The Model Theory of the Curve Graph

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Overview

1. Ivanov Metaconjecture
2. Our model-theoretic approach
3. The model theory of the curve graph
   - $\mathcal{L}$-structures
   - Definable sets
   - Interpretations of structures
   - Morley rank
   - Quantifier elimination
4. The main theorem: sketch of the proof

Simplicial actions of the mapping class group

Let S be a topological surface of finite type. The **mapping class group** of S is

$$\mathrm{Mod}(S) := \{\phi : S \to S \text{ homeos }\}/\text{isotopy}.$$

studying $G \leq \mathrm{Mod}(S) \longleftrightarrow G \curvearrowright \mathcal{K}(S)$ "nice" simplicial complex

The simplicial complex $\mathcal{K}(S)$ encodes the combinatorics of various “useful” topological objects on S: curves, arcs, triangulations...

- (Hatcher-Thurston ’80s) $\mathrm{Mod}(S)$ is finitely presented
- (Harer ’80s) homology/cohomology of $\mathrm{Mod}(S)$
- (Farb, Hamenstaedt, . . . ’00s) Coarse Geometry of $\mathrm{Mod}(S)$
- (... ’20s) **Model Theory of** $\mathrm{Mod}(S)$???

Simplicial actions of the mapping class group: the curve complex $\mathcal{C}(S)$

The **curve complex** $\mathcal{C}(S)$ is a simplicial complex encoding the combinatorics of simple essential closed curves on S (taken up to isotopy):

- each vertex corresponds to a s. e. closed curve on S (up iso.) ;
- two vertices are joined by an edge if the curves are disjoint on S (up iso);
- $k+1$ vertices span a k-simplex if the curves are pairwise disjoint (up iso).

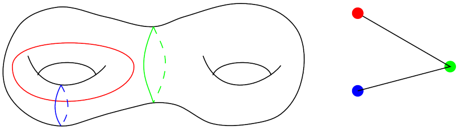


Figure: $V(\mathcal{C}(S)) = \{[$ s. e. closed curves $]\}$ and $E(\mathcal{C}(S)) = \{$ "being disjoint" $\}$

$\mathrm{Mod}(S)$ acts on $\mathcal{C}(S)$ by simplicial automorphisms.

Simplicial actions of the mapping class group: the curve complex $\mathcal{C}(S)$

The curve complex $\mathcal{C}(S)$ is connected, ∞-diameter, locally infinite.

Applications: 3-manifolds, Teichmüller theory, GGT of $\mathrm{Mod}(S)$

Ivanov '87 $\mathrm{Aut}\,\mathcal{C}(S) \cong \mathrm{Mod}(S) \cong \mathrm{Iso}(\mathrm{Teich}(S), d_{Teich})$

Masur-Minsky '99 The curve graph $\mathcal{C}(S)$ is Gromov-hyperbolic

Masur-Schleimer '13 Under "good hypothesis" $\mathcal{K}(S)$ are Gromov-hyperbolic.

Betsvina-Bromberg-Fujiwara '15 $\mathrm{Mod}(S)$ has finite asymptotic dimension

Simplicial actions of the mapping class group: Ivanov Theorem

Theorem (Ivanov '87, Luo '00)

If S is non-sporadic then $\mathrm{Mod}(S) \cong \mathrm{Aut}\,\mathcal{C}(S) \cong \mathrm{Iso}(\mathrm{Teich}(S), d_T)$ *.*

- **Many** other graphs $\mathcal{K}(S)$ such that $\mathrm{Aut}\,\mathcal{K}(S) \cong \mathrm{Mod}(S)$.

graphs $\mathcal{K}(S)$	vertices $V(\mathcal{K}(S))$	author
pants graph $\mathcal{P}(S)$	pants decomp.	Margalit
nonsep. curve graph $\mathcal{N}(S)$	non-sep. curves	Irmak-Korkmaz
multi-curve graph $\mathcal{C}_k(S)$	multi-curves	Erlandsson-Fanoni
arc graph $\mathcal{A}(S)$	simple arcs	Irmak, D.
arc-and-curve graph $\mathcal{A}(S)$	simple arcs	Korkmaz-Papadopoulos
polygonalization complex $\mathcal{P}(S)$	polygonalisations	Bell-D.-Tang
...	...	...

Simplicial actions of the mapping class group

Ivanov Metaconjecture

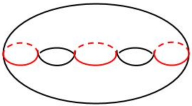


Figure: Pants graph $\mathcal{P}(S)$: each vertex is a pants decomposition

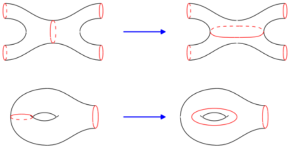


Figure: Pants graph $\mathcal{P}(S)$: each edge corresponds to an elementary move

A metaconjecture by Ivanov

Meta-conjecture (Ivanov '10s)

Every **object naturally associated** to a surface S and having a **sufficiently rich structure** has $\mathrm{Mod}(S)$ as its groups of automorphisms. Moreover, this can be proved by a **reduction** to the theorem about $\mathrm{Aut}\,\mathcal{C}(S)$.

- **object naturally associated** ???
- **sufficiently rich** ???
- **reduction** ???

A metaconjecture by Ivanov: the graph of domains $\Gamma(S)$

Counterexample (McCarthy-Papadopoulos'10): Graph of domains $\Gamma(S_{g,n})$

$V(\Gamma(S_{g,n})) = \{$ connected subsurfaces $R \subset S$ (up to isotopy) $\}$
$E(\Gamma(S_{g,n})) = \{$ being disjoint (up to isotopy) $\}$

If $n \geq 2$ then $\mathrm{Aut}\,\Gamma(S_{g,n})$ is **much larger** than $\mathrm{Mod}(S_{g,n})$.

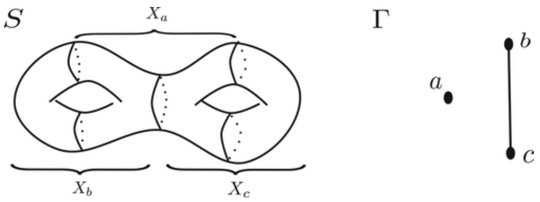


Figure: Adjacency relation on $\Gamma(S)$

Brendle-Margalit's topological approach

Theorem (Brendle-Margalit '17)

Let $\mathcal{R}(S_g)$ be a connected subgraph of $\Gamma(S_g)$ such that no vertex is a hole or a cork. There exists a constant $c(V(\mathcal{R}))$ such that for every $g \geq c(V(\mathcal{R}))$:

$$\mathrm{Aut}\ \mathcal{R}(S_g) \cong \mathrm{Mod}(S_g)$$

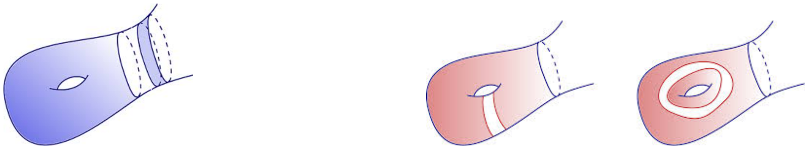


Figure: Corks and Holes

Brendle-Margalit's topological approach

Applications to normal subgroups of the mapping class group.

Open Problem: Extend Brendle-Margalit's work to other classes of complexes, which are popular in geometric group theory:

- graphs on punctured surfaces;
- graphs of (multi-)arcs;
- graphs of (multi-)curves where the edge relation is not disjointness;
- graphs of multi-regions.

Some progress: McLeay '18 , Aougab-Loving et. al '19

Our model-theoretic approach

Model theory machinery in a nutshell

1. a first-order **structure** $\mathcal{M}$ on a language $\mathcal{L}$;
2. **definable** subsets of $\mathcal{M}$, with a notion of “dimension" (the **Morley rank**);
3. **interpretation** for $\mathcal{L}$-structures $\mathcal{M} \rightsquigarrow \mathcal{N}$:
   if two structures $\mathcal{M}$ and $\mathcal{N}$ are **bi-interpretable** then

$$\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{N}) \ ;$$

4. Shelah’s **classification theory** provides invariants of interpretability:
   Morley rank of definable sets, ω-stability, etc...

Model theory machinery: the curve graph $\mathcal{C}(S)$

The curve graph $\mathcal{C}(S)$ is actually a $\mathcal{L}$**-structure** with $\mathcal{L} = \{\mathcal{E}^2\}$ be the language of "edge-adjacency".

In 1987 Ivanov actually proves that these sets are **definable** in $\mathrm{Th}(\mathcal{C}(S))$:

- $\mathcal{N} = \{\gamma \in V(\mathcal{C}(S)) \mid \gamma$ is a nonseparating curve $\}$;
- $\mathcal{S} = \{\gamma \in V(\mathcal{C}(S)) \mid \gamma$ is a separating curve $\}$;
- $\mathcal{I} = \{(\alpha, \beta) \in V(\mathcal{C}(S))^2 \mid i(\alpha, \beta) = 1\}$;
- $\mathcal{J} = \{(\alpha, \beta) \in V(\mathcal{C}(S))^2 \mid i(\alpha, \beta) = 2$ and $\mathrm{Fill}(\alpha, \beta) \cong S_{0,4}\}$.

A model-theoretical frame for Ivanov's metaconjecture

Meta-conjecture (Ivanov '06)

Every **object naturally associated** to a surface S and having a **sufficiently rich structure** has $\mathrm{Mod}(S)$ as its groups of automorphisms. Moreover, this can be proved by a **reduction** to the theorem about $\mathrm{Aut}\, \mathcal{C}(S)$.

Model-theoretic Reformulation // Exercise Every **$\mathcal{L}$-structure** $\mathcal{K}(S)$ **bi-interpretable with** $\mathcal{C}(S)$ has $\mathrm{Mod}(S)$ as its group of automorphisms. Moreover, this **follows** by a **reduction** to the theorem about $\mathrm{Aut}\, \mathcal{C}(S)$. Indeed, by bi-interpretability we have:

$$\mathrm{Aut}\, \mathcal{K}(S) \cong \mathrm{Aut}\, \mathcal{C}(S) \overset{IV.}{\cong} \mathrm{Mod}(S)\ .$$

Motivating questions in our work

Many other configurations of arcs and curves can be described with first-order formulas, in particular many graphs $X(S)$ have

$$\text{Aut } X(S) \cong \text{Mod}(S) \ .$$

- Is $X(S)$ interpretable/bi-interpretable with $\mathcal{C}(S)$?

ω-stability is a natural obstruction to (bi-)interpretability.

- Understand the stability type of $\mathcal{C}(S)$ (or $X(S)$): is $\mathcal{C}(S)$ ω-stable ?

Our results: the first-order theory of $\mathcal{C}(S)$

The **Morley rank** measures the “dimension" of definable sets of $\mathcal{M}$:

$$MR : \{ \text{ definable sets in } \mathcal{M}\} \to \{-1\} \cup \mathrm{Ord} \cup \{\infty\}$$

If every definable set X has $MR(X) \in \{-1\} \cup$ Ord then $\mathrm{Th}(\mathcal{M})$ is ω**-stable**.

Theorem (D. – Koberda – de la Nuez Gonzàlez)

Let S be a non-sporadic surface. Then $\mathrm{Th}(C(S))$ is ω-stable. If S has genus g and n punctures, then we have:

$$MR(\mathrm{Th}(\mathcal{C}(S))) \leq \omega^{3g+n-3} .$$

Our results: ω-stability and interpretability of geometric graphs

We define $X(S) = (V(X(S)); E(X(S)))$ **geometric** if $\exists N > 0$:

- Each $v \in V(X(S))$ is made by at most N curves or arcs;
- $\mathrm{Mod}(S)$ acts on $X(S)$ via its action on curves and arcs with

$$V(X(S))/\mathrm{Mod}(S) \text{ finite and } E(X(S))/\mathrm{Mod}(S) \text{ finite .}$$

Theorem (D. – Koberda – de la Nuez Gonzàlez)

Every geometric graph $X(S)$ is interpretable in $\mathcal{C}(S)$.

Corollary

Every geometric graph $X(S)$ is ω-stable.

Our results: ω-stability and interpretability of geometric graphs

Geometric graphs include the following:

- the Hatcher-Thurston graph;
- the pants graph;
- the marking graph;
- the non-separating curve graph;
- the separating curve graph;
- the arc graph;
- the flip graph;
- the polygonalization graph;
- the arc-and-curve graph;
- the Schmutz -Schaller graph.

Our results: ω-stability and interpretability of geometric graphs

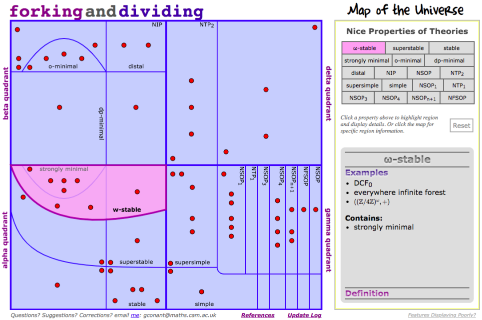


Figure: A map of the model theory universe by Gabriel Conant
`https://www.forkinganddividing.com/`

Our results: a recipe for definable sets

(Ivanov '87) $\mathcal{I} = \{(\alpha, \beta) \in V(\mathcal{C}(S))^2 \mid i(\alpha, \beta) = 1\}$ is definable.

Corollary (D. - Koberda - de la Nuez Gonzàlez)

Let X be a subset of $V(\mathcal{C}(S))^k$. If X is invariant by the diagonal action of $\mathrm{Mod}(S)$ on V^k and its projection to $\mathrm{Mod}(S)/V^k$ is finite or cofinite, then X is definable in $\mathcal{C}(S)$.

In particular, the following set is definable in $\mathcal{C}(S)$:

$$\mathcal{I}_n = \{(\alpha, \beta) \in V(\mathcal{C}(S))^2 \mid i(\alpha, \beta) = n\}$$

where $i(\cdot, \cdot)$ is the geometric intersection number between curves.

Our results: relative QE and non-definable sets of $\mathcal{C}(S)$

Theorem (D. – Koberda – de la Nuez Gonzàlez)

For all non-sporadic S the theory of $\mathcal{C}(S)$ has quantifier elimination relative to the collection of $\forall\exists$-formulas.

Corollary

The following sets are not definable:

- *the set* $X_n = \{(\alpha, \beta) \in V(\mathcal{C}(S))^2 \mid |ai(\alpha, \beta)| = n\}$ *for* $n > 1$*;*
- *the set* $Y = \{(\alpha, \beta) \in V(\mathcal{C}(S))^2 \mid ai(\alpha, \beta) = 0 \mod 2\}$*.*

where $ai(\cdot, \cdot)$ is the algebraic intersection number between curves.

Our results: a bridge between geometric topology and model theory

(Baudisch - Pizarro - Ziegler '18) Model theory of Right-Angled Buildings

- Contribute to Shelah's classification of theories with many examples coming from geometry and topology;
- Model-theoretical framework for Ivanov's metaconjecture;
- Future study the curve complex in analogy with other ω-stable theories.

The model theory of the curve graph

Relational $\mathcal{L}$-structures

A **relational language** $\mathcal{L}$ is a collection of relations symbols $\mathcal{R}^{(k)}$ with an associated arity $k \geq 1$.

A first-order **relational $\mathcal{L}$-structure** $\mathcal{M}$ consists of the following:

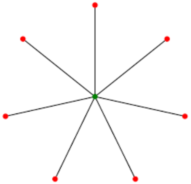
- a set M called **universe**;
- an interpretation of $\mathcal{L}$, that is, a relation $R^k_{\mathcal{M}} \subset M^k$ for each symbol $\mathcal{R}^{(k)} \in \mathcal{L}$.

Relational $\mathcal{L}$-structures: examples

1. A **graph** G is a $\mathcal{L}$-structure with language $\mathcal{L} = \{\mathcal{E}^2\}$:
   - $V(G)$ is the universe;
   - $E(G) \subset V(G)^2$ is the set of edges ($\mathcal{E}^2$ is interpreted by the edge relation).
2. A k-dimensional **simplicial complex** $\mathcal{C}$ is a $\mathcal{L}$-structure in the relational language $\mathcal{L} = \{\mathcal{E}^2, \ldots, \mathcal{E}^{k+1}\}$:
   - $V(C)$ is the universe;
   - $\Sigma^{j+1} \subset V(C)^{j+1}$ is the set of all j-simplices ($\mathcal{E}^j$ corresponds to the simplex relation)

First order theories: graphs

A $\mathcal{L}$-**sentence** is a first-order formula with no free variables. A collection of $\mathcal{L}$-sentences is called a **first-order theory**.



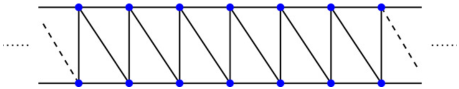
1. "The graph $\mathcal{G}$ has no isolated vertices" : $\mathcal{G} \vDash \forall u \exists v \, (E(u,v))$.
2. "All edges have a common endpoint" : $\mathcal{G} \vDash \exists v (\forall u \, E(u,v) \vee (u = v))$.

Definable sets

Let $\mathcal{M}$ be a $\mathcal{L}$-structure. A set $X \subset M^n$ is **definable** in $\mathcal{M}$ if there exists a first-order formula $\phi(x_1, \ldots, x_n)$ such that

$$X = \{(m_1, \ldots, m_n) \in M^n \ : \ \mathcal{M} \models \phi(m_1, \ldots, m_n)\} \ .$$

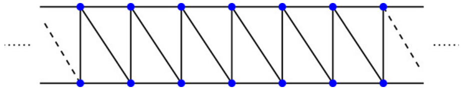
Example:



$Y = \{(x, y) \in V(G)^2 \mid d(x, y) = 2\}$ is definable via the formula:

$$\psi(x, y) \equiv (x \neq y) \wedge \neg E(x, y) \wedge (\exists z \ (\ E(x, z) \wedge E(z, y))) \ .$$

Definable sets: example



1. $Z_2 = \{(x, y, z) \in V(G)^3 \mid (x, y, z)$ is 2-clique $\}$ is definable:

$$\phi(x, y, z) \equiv (x \neq y) \wedge (x \neq z) \wedge (y \neq z) \wedge E(x, y) \wedge E(x, z) \wedge E(y, z)$$

2. $\mathrm{Lk}(v) = \{x \in V(G) \mid x$ is adjacent to $v\}$ is definable over $\{v\}$ by:

$$\phi(v, x) \equiv (x \neq v) \wedge E(v, x)\ .$$

Definable sets: examples in $\mathcal{C}(S)$

1. (Ivanov '87) $\mathcal{N} = \{\gamma \in V(\mathcal{C}(S)) \mid \gamma$ is a separating curve$\}$ is definable
2. (Ivanov '87) $\mathcal{I} = \{(\alpha,\beta) \in V(\mathcal{C}(S))^2 \mid i(\alpha,\beta) = 1\}$ is definable

Corollary (D. - Koberda - de la Nuez Gonzàlez)

Let X be a subset of $V(\mathcal{C}(S))^k$. If X is invariant by the diagonal action of $\mathrm{Mod}(S)$ on V^k and its projection to $\mathrm{Mod}(S)/V^k$ is finite or cofinite, then X is definable in $\mathcal{C}(S)$.

In particular, the following set is definable in $\mathcal{C}(S)$:

$$\mathcal{I}_n = \{(\alpha,\beta) \in V(\mathcal{C}(S))^2 \mid i(\alpha,\beta) = n\}$$

where $i(\cdot,\cdot)$ is the geometric intersection number between curves.

Interpretable structures

We say that a $\mathcal{L}'$-structure $\mathcal{N}$ is **interpretable** in an $\mathcal{L}$-structure $\mathcal{M}$ if

- there is a definable set X in $\mathcal{M}$;
- there is a definable equivalence relation R on X;
- for each symbol $\mathcal{L}'$ there is a definable R-invariant set on X

such that X/R is isomorphic to $\mathcal{N}$.

Example: When R is the identity, we say that $\mathcal{N}$ is **definable** in $\mathcal{M}$.

Interpretation (technical definition)

Given an $\mathcal{L}'$-structure $\mathcal{N}$ and a $\mathcal{L}$-structure $\mathcal{M}$, an **interpretation** $\mu : \mathcal{N} \rightsquigarrow \mathcal{M}$ consists of:

1. An integer $k \geq 0$;
2. A definable subset $X \subseteq M^k$;
3. A definable equivalence relation R on X;
4. A map $F_\mu : X \to N$ which factors through a bijection

$$\bar{F}_\mu : X/R \cong N.$$

such that $F_\mu^{-1}(E_{\mathcal{N}}) \subseteq X^r$ is definable in $\mathcal{M}$ for any relation symbol $\mathcal{E}^{(r)} \in \mathcal{L}'$.

Interpretable structures: examples

The nonseparating curve graph $\mathcal{N}(S)$ is definable in the curve graph $\mathcal{C}_1(S)$.

- (Ivanov) $X = \{\gamma \in V(\mathcal{C}_1(S)) \mid \gamma \text{ nonseparating } \} = V(\mathcal{N}(S))$ definable;
- language $\mathcal{L}'$: the same as $\mathcal{L}$.

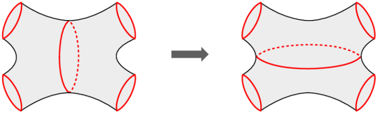
The curve complex $\mathcal{C}(S)$ is definable in the curve graph $\mathcal{C}_1(S)$.

- $X = V(\mathcal{C}_1(S))$ (same universe);
- language $\mathcal{L}' = \{\mathcal{E}^2, \dots, \mathcal{E}^{3g+n-3}\}$ for $\mathcal{C}(S)$: each $\mathcal{E}^{i+1}$ is a "i-clique" in $\mathcal{C}_1(S)$ the k-cliques are definable in $\mathcal{C}_1(S)$ by a first-order formula:

$$\mathcal{E}^k(v_1, \dots, v_k) \iff \forall v_i \forall v_j \ \ (v_i = v_j) \vee ((v_i \neq v_j) \wedge E(v_i, v_j)) \ .$$

Interpretable structures: examples

The pants graph $\mathcal{P}(S_{0,n})$ is interpretable in the curve graph $\mathcal{C}(S_{0,n})$.



- $X = \{(c_1, \ldots, c_N) \in V(C(S))^{3g+n-3} \mid (3g+n-3)$–cliques $\}$ definable in $\mathcal{C}(S)$;
- Definable equivalence relation R: permutation of the components;
- (Ivanov) Definable R-invariant set corresponding to $\mathcal{E}(P_1, P_2)$:

$$\exists \alpha \in P_1 \ \exists \beta \in P_2 \ ((P_1 \smallsetminus \{\alpha\}) = (P_2 \smallsetminus \{\beta\})) \wedge (\alpha, \beta) \in \mathcal{J}$$

where $\mathcal{J} = \{(\alpha, \beta) \in V(\mathcal{C}(S))^2 \mid i(\alpha, \beta) = 2 \text{ and } \mathrm{Fill}(\alpha, \beta) \cong S_{0,4}\}$.

Interpretable structures: examples

We define $X(S) = (V(X(S)); E(X(S)))$ **geometric** if $\exists N > 0$:

- Each $v \in V(X(S))$ is made by at most N curves or arcs;
- $\mathrm{Mod}(S)$ acts on $X(S)$ via its action on curves and arcs such that

$$V(X(S))/\mathrm{Mod}(S) \text{ finite and } E(X(S))/\mathrm{Mod}(S) \text{ finite .}$$

Corollary (D. – Koberda – de la Nuez Gonzàlez)

Every geometric graph $X(S)$ is interpretable in $\mathcal{C}(S)$.

Interpretable structures: bi-interpretability

An interpretation $\mu : \mathcal{N} \rightsquigarrow \mathcal{M}$ induces a homomorphism

$$\hat{\mu} : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{N}) \ .$$

We say that $\mathcal{N}$ and $\mathcal{M}$ are **bi-interpretable** if there are $\mu : \mathcal{N} \rightsquigarrow \mathcal{M}$ and $\eta : \mathcal{M} \rightsquigarrow \mathcal{N}$ such that $\mu \circ \zeta$ and $\zeta \circ \mu$ are definable.

- The curve complex $\mathcal{C}(S)$ is bi-interpretable to the curve graph $\mathcal{C}_1(S)$.

Fact: If $\mathcal{N}$ and $\mathcal{M}$ are bi-interpretable, then $\mathrm{Aut}(\mathcal{N}) \cong \mathrm{Aut}(\mathcal{M})$.

Morley rank

Given a complete theory T, the **Morley rank** is a class function

$$MR : \{\text{ formulas in } T\} \to \{-1\} \cup \mathrm{Ord} \cup \{\infty\} ,$$

defined recursively that serves as a notion of “dimension". The Morley rank of a theory T is defined as $MR(x = x)$.

If $\mathcal{M}$ is a $\mathcal{L}$-structure that models T, then the Morley rank also serves as a measure of “dimension" for the definable sets of $\mathcal{M}$:

$$MR : \{\text{ definable sets in } \mathcal{M}\} \to \{-1\} \cup \mathrm{Ord} \cup \{\infty\} .$$

(ACF) If K an algebraically closed field and $V \subset K^n$ is an algebraic set, $MR(V) = \mathrm{Krulldim}(V)$.

Morley rank: classification theory

(Shelah's Classification Theory '70s) Theories on countable languages can be classified according to the Morley rank of the first-order sentences.

If every definable set X has $MR(X) \in \{-1\} \cup$ Ord then T is ω-**stable**.

ω-stable theories include so far:

- algebraically closed fields;
- algebraic groups over algebraically closed fields;
- groups of finite Morley rank.

(Sela '06) The theory of free groups and torsion-free hyperbolic groups is stable but not ω-stable.

Morley rank: classification theory

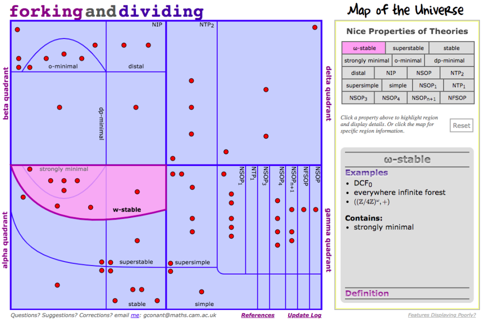


Figure: A map of the model theory universe by Gabriel Conant
`https://www.forkinganddividing.com/`

Morley rank as an obstruction to interpretability

Theorem (D. - Koberda - de la Nuez Gonzàlez)

Let S be a non-sporadic surface. Then $\mathrm{Th}(C(S))$ is ω-stable. If S has genus g and n punctures, then we have:

$$MR(\mathrm{Th}(\mathcal{C}(S))) \leq \omega^{3g+n-3} .$$

If $\mathrm{Th}(\mathcal{M})$ is ω-stable and $\mathcal{N} \rightsquigarrow \mathcal{M}$, then $\mathrm{Th}(\mathcal{N})$ is ω-stable.

Corollary (D. - Koberda - de la Nuez Gonzàlez)

Every geometric graph $X(S)$ is ω-stable.

Morley rank as an obstruction to interpretability: $\mathcal{C}(S) \not\to X(S)$

Is $\mathcal{C}(S)$ interpretable in $X(S)$?

The Morley rank of a theory is a natural obstruction to interpretability:

- If for every $k \geq 1$ we have $MR_{Th(\mathcal{N})}(N) > MR_{Th(\mathcal{M})}(M^k)$, then $\mathcal{N} \not\to \mathcal{M}$.

Corollary (D. – Koberda – de la Nuez Gonzàlez)

Let S be a surface with genus g with n punctures. Then the curve complex $\mathcal{C}(S)$ is not interpretable in any the following graphs:

- *the pants graph $\mathcal{P}(S)$ (when $3g + n > 4$);*
- *the separating curve graph $\mathcal{S}(S)$ (when $g \geq 2$ and $n \leq 1$);*
- *the arc complex $\mathcal{A}(S)$ (when $g \geq 2$ and $n = 1$).*

Quantifier elimination

A first-order theory T has **quantifier elimination** if every formula $\phi(\mathbf{x})$ is equivalent modulo T to some quantifier-free formula $\psi(\mathbf{x})$.

Example: Let $\phi(a, b, c)$ be the formula $\exists x \;\; ax^2 + bx + c = 0$. We have:

$$\mathbb{C} \vDash \phi(a, b, c) \leftrightarrow (a \neq 0 \vee b \neq 0 \vee c = 0)$$

The fomula ϕ is not equivalent to a quantifier-free formula over $\mathbb{Q}$.

Example: The theory of every algebraically closed field (ACF) has quantifier elimination.

If T has quantifier elimination then every definable set in T is definable using a formula *without quantifiers*.

Relative Quantifier Elimination

We say that a theory T has **quantifier elimination relative to the class of $\forall\exists$– formulas** if any formula is equivalent modulo T to a Boolean combination of $\forall\exists$-formulae.

Theorem (D. - Koberda - de la Nuez Gonzàlez)

For all non-sporadic S the theory of $\mathcal{C}(S)$ has quantifier elimination relative to the collection of $\forall\exists$-formulas.

Relative Quantifier Elimination: applications in $\mathcal{C}(S)$

As a consequence of relative quantifier elimination for $\mathcal{C}(S)$, we have:

Corollary (D. - Koberda - de la Nuez Gonzàlez)

Suppose S has positive genus and is not a torus with ≤ 3 boundary components. The following sets are not definable:

- *the set* $X_n = \{(\alpha, \beta) \in V(\mathcal{C}(S))^2 \mid |ai(\alpha, \beta)| = n\}$ *for each* $n > 1$*;*
- *the set* $Y = \{(\alpha, \beta) \in V(\mathcal{C}(S))^2 \mid ai(\alpha, \beta) = 0 \mod 2\}$*.*

where $ai(\cdot, \cdot)$ is the algebraic intersection number between curves.

Sketch of the proof

Theorem (D. - Koberda - de la Nuez Gonzàlez)

Let S be a non-sporadic surface. Then $\mathrm{Th}(C(S))$ is ω-stable. If S has genus g and has n punctures, then

$$MR(\mathrm{Th}(\mathcal{C}(S))) \leq \omega^{3g+n-3} .$$

In addition, $\mathrm{Th}(\mathcal{C}(S))$ has quantifier elimination with respect to $\forall\exists$-formulas.

Sketch

(Baudisch – Ziegler – Martin Pizarro '18) Study of Right-Angled buildings

1. Define an auxiliary structure $\mathcal{M}(S)$ encoding the geometry of $\mathrm{Mod}(S)$;
2. Prove that the structures $\mathcal{C}(S)$ and $\mathcal{M}(S)$ are bi-interpretable; (*)
3. Prove that the structure $\mathcal{M}(S)$ has the quantifier elimination property; (**)
4. Compute the Morley rank of $\mathcal{M}(S)$ and prove that $\mathcal{M}(S)$ is ω-stable.
5. Use the interpretation of $\mathcal{C}(S)$ in $\mathcal{M}(S)$ to deduce the ω-stability of $\mathcal{C}(S)$, get upper bounds on the Morley Ranks, and relative QE.

(*) relies on strongly finite rigid exhaustions for $\mathcal{C}(S)$ (Aramayona-Leininger)
(**) relies on HHS structure of $\mathrm{Mod}(S)$ (Behrstock's inequality)

Proof of Theorem: Step 1

1. Define an auxiliary structure $\mathcal{M}(S)$ encoding the geometry of $\mathrm{Mod}(S)$:

- universe: $\mathrm{Mod}(S)$ as a set
- Two types of binary relations:
  - R_g for $g \in \mathrm{Mod}(S)$ with $R_g(x, y)$ if and only if $x^{-1}y = g$;
  - R_D for any region $D \subseteq S$ so that $R_D(x, y)$ if and only if $x^{-1}y$ is supported on D.

Let $\mathcal{W}$ the collection of finite words in $\mathcal{A} = \{$ subsurface $D\ \}_{D \in S} \cup (\mathrm{Mod}(S) \smallsetminus \mathrm{id})$.
If $w = \delta_1 \ldots \delta_k$ is a word in $\mathcal{W}$, we write

$$R_w := R_{\delta_1} \circ \ldots \circ R_{\delta_k} \ .$$

Proof of Theorem: Step 2

2. The structures $\mathcal{C}(S)$ and $\mathcal{M}(S)$ are bi-interpretable.

A set $\chi \subset \mathcal{C}(S)$ is **strongly rigid** if any isomorphism between χ and another subgraph of $\mathcal{C}(S)$ extends to a unique automorphism of $\mathcal{C}(S)$.

(Aramayona – Leininger) There exists an exhaustion of the curve complex $\mathcal{C}(S)$ by strongly finite rigid sets.

Proof of Theorem: Step 3

3. $\mathcal{M}(S)$ has the quantifier elimination property.

$$\text{Back-and-forth Property} \implies \text{Quantifier Elimination}$$

Back-and-forth property. Every isomorphism $\phi : A \to B$ between a substructure $A \subset \mathcal{C}(S)$ and a substructure $B \subset \mathcal{M}(S)$ admits a “suitable extension":

- $\forall a \in \mathcal{C}(S)\ \exists b \in \mathcal{M}(S)$ such that ϕ extends to an iso. $\overline{\phi} : A \cup \{a\} \to B \cup \{b\}$
- $\forall b \in \mathcal{M}(S)\ \exists a \in \mathcal{C}(S)$ such that ϕ extends to an iso. $\overline{\overline{\phi}} : A \cup \{a\} \to B \cup \{b\}$.

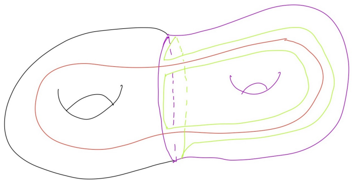
Proof of Theorem: Step 3

3. $\mathcal{M}(S)$ has the quantifier elimination property.

   HHS structure on $\mathrm{Mod}(S) \Longrightarrow$ Back-and-Forth Property

HHS structure of $\mathrm{Mod}(S)$ (Berhstock – Hagen – Sisto):

- Subsurface Projections $\pi_D : \mathcal{C}(S) \to \mathcal{C}(D)$ (Masur – Minsky);
- Berhstock Inequality for projections π_D.



Proof of Theorem: Step 4

4. $MR(\mathcal{M}(S)) = \omega^{3g+n-3}$, therefore $\mathcal{M}(S)$ and $\mathcal{C}(S)$ are ω-stable.

Since $\mathcal{M}(S)$ has QE, every definable set has a canonical form as a Boolean combination of definable sets described by "simple" formulas:

$$R_w(a, x) \text{ and } \neg R_w(a, x)$$

where the w's are words with letters in $\mathcal{A}$. We find:

$$RM(R_w(a, x)) = \omega^{k(S)}$$

where $k(S) = 3g + n - 3$ is the length of the longest chain of subsurfaces:

$$\emptyset \subset D_0 \subset \ldots \subset D_k = S \, .$$

Future questions

Conjectural picture for ω-stability of other analogue graphs:

RAAGs	$\mathrm{Mod}(S)$	handlebody groups $\mathcal{H}_g$	$\mathrm{Out}(\mathbb{F}_n)$	$\mathrm{Aff}(S,q)$
buildings ...	$\mathcal{C}(S)$ $X(S)$	$\mathcal{D}(V_g)$ ...	$\mathcal{FF}_n$ ...	$\mathcal{SC}(S,q)$ ...

The key ingredients for in our proof are :

- Analogies between $\mathcal{C}(S)$ and Right-Angled Buildings;
- Good understanding of the coarse geometry of $\mathrm{Mod}(S)$;
- Rigidity results for $\mathcal{C}(S)$ (finite strongly rigid exhaustion).

THANKS!