Characterizing graph properties via RAAGs

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Goal

In the last years there has been great interest in characterizing graph properties via right-angled Artin groups.

In this talk we will give a general overview of this problem, and some very recent results about the subject that mainly concern properties related to planarity.

This is a long-term project with Delaram Kahrobaei and Thomas Koberda.

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Outline

- Right-angled Artin groups.
- A bridge between Graph Theory and Algebra.
- Algebraic characterization of graph properties.
- The cohomology basis graph.
- Minors and Colin de Verdière invariant.
- Main results.
- Prospective work.

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Right-angled Artin groups

Definition

Let Γ be a finite simple graph, V its vertex set and E its edge set. The right-angled Artin group (RAAG) over Γ is the group:

$$A(\Gamma) = \{V \mid [v_i, v_j] = 1 \text{ if } e_{ij} \in E\}.$$

It can be proved that two graphs Γ and Γ' are isomorphic as graphs if and only if $A(\Gamma)$ and $A(\Gamma')$ are isomorphic as groups.

The generators given by the vertices are usually called Artin generators, or just vertices.

These groups are also called partially commutative groups, semifree groups or graph groups in the literature.

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Examples of RAAG's

When Γ is the complete graph in n vertices, $A(\Gamma)$ is the free abelian group in n generators.

At the other end, if Γ is the empty graph in n vertices, $A(\Gamma)$ is the free group in n generators.

If Γ is a graph with four vertices and two edges without common vertices, $A(\Gamma)$ is isomorphic to $\mathbb{Z}^2 * \mathbb{Z}^2$.

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Special subgroup

Given a presentation of a RAAG A with Artin generators and relations, every subset of the set of generators define a subgroup of A, called special subgroup.

These subgroups are again RAAG's.

The RAAG's possess a very rich subgroup structure, as they can contain for example surface groups, braid groups of graphs, Bestvina-Brady groups... This is a hot research topic nowadays.

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- There is a good description of centralizers.
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Example. Let us consider the RAAG

$$\{x_1, x_2, x_3, x_4 \mid [x_1, x_2] = [x_2, x_3] = [x_1, x_3] = [x_1, x_4] = 1\}$$

Then its integer (reduced) cohomology is given by:

- $H^1(A(\Gamma), \mathbb{Z}) = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus \mathbb{Z}v_3 \oplus \mathbb{Z}v_4$.
- $H^2(A(\Gamma), \mathbb{Z}) = \mathbb{Z}v_1v_2 \oplus \mathbb{Z}v_2v_3 \oplus \mathbb{Z}v_1v_3 \oplus \mathbb{Z}v_1v_4$.
- $H^3(A(\Gamma), \mathbb{Z}) = \mathbb{Z}v_1v_2v_3$.
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For $1 \leq i \leq 4$, v_i is the dual of the image of x_i in $A(\Gamma)^{ab}$.

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Cohomology of RAAG's

Let $A(\Gamma)$ be a RAAG. Then $H^*(A(\Gamma), \mathbb{Z})$ is generated, as a ring, for the dual classes of the Artin generators, and we have

$$H^*(A(\Gamma), \mathbb{Z}) = (\bigoplus_{V_0^* \subset V^*} \Lambda_{\mathbb{Z}}(V_0^*))/\sim .$$

In this expression:

- V^* denote the set of dual classes of the Artin generators.
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In practice, this means that a subset of vertices of Γ generates a complete graph (or equivalently, its Artin generators generate a free abelian group) if and only if the product of their dual classes is non-trivial in the corresponding cohomology group.

Moreover, this information describes the whole cohomology ring.

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The bridge

A graph Γ is identified by its vertices, edges and incidence relations. Because of these, the graph properties use to be easily translated, via the Artin generators, to the associated RAAG.

For example, if a graph is disconnected, it is immediately deduced from the relations between the corresponding Artin generators that the group $A(\Gamma)$ breaks as a free product of non-trivial subgroups.

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Conversely, if we do not select a priori an Artin presentations, the reverse questions can be interesting and difficult.

In the previous example, the reverse question would be: given a RAAG that breaks as a free product of non-trivial subgroups, is it true that the associated graph is disconnected?

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We should remark two important facts in this context, which are related:

- We cannot hope that all the algebraic structure of the group is reflected in properties of the graph. For example, as said above, the subgroup structure of the RAAG's can be extremely complex.
- Conversely, the richness of the algebraic structure of the groups suggests that it should be possible to translate many properties of the graph.

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A graph Γ is disconnected if and only if it decomposes as a free product of non-trivial groups.

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Now we present some graph properties that have been characterized intrinsecally in terms of RAAG's:

- Being a join of two graphs (Servatius 1989).
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- Having nontrivial automorphisms.
- Colorability.
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Hamiltonicity to start

The starting point of our research about planarity was the characterization of hamiltonicity, that we next state. Although the next developments are in general valid for any field, in this talk we will assume that the cohomology is taken with coefficients in $\mathbb{F}_2$.

Hamiltonicity in cohomology

Let $A(\Gamma)$ be a RAAG. We say that a basis $\{w_i\}$ of $H^1(A(\Gamma))$ is hamiltonian if for some ordering of the basis the cup products $w_1 w_2, w_2 w_3, \ldots w_{n-1} w_n, w_n w_1$ are non-trivial.

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Proposition

Let Γ be a graph and $A(\Gamma)$ the corresponding RAAG. Then Γ is hamiltonian if and only if every basis of $H^1(A(\Gamma))$ is so.

In the same way it is defined the notion of hamiltonian path in the cohomology, and the corresponding result is valid for hamiltonian paths.

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Basis cohomology graph

Definition

Let Γ be a graph, $A(\Gamma)$ the corresponding RAAG, $\mathcal{B} = \{w_1, \dots, w_n\}$ a basis of $H^1(A(\Gamma))$. Then the cohomology basis graph associated to $\mathcal{B}$ is the graph $\Gamma_{\mathcal{B}}$ such that:

- The vertices of $\Gamma_{\mathcal{B}}$ are given by the basis elements $\{w_i\}$.
- There is an edge between w_i and w_j if and only if the cup product $w_i w_j$ is nonzero.

When $\mathcal{B} = \{w_i\}$ is the dual basis of the basis of $H_1(A(\Gamma))$ induced by a system of Artin generators of Γ, then $\Gamma_{\mathcal{B}} = \Gamma$. In this context we will call Γ the defining graph and denote this basis $\{e_i\}$ instead of $\{w_i\}$. .

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Example

Consider the graph $\Gamma = P_3$ with vertices $\{e_1, e_2, e_3\}$ and edges (e_1, e_2) and (e_2, e_3). Also consider the invertible matrix:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Abusing notation, let us call also e_1, e_2, e_3 the basis defined by the vertices in $H^1(A(\Gamma))$, and $\mathcal{B} = \{w_1, w_2, w_3\}$ the new basis defined via the matrix A.

As $w_1 w_2 = e_2 e_3$, $w_1 w_3 = e_1 e_2 + e_2 e_3$ and $w_2 w_3 = e_1 e_2$, all the possible cup products of elements of $\mathcal{B}$ are non-trivial.

Hence, $\Gamma_{\mathcal{B}} = K_3$, complete graph in three vertices.

Example

Consider the graph $\Gamma = P_3$ with vertices $\{e_1, e_2, e_3\}$ and edges (e_1, e_2) and (e_2, e_3). Also consider the invertible matrix:

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Abusing notation, let us call also e_1, e_2, e_3 the basis defined by the vertices in $H^1(A(\Gamma))$, and $\mathcal{B} = \{w_1, w_2, w_3\}$ the new basis defined via the matrix A.

As $w_1 w_2 = e_2 e_3$, $w_1 w_3 = e_1 e_2 + e_2 e_3$ and $w_2 w_3 = e_1 e_2$, all the possible cup products of elements of $\mathcal{B}$ are non-trivial.

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Hence, $\Gamma_{\mathcal{B}} = K_3$, complete graph in three vertices.

Basis cohomology graph

It is important to point out that given any basis of the first cohomology group of a RAAG it is possible to decide which cup products of the generators are non-trivial without appealing to the duals of the Artin generators.

This is in particular the information needed to construct the cohomology graph.

Moreover, given any presentation of the RAAG whose number of generators is equal to the rank, it is possible to construct out of it the cohomology graph associated to the duals of the generators.

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De Verdière invariant: preliminaries

Now the question is how to use this structure to extract graph properties from the RAAG. Here a powerful invariant enters into the picture.

The matrices

We represent a finite simple connected graph Γ by its vertex set $V = \{1, \ldots, n\}$ and its edge set E. We consider symmetric real $n \times n$ matrices M such that the following three conditions hold:

- For all distinct indices $1 \leq i, j \leq n$, we have $M_{ij} < 0$ if $\{i, j\} \in E$, and $M_{ij} = 0$ otherwise;
- M has exactly one negative eigenvalue of multiplicity one;
- There is no nonzero symmetric real $n \times n$ matrix X such that $MX = 0$ and such that $X_{ij} = 0$ whenever $i = j$ or $M_{ij} \neq 0$.

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Now we can define the Colin de Verdière invariant of a graph, which arises from spectral graph theory, and gives a vast generalization of classical planarity criteria for graphs:

The invariant

The Colin de Verdière invariant $\mu(\Gamma)$ is the largest corank of any M satisfying these conditions.

Recall that for a symmetric matrix $m \times m$ of rank r, the corank is equal to $m - r$.

The definition is extended to non-connected nonempty graphs as the maximum of the value of the invariant in the components. For empty graphs it is assumed to be zero.

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The importance of the Colin de Verdière invariant comes in part from the fact that it permits to characterize in the same way different properties of graphs. Let us recall some of these properties:

Planarity

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More graph properties

Outerplanarity

A graph is outerplanar if it can be embedded in the plane in such a way that every vertex is adjacent to the unbounded component of the complement.

Linklessly embeddability

A graph is linklessly embeddable in $\mathbb{R}^3$ if there is an embedding of the graph in $\mathbb{R}^3$ such that no pair of cycles are linked after being embedded.

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Graph properties via the invariant

The characterization of these properties (and others) via this invariant is as follows:

Theorem

Let Γ be a finite simple graph such that $\mu(\Gamma) \leq k$. Then:

- $k = 0$ if and only if Γ has no edges.
- $k = 1$ if and only if Γ is a union of disjoint paths.
- $k = 2$ if and only if Γ is outerplanar.
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Graph minors

The key property of the Colin de Verdière invariant in our framework is its good behavior with respect to minors. Let us first recall the definition.

Minors

A graph Γ is an elementary minor of another graph Λ if it can be obtained from Λ, deleting an edge, deleting a vertex or contracting an edge.
Γ is said a minor of Λ if there exists a finite sequence of graphs $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ such that $\Gamma = \Gamma_0$, $\Lambda = \Gamma_n$ and Γ_i is an elementary minor of Γ_{i+1} if $0 \leq i \leq n-1$.

Note that a subgraph is in particular a minor.

Minor monotonicity

If Γ is a minor of Γ', then $\mu(\Gamma) \leq \mu(\Gamma')$.

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The key result

The following is probably the most interesting result of this talk. Combined with the previous material will permit to identify different graph properties in terms of (cohomology of) groups.

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Let Γ be a graph, $A(\Gamma)$ the associated RAAG, $\mathcal{B}$ any basis of $H^1(A(\Gamma))$. Then $\Gamma \leq \Gamma_{\mathcal{B}}$.

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Remarks about the result

It is tempting to approach this result using inductive arguments, but they seem not useful in this context, as the inclusion is highly not canonical.

In general, assume that $\Gamma^1 < \Gamma^2$ is an inclusion of graphs of n vertices, and $\mathcal{B}_1$ a cohomology basis. Let $\mathcal{B}_2$ be another cohomology basis constructed out of $\mathcal{B}_1$ using the extra edges of Γ_2. Then a concrete inclusion $\Gamma^1 < \Gamma_{\mathcal{B}_1}$ does *not* extend in general to an inclusion $\Gamma^2 < \Gamma_{\mathcal{B}_2}$.

It is also interesting to observe that the key result admits equivalent formulations in other contexts, as Commutative Algebra and Graph Theory, whose proofs seemed unknown so far.

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The main characterization

Armed with the previous results, we can establish the following characterization.

Theorem

Let P one of the following graph properties:

- Emptiness (having no edges).
- Being a linear forest (union of disjoint paths).
- Outerplanarity.
- Planarity.
- Linkless embeddability.

Then graph Γ has property P if and only if there exists a basis $\mathcal{B}$ of $H^1(A(\Gamma))$ such that $\Gamma_{\mathcal{B}}$ has property P.

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Proof

The proof of this result is easy using the previous work. For every basis $\mathcal{B}$ of $H^1(A(\Gamma))$, $\Gamma_{\mathcal{B}}$ has property P if and only if $\mu(\Gamma_{\mathcal{B}}) \leq k$ for a certain fixed k which depends on P.

Now (tautologically), if $\mu(\Gamma) \leq k$, then $\mu(\Gamma_{\mathcal{B}}) \leq k$ for the basis $\mathcal{B}$ of the duals of the special generators of the group $A(\Gamma)$.

Conversely, if $\mu(\Gamma_{\mathcal{B}}) \leq k$ for some basis $\mathcal{B}$ of $H^1(A(\Gamma))$, the previous result and the minor monotonicity of the Colin de Verdière invariant imply that $\mu(\Gamma) \leq k$, and then Γ has property P.

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Comments to the characterization

This method also allows us to characterize planarity properties of the complement of the graph, as well as its crossing number, in terms of the cohomology basis graph.

Very recently, M. Gheorghiu has described a different characterization of the (outer)planarity, using the notion of ear decomposition.

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Using our method, it is possible to test effectively if the defining graph possesses any of the previous properties, out of a presentation of the associated RAAG.

Example

Consider the group

$$G = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_1x_2x_5x_4 = x_2x_5x_4x_1,$$

$$x_3x_2x_5x_4 = x_2x_5x_4x_3, x_4x_5 = x_5x_4, x_2x_5x_4 = x_6x_2x_4x_5x_6^{-1}\rangle.$$

This group is abstractly isomorphic to a right-angled Artin group on a graph with six vertices.

If we consider the dual generators

$$\{x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*\}$$

in $H^1(G; \mathbb{F}_2)$, the only non-trivial products of two of these elements are $x_2^*x_i^*$ for every $i \neq 2$, and $x_4^*x_5^*$.

The cohomology basis graph is a star with one additional edge, which is planar. Thus, the defining graph is also planar.

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Work in progress: self-complementarity

Definition

Let Γ be a finite simple graph of n vertices. Then Γ is said to be self-complementary if it is isomorphic to its complement in the complete graph K_n.

Characterization of self-complementary

Proposition

Let Γ be a finite simple graph on n vertices with $\binom{n}{2}/2$ edges. Then Γ is self-complementary if and only if there exists a basis $\mathcal{B}$ of $H^1(A(\Gamma);\mathbb{F}_2)$ such that $\Gamma_{\mathcal{B}}$ is self-complementary.

We currently intend to get another characterization in terms of subspaces of the cohomology algebra of the concrete graph.

This is joint work with D. Kahrobaei, T. Koberda and K. Li.

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Work in progress: existence of surjections

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Let Γ and Λ finite simple graphs. Then there exists a surjective homomorphism $A(\Gamma) \twoheadrightarrow A(\Lambda)$ if and only if there is a sequence

$$\Gamma = \Gamma_0, \Gamma_1, \ldots, \Gamma_n = \Lambda$$

of finite simple graphs such that for all i, the graph Γ_{i+1} is obtained from the graph Γ_i by either adding an edge, or by deleting a vertex.

However, we can prove that a surjection $A(\Gamma) \twoheadrightarrow A(\Lambda)$ cannot be obtained in general, up to RAAG-isomorphism, by adding edges and deleting vertices.

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Sketch proof of the key result

Given a graph Γ of n vertices, two rows of a $(n \times n)-$matrix A are said Γ-null connected if every (2×2)-minor contained in A and defined by columns i and j of A is singular whenever the edge (i,j) belongs to Γ.

For example, if we consider the graph Γ of four vertices with edges (v_1, v_2) and (v_1, v_3) and the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

the rows 1 and 2 are Γ-null-connected.

Observe that if A is a change of basis matrix in $H^1(A(\Gamma))$ from $\{e_i\}$ to $\{w_i\}$, the fact that the rows i and j are Γ-null-connected imply that that $w_i w_j = 0$.

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Sketch proof of the key result

The proof of the key result ($\Gamma \leq \Gamma_{\mathcal{B}}$ for every $\mathcal{B}$) is based on the concepts of 1-block and 1-track.

Given a graph Γ of n vertices and an $(n \times n)$-matrix A, a Γ-1-block is a $(k \times k)$-submatrix of A with $k \geq 2$ and nonzero entries, whose structure depends on the graph Γ.

It can be defined a Γ-1-track as a sequence of square submatrices $\{A_1, \ldots, A_r\}$ of A that meet every column of A at least once, such that every A_i is (1×1) or a 1-block, and for $i \neq j$ A_i, A_j never belong to a common minor in a 1-block.

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Sketch proof of the key result

The 1-tracks enjoy two crucial properties:

- Every string $(a^1_{\sigma(1)}, \ldots, a^n_{\sigma(n)})$ belongs to just *one* 1-track.
- If at least one of the matrices A_i is $n \times n$ for $n \geq 2$, the number of strings included in the 1-track is even.

Consider the graph Γ given by the edge (e_2, e_3). The colored entries represent a 1-track respect to this graph:

$$A = \begin{pmatrix} {\color{red}1} & 0 & 0 & 0 \\ 0 & {\color{green}1} & {\color{blue}1} & 1 \\ 1 & {\color{blue}1} & {\color{green}1} & 0 \\ 0 & 0 & 0 & {\color{red}1} \end{pmatrix},$$

The string red-green and the string red-blue are "paired", and according to the Leibniz rule have no influence in the determinant of A.

Sketch proof of the key result

Proposition

Suppose A is invertible. Then there exists a reordering of the rows of A such that for all edges $\{i, j\}$ of Γ, we have that the rows a_i and a_j are not Γ–null-connected.

The proof is by contradiction. If such a reordering does not exist, then it can be seen that every string is contained in a 1-track with at least a 2×2 matrix, and now the previous properties and the Leibniz rule imply that the determinant of A must be zero.

From this proposition it is clear that after perhaps a reordering of rows, it is always possible to find a copy of the defining graph inside the cohomology basis graph, for every basis of $H^1(A(\Gamma))$.

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THANK YOU!!!