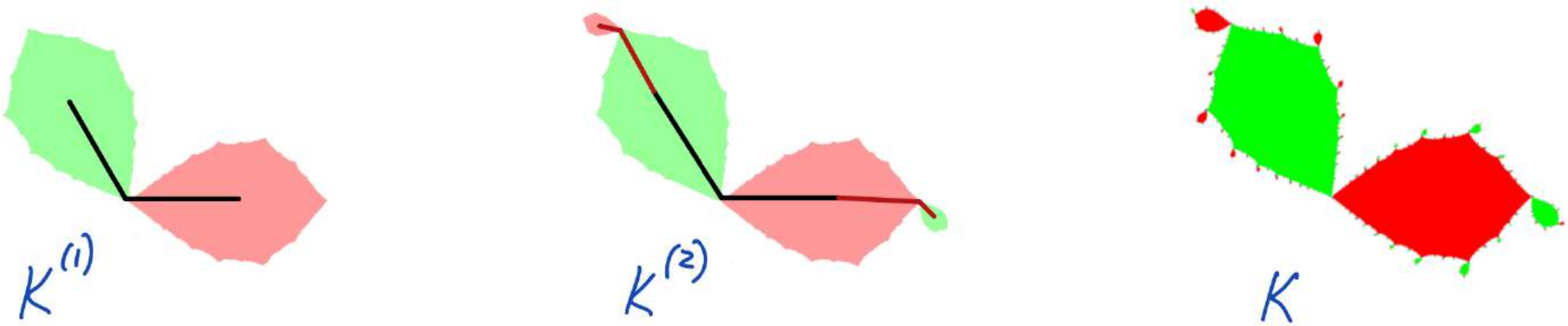

Set $K^{(n)} := \overline{\bigcup_{\Omega \cap \mathcal{G}^n \neq \emptyset} \Omega}$ and $K := \overline{\bigcup_n K^{(n)}}$ – the cluster of $\mathcal{G}$.

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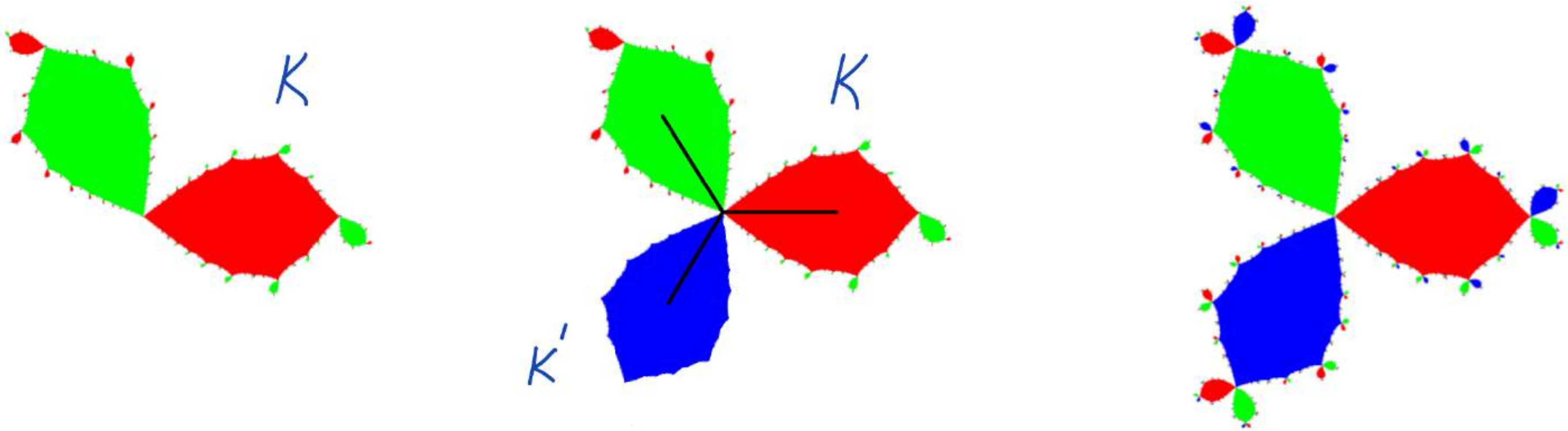
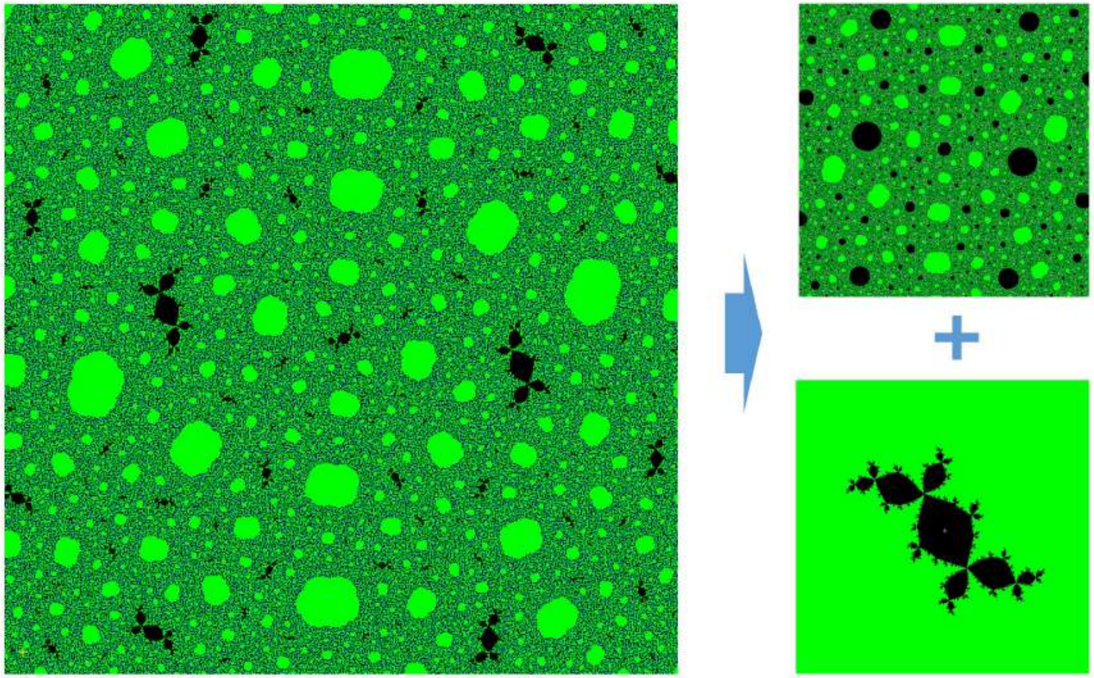


Set $K^{(n)} := \overline{\bigcup_{\Omega \cap \mathcal{G}^n \neq \varnothing} \Omega}$ and $K := \overline{\bigcup_n K^{(n)}}$ – the cluster of $\mathcal{G}$.

- If $x \in K$ is a (pre)periodic point in the cluster K then there is a finite f-invariant 0-entropy connected graph $\mathcal{G}_x \supset \mathcal{G} \cup \{x\}$.
- If K' is another cluster and $K' \cap K \neq \varnothing$ then $K' \cap K$ contain a (pre)periodic point. Thus, we may "combine clusters".

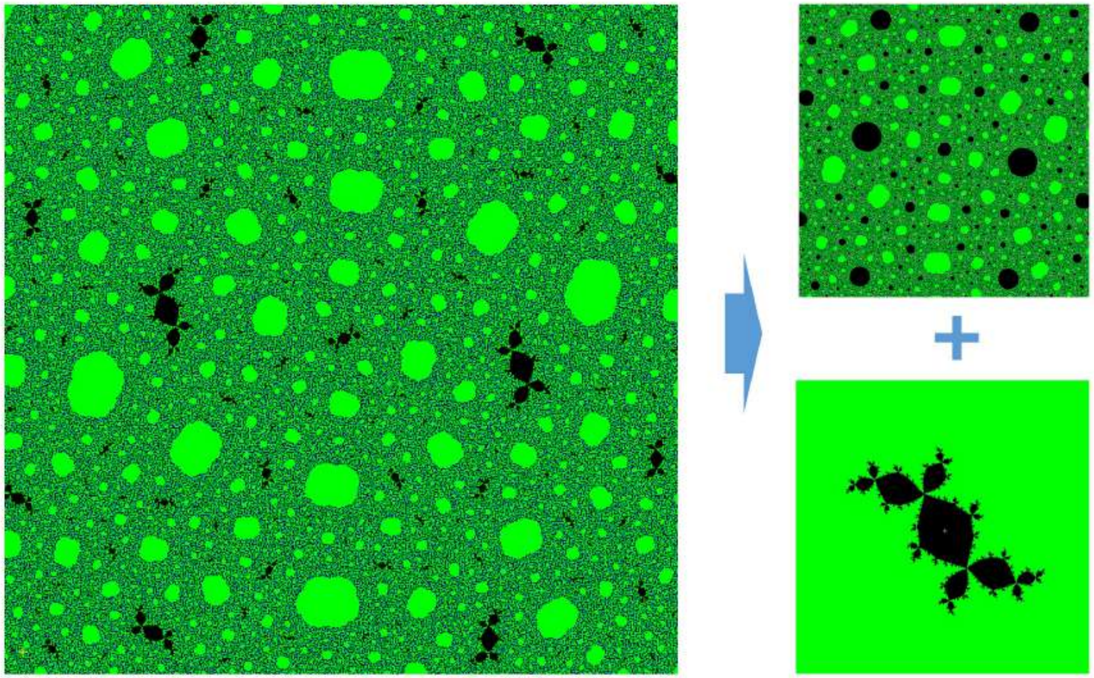
Proof ingredients III — Crochet algorithm



Crochet algorithm

(1) Compute maximal clusters of touching Fatou components.

Proof ingredients III — Crochet algorithm



Crochet algorithm

(1) Compute maximal clusters of touching Fatou components.

(2) Decompose the map with respect to the boundary multicurve of the clusters.

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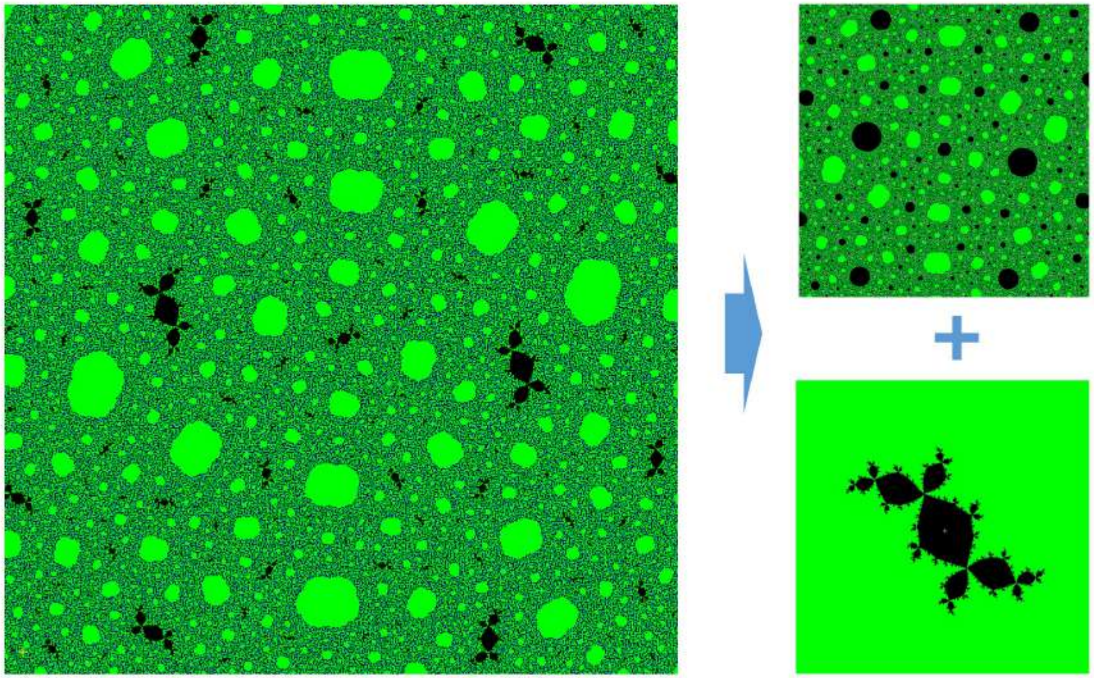

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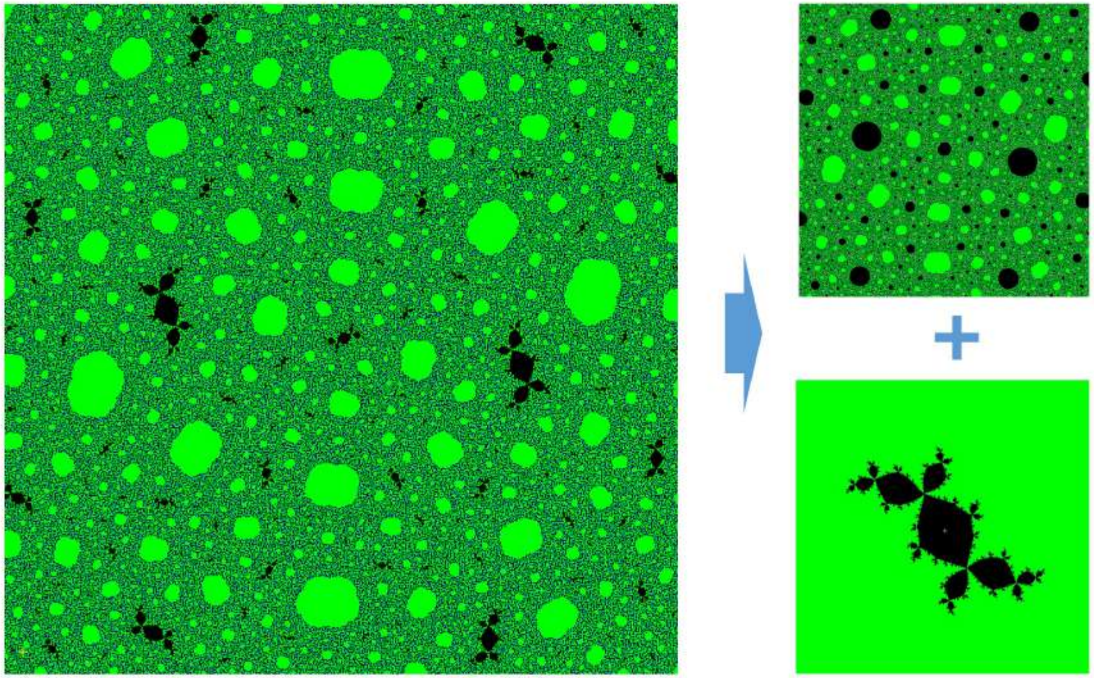

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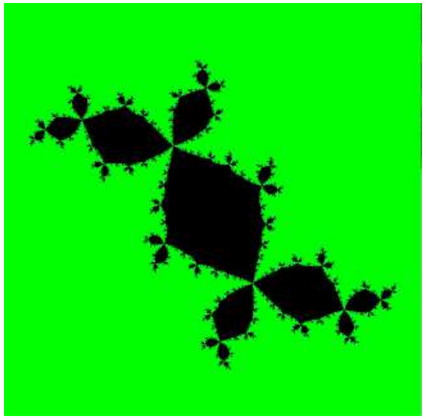
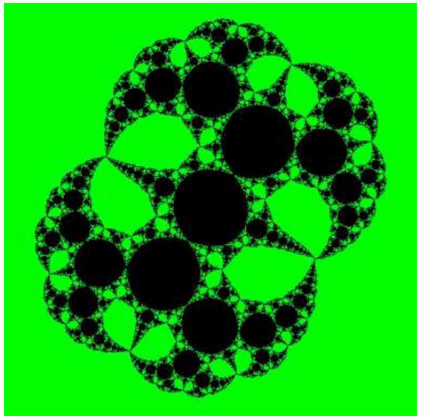
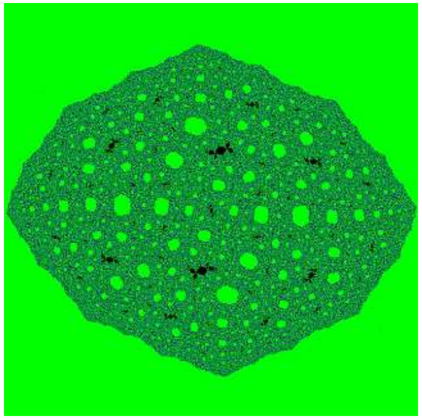
(3) Iterate (1) and (2) for each small map until all small maps are crochet or Sierpiński.

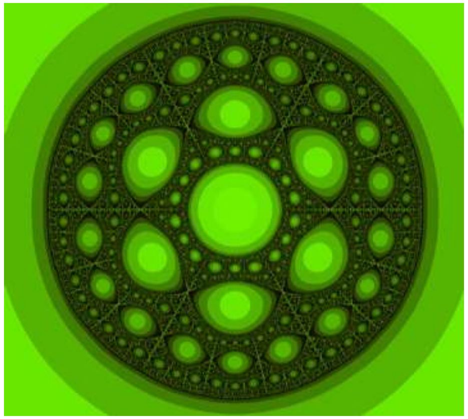
(4) Glue small crochet maps that correspond to the same point in the cactoid $\widehat{\mathbb{C}}/_{\sim\mathcal{F}}$.

Proof ingredients IV — Maximal cactoid quotients









Consider the equivalence relation $\sim_{\mathcal{F}}$ on $\widehat{\mathbb{C}}$ that collapses all Fatou components, i.e., $\sim_{\mathcal{F}}$ is the smallest closed equivalence relation on $\widehat{\mathbb{C}}$, s.t.,

$$\sim_{\mathcal{F}} \quad \supset \quad \{(x_1, x_2) : x_1, x_2 \in \overline{\Omega} \text{ for a Fatou component } \Omega\}.$$

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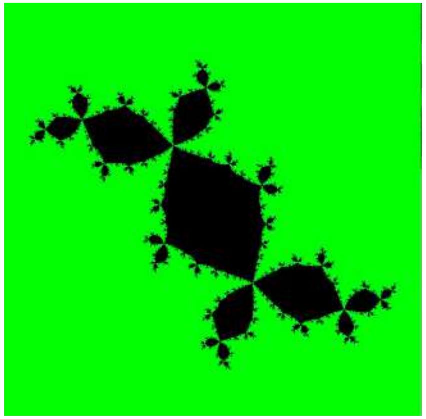

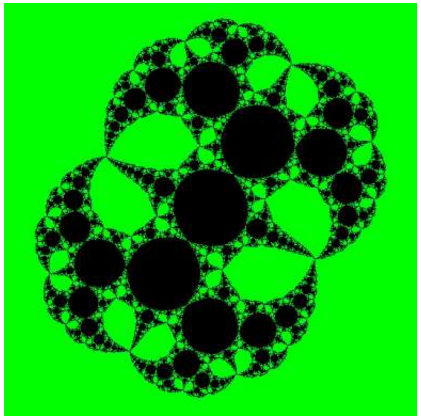

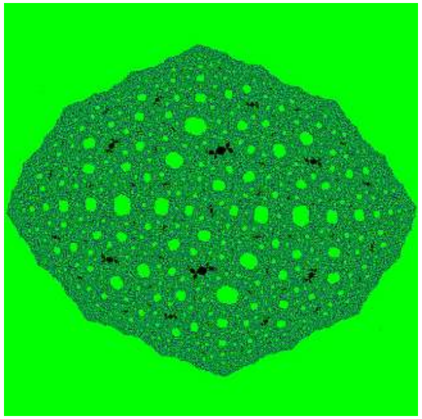

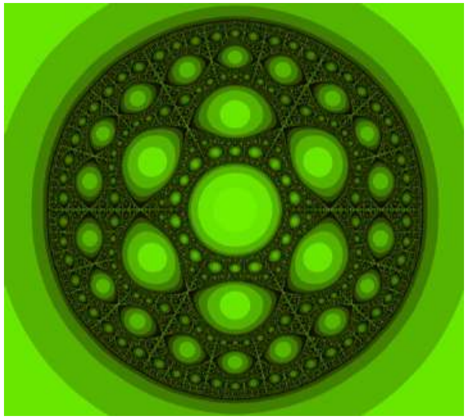

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"squeeze the fractal sponge"

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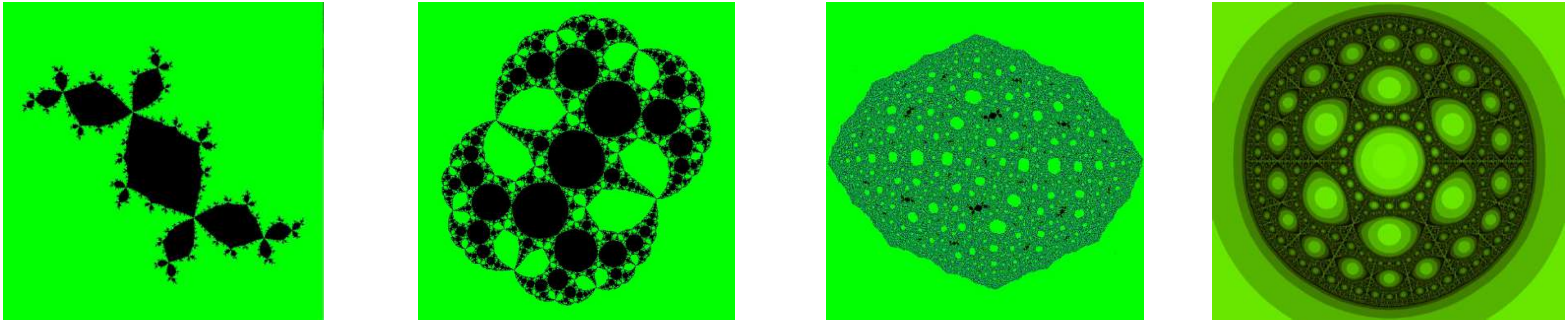

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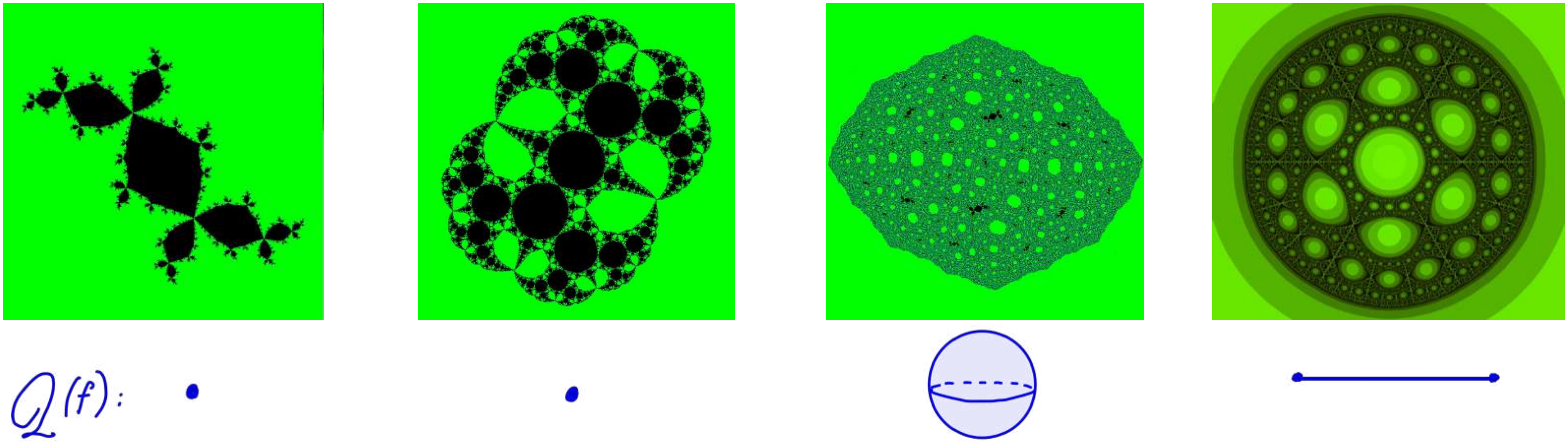

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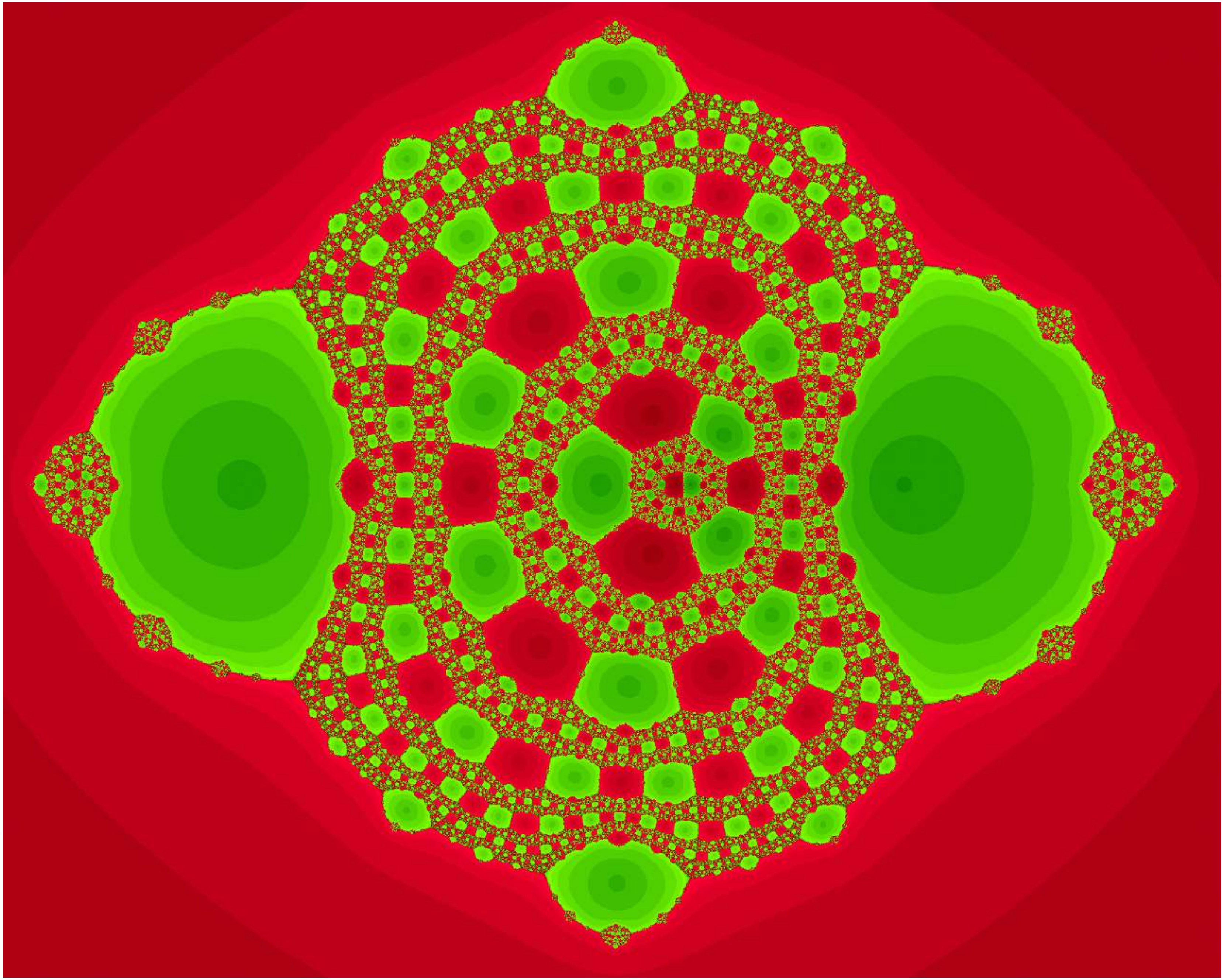
The quotient $\widehat{\mathbb{C}}/_{\sim_{\mathcal{F}}}$ is a sphere cactoid – (the closure of) a tree-like countable union of

- spheres – small Sierpiński carpet maps;
- segments – Cantor-like multicurves, i.e., Cantor set $\times\, \mathbb{S}^1 \hookrightarrow \mathcal{J}_f$;
- points – small crochet maps.

Dendrite quotient $\widehat{\mathbb{C}}/_{\sim_{\mathcal{F}}}$



Characterization of decomposition [Dudko-H.-Schleicher]

Theorem

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and

(ii) the following are true for the quotient map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}/_{\sim_{\mathcal{F}}}$:

- *Julia sets of small Sierpiński maps project onto spheres;*
- *Julia sets of small crochet maps project to points;*
- *different small crochet Julia sets project to different points in $\widehat{\mathbb{C}}/_{\sim_{\mathcal{F}}}$.*

Topological complexity of Julia sets [Dudko-H.-Schleicher]

The following are equivalent:

(i) f is a crochet map;

(ii) $\widehat{\mathbb{C}}/_{\sim_{\mathcal{F}}}$ is a single point;

(iii) there is a finite f-invariant connected graph $\mathcal{G}$ with $P_f \subset \mathcal{G}$ such that $\mathcal{G} \cap \mathcal{J}_f$ is countable.

(iv) there is a finite f-invariant connected graph $\mathcal{G}$ with $P_f \subset \mathcal{G}$ such that the topological entropy of $f|\mathcal{G}$ is 0.

(v) there is a countable set $S \subset \mathcal{J}_f$ such that $\mathcal{J}_f \setminus S$ is totally disconnected.

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The following are equivalent:

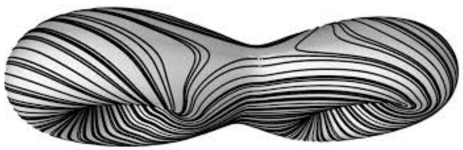
(i) f is Sierpiński-free, that is, no small Sierpiński carpet maps in any decomposition;

(ii) $\widehat{\mathbb{C}}/_{\sim_{\mathcal{F}}}$ is a dendrite;

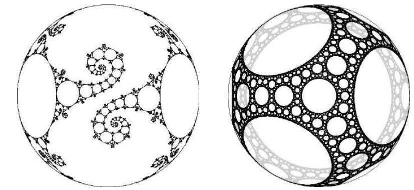
(iii) $\mathcal{J}_f$ has countable separation property, that is, for each $x, y \in \mathcal{J}_f$ there is a countable subset $S \subset \mathcal{J}_f$ such that x and y belong to different connected components of $\mathcal{J}_f \setminus S$.

Dessert menu: Connections of the decomposition theorem

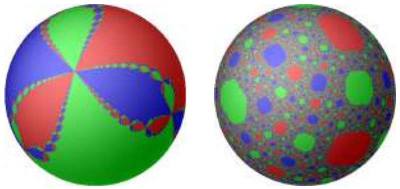
Mapping Class Groups



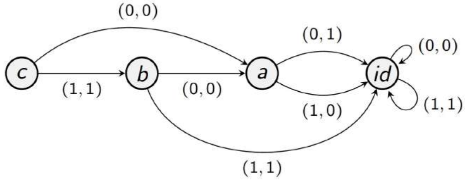
Geometric Group Theory



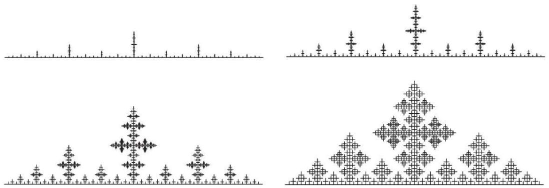
Rational Dynamics



Self-similar Groups



Fractal Geometry



Pictures courtesy of C. Bishop, D. Calegary, and C. McMullen

Connections to fractal geometry

Question: How can we measure "geometric complexity" of a fractal?

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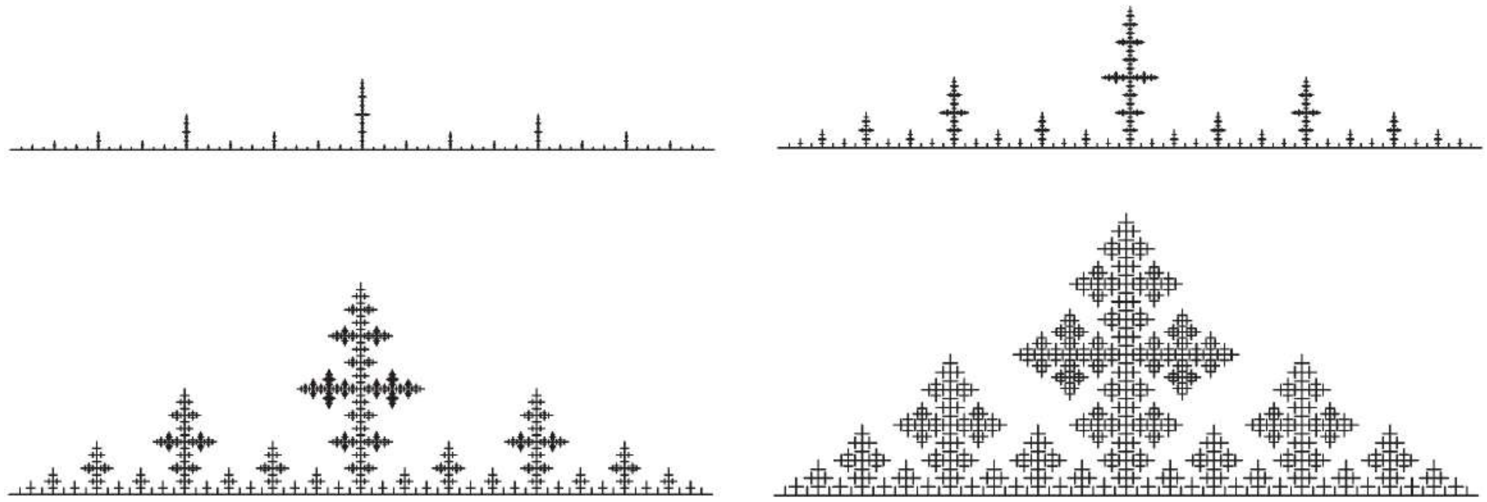
Conformal dimension of a metric space $\mathcal{X}$:

$$\mathrm{ConfDim}(\mathcal{X}) := \inf\{\dim_{\mathrm{H}}(\mathcal{Y}) : \text{metric spaces } \mathcal{Y} \text{ quasisymmetric to } \mathcal{X}\}$$



Picture by C. Bishop

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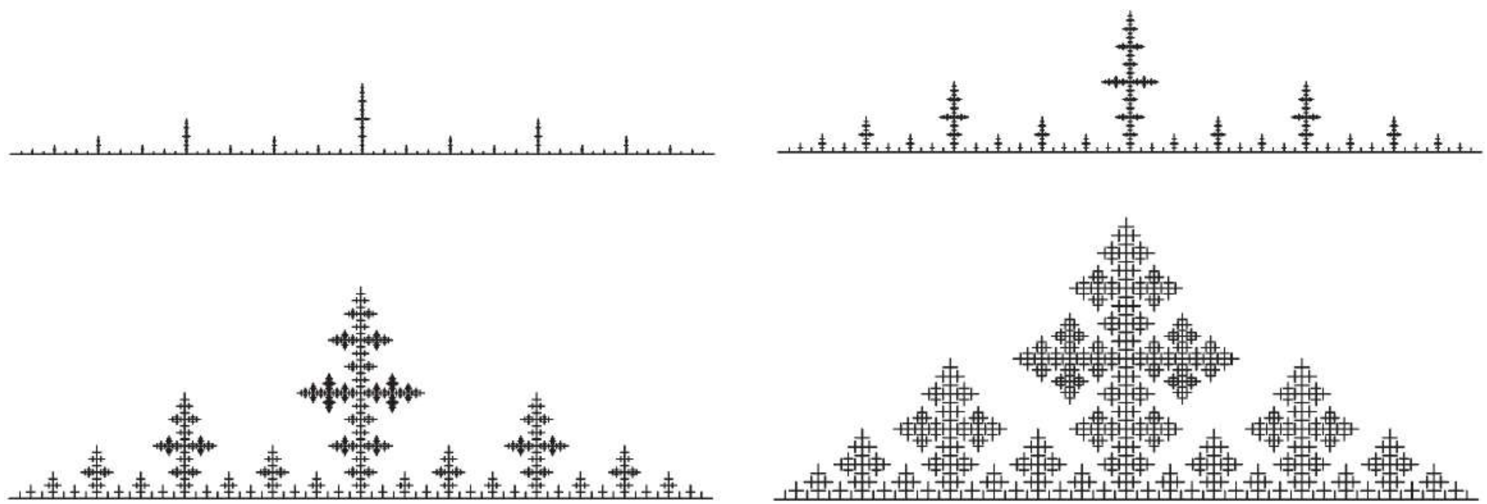


Picture by C. Bishop

Similarly, one defines Ahlfors-regular conformal dimension ARConfDim.

(provides natural invariants for boundaries of Gromov hyperbolic groups)

Geometric complexity of Julia sets

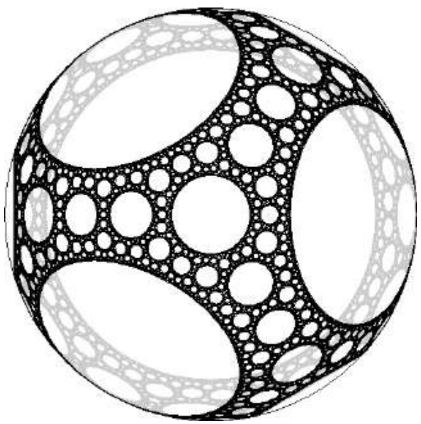
Theorem [Park'22]

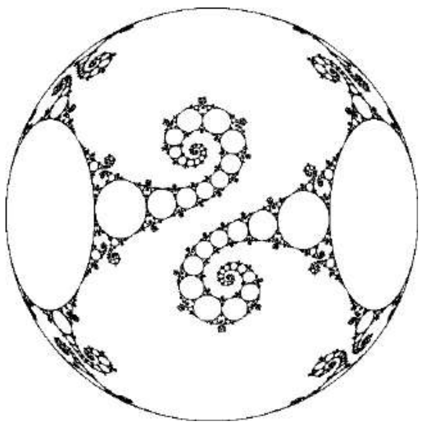
A hyperbolic pcf rational map f is crochet if and only if $\mathrm{ARConfDim}(\mathcal{J}_f) = 1$.

- $\Leftarrow$ If f is not crochet, then the Julia set $\mathcal{J}_f$ contains Cantor set $\times\, \mathbb{S}^1$.
- $\Rightarrow$ If f is crochet, then there is a 0-entropy f-invariant graph $\mathcal{G} \supset P_f$. $\mathcal{G}$ provides a basis for a criterion by [Pilgrim-D.Thurston'21].

Connections to geometric group theory

Sullivan's dictionary'85:
a framework relating dynamics of rational maps and Kleinian groups.





Limit spaces Λ_G of Kleinian groups G

(Λ_G is the closure of the set of repelling fixed points of $g \in G$)

Pictures by C. McMullen

- Similar objects, results, and even proofs (e.g., Sullivan's no-wandering-domain theorems);

Theorem [Carrasco-Mackay'21]

TFAE for a Gromov hyperbolic group G with no 2-torsion and not virtually free:

- $\mathrm{ARConfDim}(\partial_\infty G) = 1$ if and only if
- G has a hierarchical decomposition with elementary edge groups and elementary or virtually Fuchsian vertex groups.

Theorem [Carrasco-Mackay'21]

Let G be a Gromov hyperbolic group that is not virtually free.
Suppose G has a graph of groups decomposition with elementary edge groups and vertex groups G_i. Then

$$\mathrm{ARConfDim}(\partial_\infty G) = \max\left(\mathrm{ARConfDim}(\partial_\infty G_i),\ 1\right).$$

Question

For maps f with $\mathcal{F}_f \neq \varnothing$ and the decomposing curve Γ_{cro}:

$$\text{Is} \quad \mathrm{ARConfDim}(\mathcal{J}_f) = \max\left(\mathrm{ARConfDim}(\text{small Julia sets}),\ Q(\Gamma_{\mathrm{cro}})\right)\ ?$$

Conjecture [Bonk-Geyer-Pilgrim]

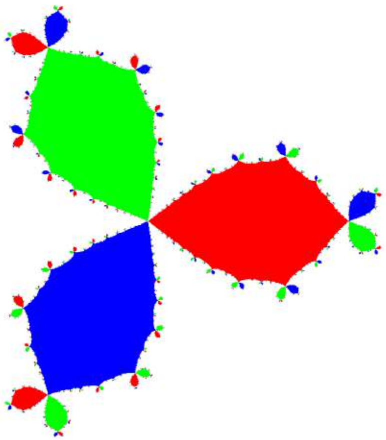
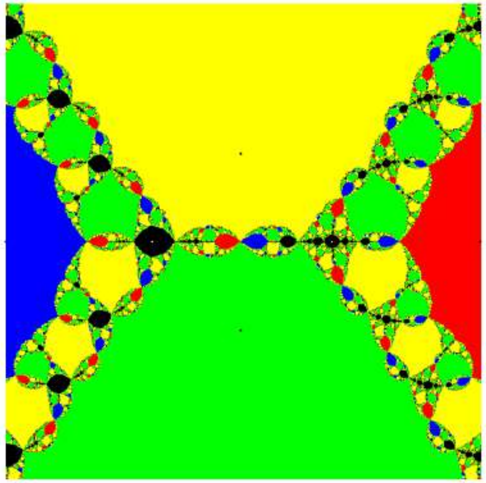
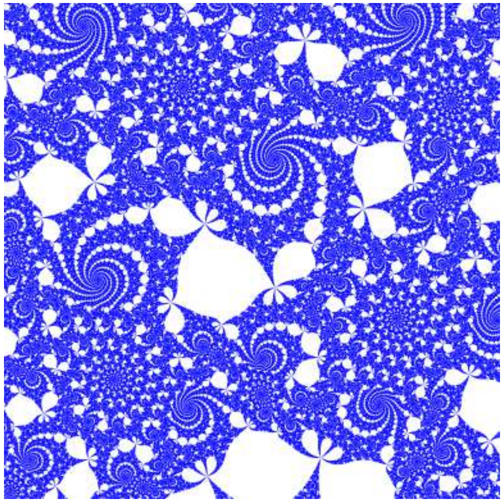
Let f be an obstructed expanding map. Then $\mathrm{ARConfDim}(\mathcal{J}_f) = Q(\Gamma_{\mathrm{Th}})$.

Theorem [Bonk-H.-Meyer]

For each symmetric blown-up $(n \times n)$-Lattès map f we have $\mathrm{ARConfDim}(\mathcal{J}_f) = 2$.

Some further questions

- Is there a natural way to measure “complexity” of crochet maps?

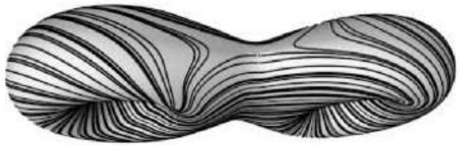
Picture by V. Nekrashevych

  - Cantor-Bendixson rank of (local) separating sets;
  - minimal polynomial growth of generating automata.

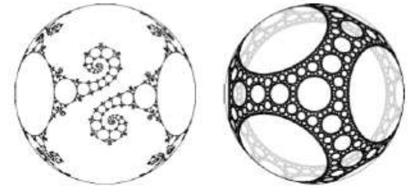
- Is there a natural decomposition of crochet maps?

- Is there a natural decomposition for the limit spaces of contracting self-similar groups?

Mapping Class Groups



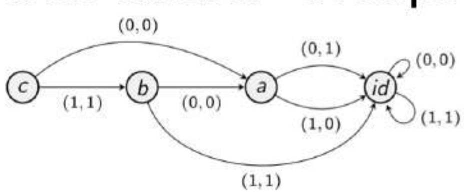
Geometric Group Theory



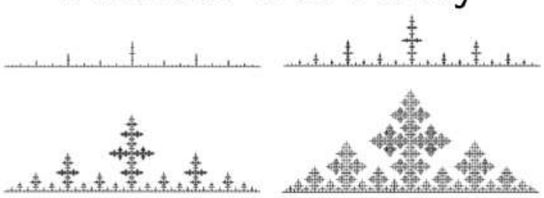
THANK YOU

for your attention!

Self-similar Groups



Fractal Geometry



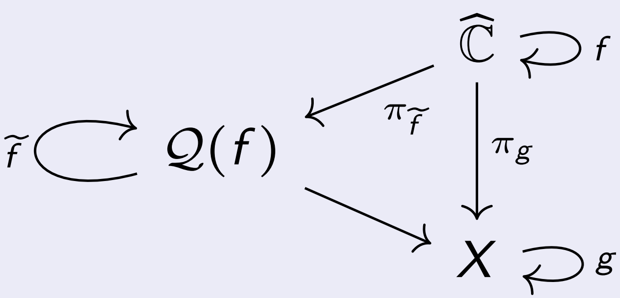
Pictures courtesy of C. Bishop, D. Calegary, and C. McMullen

Proof ingredients — Maximal cactoid quotients

Properties of the cactoid quotient $\mathcal{Q}(f)$

Consider $\widetilde{f}\colon \mathcal{Q}(f) \to \mathcal{Q}(f)$ and the semi-conjugacy $\pi_{\widetilde{f}} : \widehat{\mathbb{C}} \to \mathcal{Q}(f)$.

- Small crochet spheres project under $\pi_{\widetilde{f}}$ to (marked) points in $\mathcal{Q}(f)$ and small Sierpiński spheres project to spheres in $\mathcal{Q}(f)$.
- The quotient map $\widetilde{f}\colon \mathcal{Q}(f) \to \mathcal{Q}(f)$ is topologically expanding with $\mathcal{J}_{\widetilde{f}} = \mathcal{Q}(f)$.
- Let $g\colon X \to X$ be another expanding quotient of f with $\mathcal{J}_g = X$ and the semi-conjugacy $\pi_g\colon \widehat{\mathbb{C}} \to X$. Then π_g factors through $\pi_{\widetilde{f}}$.



That is, $\mathcal{Q}(f)$ is maximal compact expanding quotient.

Connections to mapping class groups

Let $f: X \to X$ be a homeomorphism of a compact metric space X.

- $x, y \in X$ are zero-entropy equivalent if there is a (not necessarily invariant) continuum $\mathcal{C} \ni x, y$ that carries zero entropy.
- f is tight if every continuum in X carries positive entropy.
  Example: pseudo-Anosov maps on closed surfaces.

Theorem [de Carvalho-Paternain'02]

Every $C^{1+\alpha}$-diffeomorphism of a closed surface factors to a tight homeomorphism on a generalized cactoid by a semi-conjugacy whose fibers carry zero entropy.

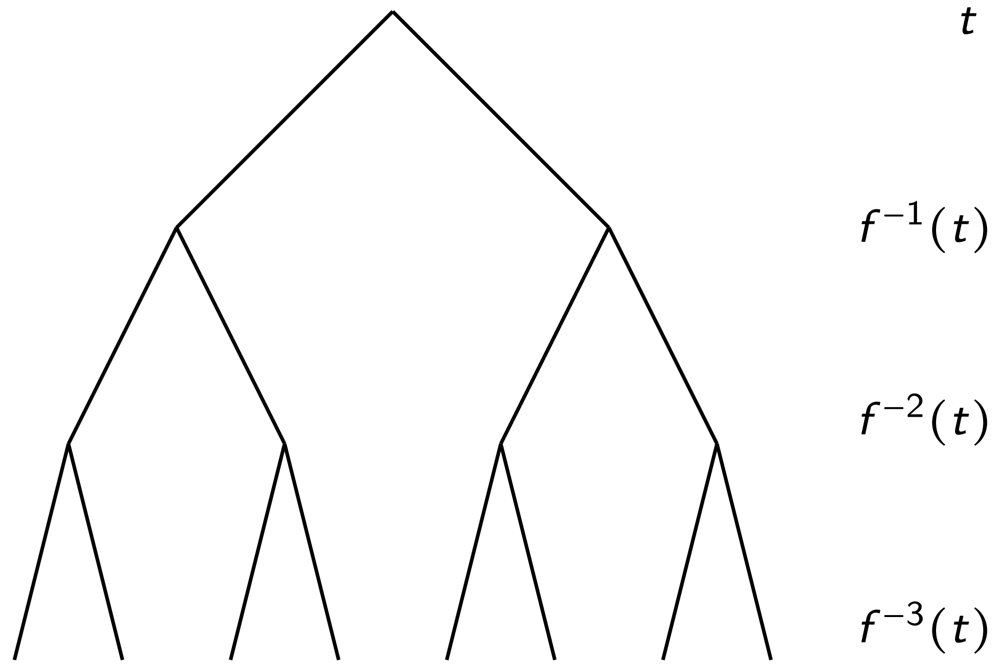
Connections to self-similar groups

Iterated monodromy groups [Nekrashevych'00s]

Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a pcf rational map and $t \in \widehat{\mathbb{C}} \setminus P_f$ be a basepoint.

Preimage tree of t:

$\bigsqcup_{n=0}^{\infty} f^{-n}(t)$



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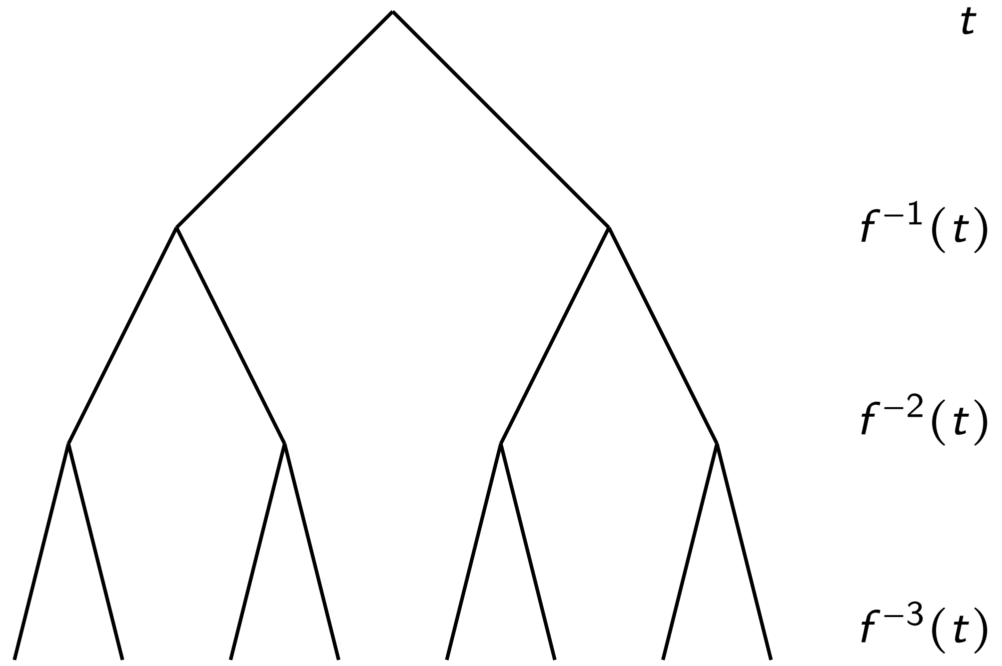

$\pi_1(\widehat{\mathbb{C}} \setminus P_f, t) \curvearrowright f^{-1}(t)$ by the monodromy action

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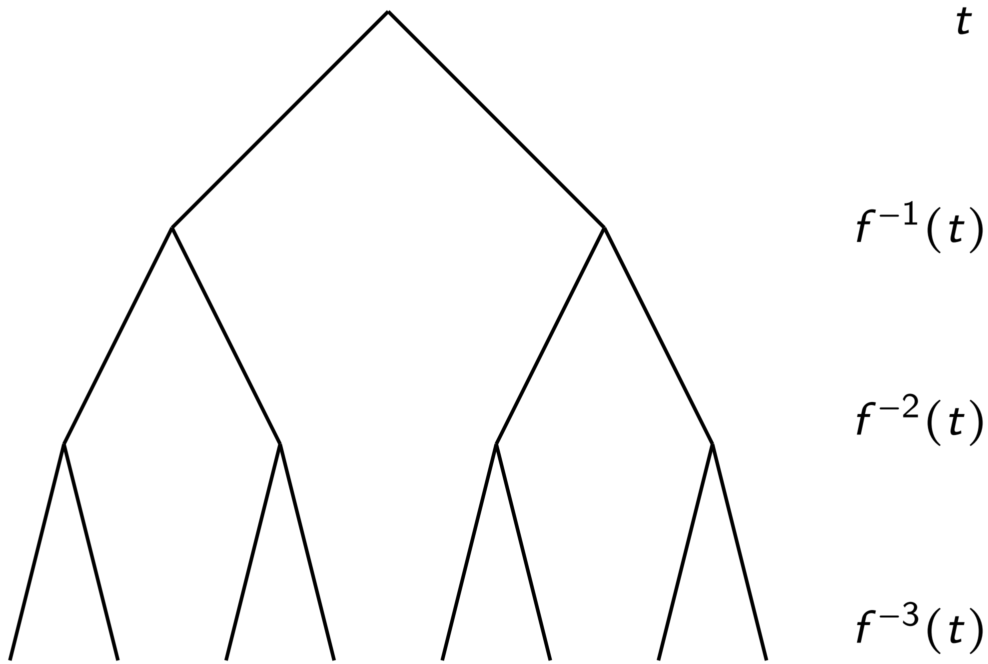

$\pi_1(\widehat{\mathbb{C}} \setminus P_f, t) \curvearrowright f^{-1}(t)$ by the monodromy action

$\pi_1(\widehat{\mathbb{C}} \setminus P_f, t) \curvearrowright \bigsqcup_{n=0}^{\infty} f^{-n}(t)$ — iterated monodromy action

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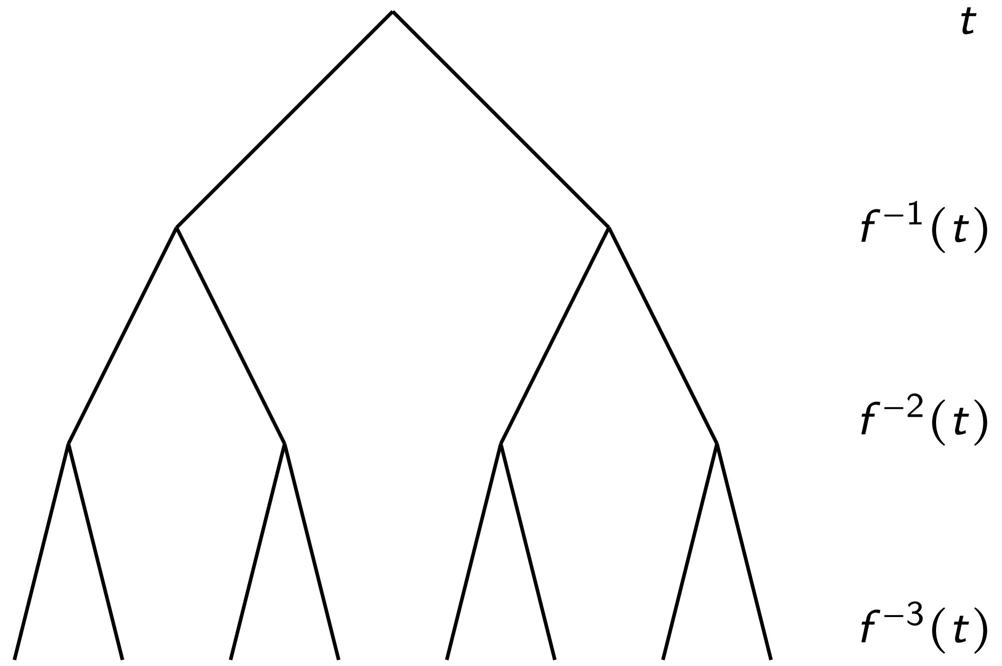

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$\pi_1(\widehat{\mathbb{C}} \setminus P_f, t) \curvearrowright \bigsqcup_{n=0}^{\infty} f^{-n}(t)$ — iterated monodromy action

$\mathrm{IMG}(f) := \pi_1(\widehat{\mathbb{C}} \setminus P_f, t)/\ker$ — iterated monodromy group

Iterated monodromy groups

- IMG's provide a useful computable algebraic invariant in complex dynamics.

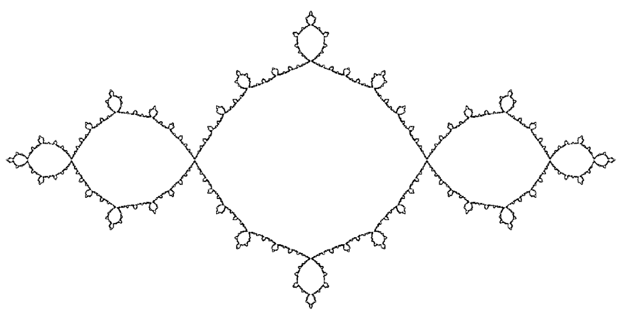
Iterated monodromy groups

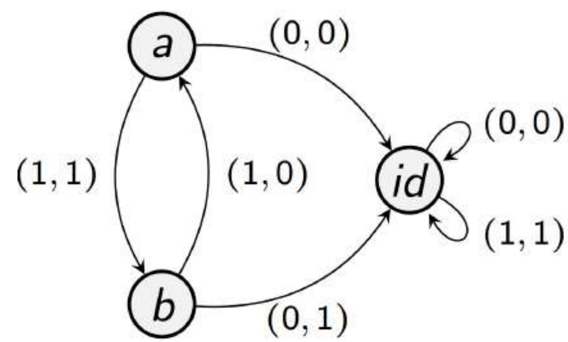
- IMG's provide a useful computable algebraic invariant in complex dynamics.
- IMG's frequently have "exotic" algebraic properties:
  - $\mathrm{IMG}(z^2 + i)$ is of intermediate growth [Bux-Perez'06];
  - $\mathrm{IMG}(z^2 - 1)$ is an amenable but not subexponentially amenable group [Bartholdi-Virag'05].

Iterated monodromy groups

- IMG's provide a useful computable algebraic invariant in complex dynamics.
- IMG's frequently have "exotic" algebraic properties:
  - IMG($z^2 + i$) is of intermediate growth [Bux-Perez'06];
  - IMG($z^2 - 1$) is an amenable but not subexponentially amenable group [Bartholdi-Virag'05].
- IMG(f) is a contracting self-similar group, so its action may be described by a finite automaton.

$z^2 - 1$:





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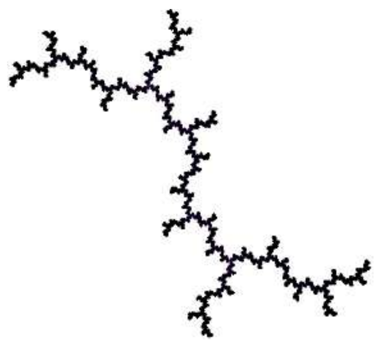

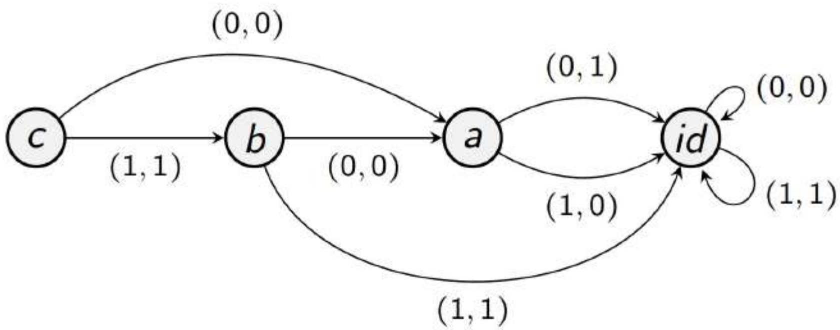

Self-similar groups

Let X be a finite alphabet, $T = X^*$ be the tree of words in X, and $G < \mathrm{Aut}(T)$.

- G is self-similar if $\quad \forall g \in G, \forall v \in T:\ g|_v \in G$.

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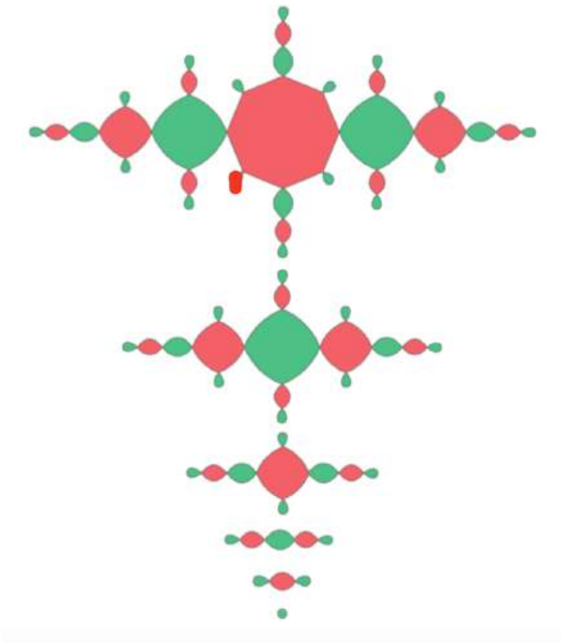

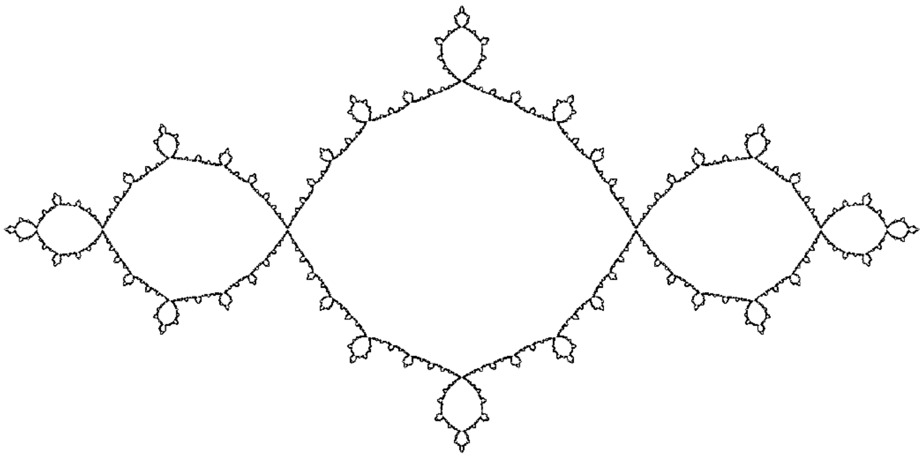

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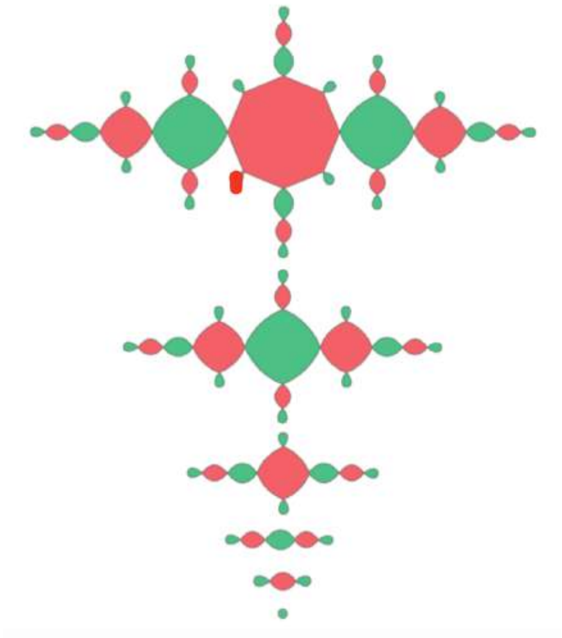

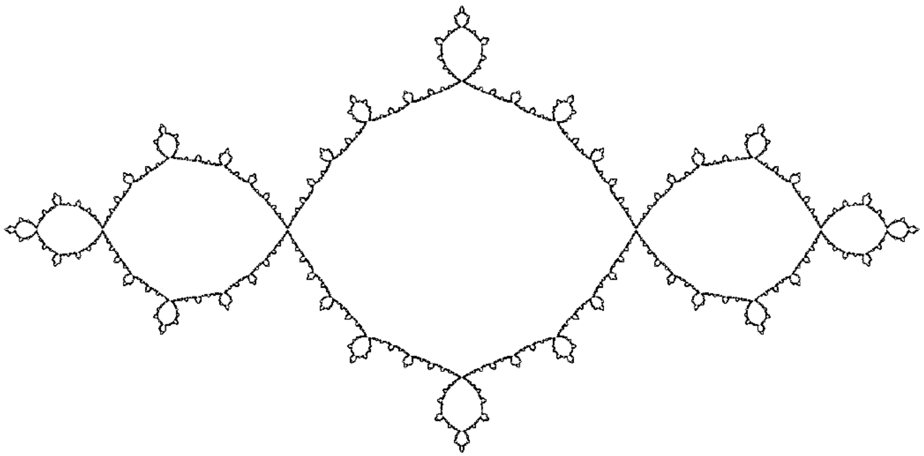

Theorem [Nekrashevych'00s] For a pcf rational map f we have:

- $\mathrm{IMG}(f)$ is contracting and generated by its nucleus;
- the Julia set $\mathcal{J}_f$ is homeomorphic to the limit set $\mathcal{J}_{\mathrm{IMG}(f)}$.

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Let f be a crochet map. Then IMG(f) is amenable.

Based on a criterion for amenability [Juschenko-Nekrashevych-de la Salle], which needs:

(1) recurrence of the simple random walk on the orbital Schreier graphs of IMG(f) [Nekrashevych-Pilgrim-D. Thurston];

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Theorem [Matte Bon-Nekrashevych-Zheng'23]

Let f be a pcf rational map with $\mathcal{F}_f \neq \varnothing$. Then IMG($f$) is amenable.

Decomposition theory [Pilgrim]

Let $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ be a pcf branched covering map.

A multicurve Γ is *f-invariant* if:

(i) $f^{-1}(\Gamma) \subset \Gamma$: each essential component of $f^{-1}(\Gamma)$ is isotopic rel. P_f to a curve in Γ.

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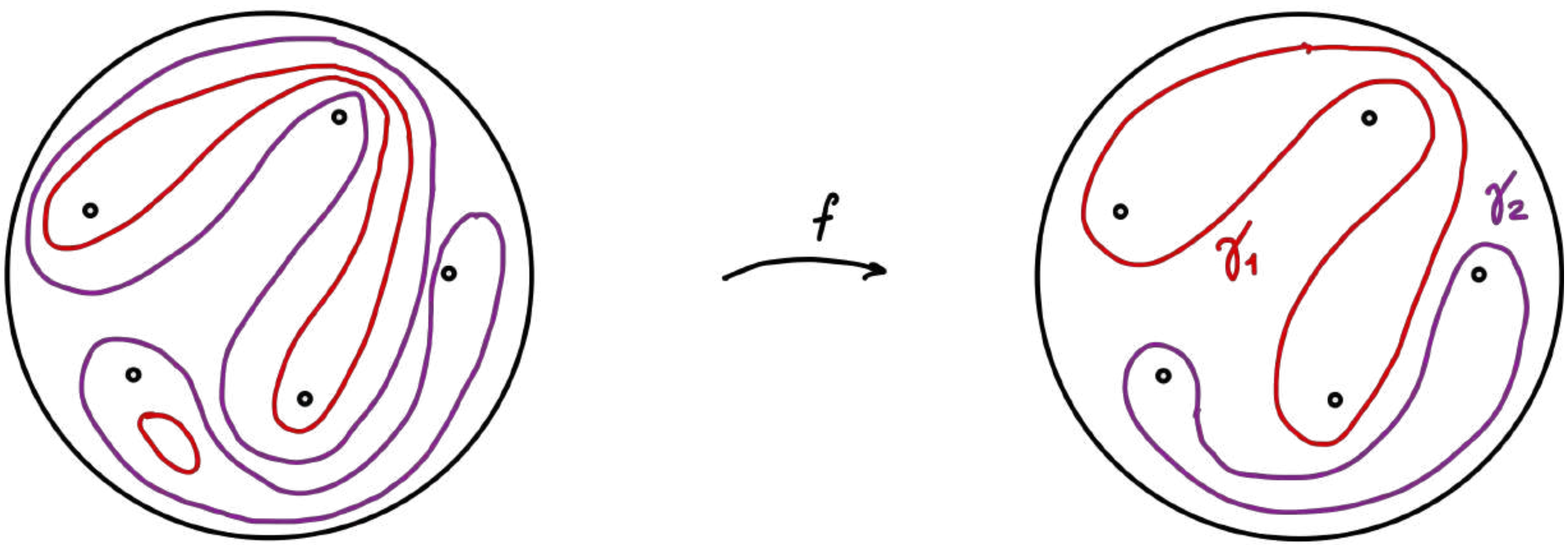

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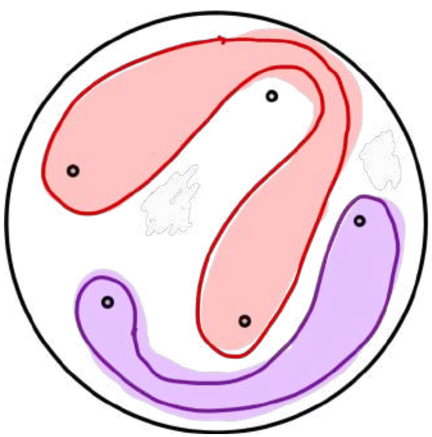

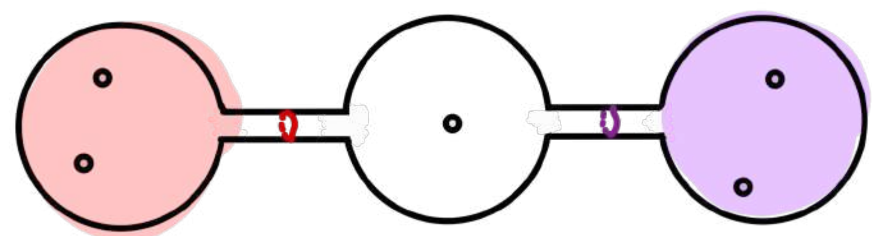

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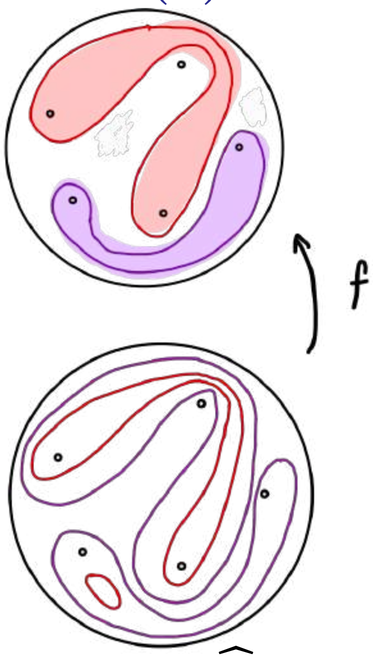

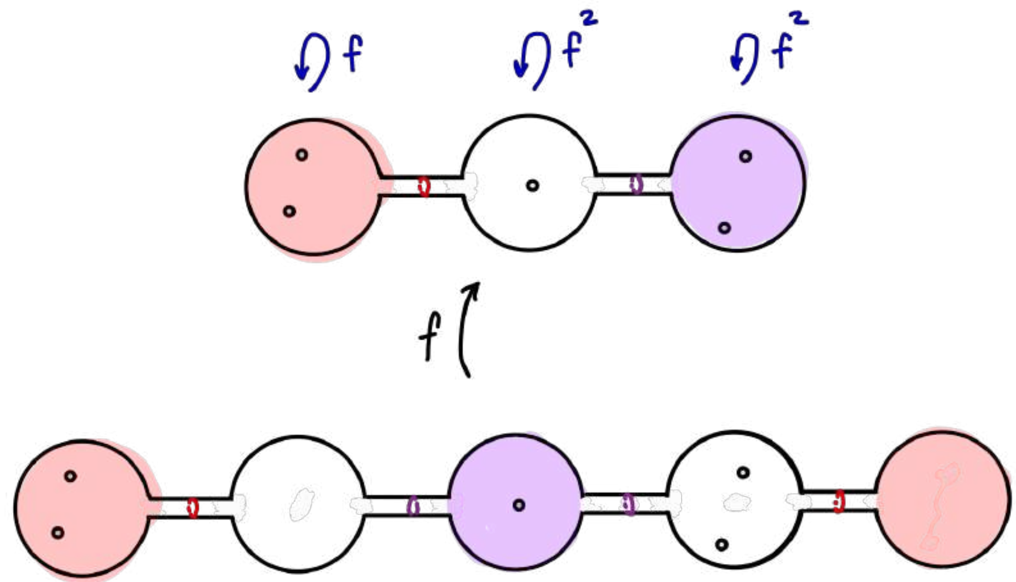

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For a periodic (up to isotopy rel. P_f) small sphere $\widehat{S}^2$, the *first return map* $f^k\colon \widehat{S}^2 \to \widehat{S}^2$ of f to $\widehat{S}^2$ is called a small map.