Free growth, free counting

joint with M. Kambites, N. Szakács & R. Webb (Manchester)

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KIAS 고등과학원 KOREA INSTITUTE FOR ADVANCED STUDY

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$$ww^{-1}w = w, \quad uu^{-1} \cdot vv^{-1} = vv^{-1} \cdot uu^{-1} \quad (\forall u, v, w \in (A \cup A^{-1})^*)$$

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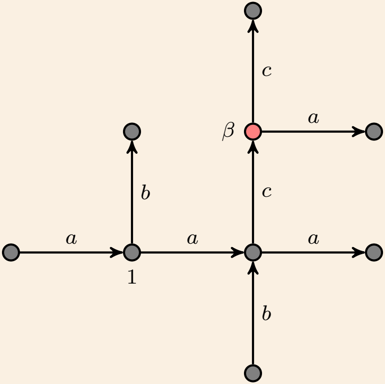
Given $M = \langle A \mid u_1 = v_1, \ u_2 = v_2, \dots\rangle$, natural to ask:

The Word Problem for M
Does there exist an algorithm which does the following:

Input : two words $u, v \in A^*$.

Output : is $u = v$ in M?

2. Munn tree $\mathrm{MT}(u)$ for $u \equiv a^2a^{-3}abb^{-1}ab^{-1}bcaa^{-1}cc^{-1}$



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Let M be the monoid with five generators $\{a, b, c, d, e\}$ and 7 defining relations:

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Then M has undecidable word problem (Tseytin, 1958), cf. N.-B. `arXiv:2401.11757`.

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Type of growth does not depend on generating set; but value of γ does.

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8. (Okninski 1993) $\mathrm{Sgp}\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle$ has intermediate growth.

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Let FIM_r be the free inverse monoid of rank $r > 1$. Let $p = 2r - 1$. Then γ_r, the exponential growth rate of FIM_r, is the largest real root of the polynomial equation

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E.g. when $r = 2$, we have $\gamma_2 = \frac{11}{6} + \frac{\sqrt{13}}{2} \approx 3.6361\ldots$. For large r we have $\gamma_r \to 2r$.

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$$|S(K)| = \frac{p+1}{\frac{1}{2}Kp + p + 1}\binom{\frac{1}{2}Kp + p + 1}{\frac{1}{2}K} + \\ + \sum_{\substack{t+2k=K \\ t\geq 1,\ k\geq 0}} (p+1)p^{t-1}\frac{2p + (t-1)(p-1)}{kp + 2p + (t-1)(p-1)}\binom{kp + 2p + (t-1)(p-1)}{k},$$

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and when K is odd, we have

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Looks terrifying, but lends itself to asymptotic analysis; and find the growth rate as the largest root of

$$p^p x^{p-2} - (px-1)^{p-1} = 0.$$

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2. $\mathbf{N} \times \mathbf{N}$: $ab \to ba$.
3. Free groups: $xx^{-1} \to 1, \quad x^{-1}x \to 1$.

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Theorem (Schein, 1975[a])

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Corollary (NB, 2024)

The following properties are equivalent for monogenic inverse monoids: (1) being finitely presented; (2) FP_2; (3) FP_∞; (4) admitting a FCRS; (5) being non-free.

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Note: some evidence that $H_2(\mathrm{FIM}_1, \mathbf{Z}) = 0$.

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Some questions:

1. What is the growth rate of F_r^*?
2. What are the finitely presented quotients of F_1^*?
3. Is the word problem decidable in any monogenic $*$-semigroup?
4. What is $H_n(FIM_r, \mathbf{Z})$ for $n \geq 2$?
5. What is $H_n(F_r^*, \mathbf{Z})$ for $n \geq 3$?

Thank you!