Faces of the Thurston norm ball up to isotopy

GaTO
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Michael Landry
Washington University in St. Louis
mlandry@wustl.edu
• [Lan20] *Veering triangulations and the Thurston norm: homology to isotopy.*
  arxiv:2006.16328 or math.wustl.edu/~landry

• [LMT20] *A polynomial invariant for veering triangulations* (joint w/ Yair Minsky and Samuel Taylor).
  Available soon.
$M$ is a: connected, oriented, closed, irreducible, atoroidal 3-manifold
Some motivating questions (background to come)

1. Can we organize all essential surfaces in $M$?

2. What does a face of the Thurston norm ball mean?
   
   2a. Given an object associated to fibered faces (flow, veering triangulation, Teichmüller polynomial...), is there a generalization for non-fibered faces?

3. Given a face $F$ of the Thurston norm ball, can we organize all the essential surfaces in $M$ whose homology classes lie over $F$?
First goal today: explain statement of, and give context for, the main result from [Lan20].

Main Theorem. Let $\tau$ be a veering triangulation of a compact 3-manifold $\hat{M}$. If $M$ is obtained by Dehn filling each component of $\partial \hat{M}$ along slopes with $\geq 3$ prongs then $M$ is irreducible and atoroidal. Let $\sigma_\tau$ be the face of the Thurston norm ball $B_x(M)$ determined by the Euler class $e_\tau$. Then the following hold:

(i) $\text{cone}(\sigma_\tau) = \mathcal{C}_x^\vee$, and the codimension of $\sigma_\tau$ in $\partial B_x(M)$ is equal to the dimension of the largest linear subspace contained in $\mathcal{C}_x$.

(ii) If $S \subset M$ is a surface, then $S$ is taut and $[S] \in \text{cone}(\sigma_\tau)$ if and only if $S$ is carried by $\tau^{(2)}$ up to isotopy.

Second goal: tell you some of what is in [LMT20]
Thurston norm on $H_2(M; \mathbb{R})$

- Let $\alpha \in H_2(M; \mathbb{R})$ be a $\mathbb{Z}$-lattice point

- define $x(\alpha) = \min \left\{ -\chi(S) \mid S \hookrightarrow M \text{ sphereless}, [S] = \alpha \right\}$

- Thurston: $x$ extends to a norm on $H_2$

- unit ball $B_x$ is a finite sided polyhedron w/ rational vertices

Thurston norm ctd.

- If $M$ fibers over the circle with fiber $S$, then $[S] \in \text{int}(\text{cone}(F))$ for a top dimensional face $F$ of $B_x$
- further, any lattice point in $\text{int}(\text{cone}(F))$ is represented by a fiber of some fibration $M \to S^1$
- such an $F$ is called a fibered face of $B_x$

[William Thurston, A norm for the homology of 3-manifolds, Memoirs AMS, 1986]
Fried: Let $F$ be a fibered face. There is a pseudo-Anosov flow $\varphi$ on $M$ such that every lattice point in $\text{int} \left( \text{cone}(F) \right)$ is represented by a cross section to $\varphi$. (cross section: transverse, intersects every orbit)

The flow $\varphi$ can be constructed as the suspension flow of any of the fibers corresponding to $F$. Unique up to isotopy, reparametrization.

Let $e_\varphi \in H^2(M)$ be the Euler class of $\varphi$. Then $x = -e_\varphi$ on cone($F$).

[David Fried, "Fibrations over $S^1$ with pseudo-Anosov monodromy," Thurston's work on Surfaces (FLP) Ch. 14]
Fried's picture:

- let $\mathcal{C}_\varphi = \text{cone in } H_1(M)$ generated by periodic orbits of $\varphi$
- let $\mathcal{C}_\varphi^\lor \subset H_2(M)$ be classes pairing nonnegatively with $\mathcal{C}_\varphi$
- then $\mathcal{C}_\varphi^\lor = \text{cone}(F)$.

[David Fried, "Fibrations over $S^1$ with pseudo-Anosov monodromy," Thurston's work on Surfaces (FLP) Ch. 14]
Another perspective (McMullen):

- any point $\alpha \in \text{int}(\text{cone}(F))$ assigns a positive "length" to each periodic orbit.

- set $h(\alpha) =$ exponential growth rate of periodic orbits w.r.t this length. Then $h$ is an analytic function on $\text{int}(\text{cone}(F))$, tends to infinity at boundary of cone.

- Set $G = H_1(M; \mathbb{Z})/\text{torsion}$. There exists an element $\Theta_F \in \mathbb{Z}[G]$ called the Teichmüller polynomial that packages all these growth rates.
Q: what about classes in \( \partial \text{cone}(F) \)?

- each one pairs trivially with some closed orbit of \( \varphi \)
- **Mosher:** any lattice point \( \alpha \in \partial \text{cone}(F) \) is represented by a surface which is *almost transverse* to \( \varphi \).

(almost transverse: there is a "dynamic blowup" of \( \varphi \) to which the surface is transverse)

**Definition**: an oriented surface $S \hookrightarrow M$ is **taut** if:
- no components are nullhomologous, and
- $\chi([S]) = -\chi(S)$

**Example 1**: fiber of a fibration $M \to S^1$

**Example 2**: more generally, compact leaf of a taut foliation (Thurston)

**Example 3**: surface almost transverse to a pseudo-Anosov flow (Mosher)

**Fact**: If $S \hookrightarrow M$ is a fiber, then $S$ is the unique taut representative of its homology class up to isotopy (Thurston).

**However**: taut surfaces are not necessarily unique up to isotopy in their homology classes.
Combining Fried, Mosher, Thurston:

Given a fibered face $F$ of $B_x$, the flow $\varphi$ sees every isotopy class of taut surface lying over $\text{int}(F)$, and sees one taut representative of every class lying over $\partial F$.

Questions

- What about the missing isotopy classes of taut surfaces over $\partial F$?
- What about other faces? (non-fibered and/or lower dimensional)
Theorem (Mosher). Let $\varphi$ be a pseudo-Anosov flow on $M$ with no dynamically parallel closed orbits. Then $\varphi$ dynamically represents a face $F$ of $B_x$, and every integral class in $\text{cone}(F)$ is represented by a surface almost transverse to $\varphi$.

we will gloss over "no dynamically parallel closed orbits"

"dynamically represents" means $\text{cone}(F)$ is equal to both:
1. set on which $x = -e_\varphi$
2. set of all classes pairing nonnegatively with closed orbits of $\varphi$

[Lee Mosher, Dynamical systems and the homology norm of a 3-manifold II, Invent. Math. 1992]
back to our main theorem:

**Main Theorem.** Let $\tau$ be a veering triangulation of a compact 3-manifold $\hat{M}$. If $M$ is obtained by Dehn filling each component of $\partial \hat{M}$ along slopes with $\geq 3$ prongs then $M$ is irreducible and atoroidal. Let $\sigma_\tau$ be the face of the Thurston norm ball $B_x(M)$ determined by the Euler class $e_\tau$. Then the following hold:

(i) $\text{cone}(\sigma_\tau) = C_\tau$, and the codimension of $\sigma_\tau$ in $\partial B_x(M)$ is equal to the dimension of the largest linear subspace contained in $C_\tau$.

(ii) If $S \subset M$ is a surface, then $S$ is taut and $[S] \in \text{cone}(\sigma_\tau)$ if and only if $S$ is carried by $\tau^{(2)}$ up to isotopy.

We will elide the $\geq 3$ prongs condition (it's a mild restriction) and briefly explain the other terms.
A **veering triangulation** $\tau$ is a cellular decomposition of a torally bounded compact 3-manifold $\hat{M}$ which satisfies a combinatorial condition called **veering**. (Defined by Agol).

The 2-skeleton $\tau^{(2)}$ is a cooriented branched surface. Its branch locus looks like this:

Let $M$ be a closed Dehn filling of $\hat{M}$. Then $\tau^{(2)}$ is not quite a branched surface in $M$ (it stops at $\partial \hat{M}$).

There is an Euler class
\[ e_\tau \in H_1(M) \]
naturally associated to \( \tau \).
- \( e_\tau \) is a weighted sum of the cores of the filling tori
- the weights depend on the filling slopes.

Let \( W \subset H_2(M) \) be the set on which \( x = \langle -e_\tau, \cdot \rangle \). If this is nonempty then it equals \( \text{cone}(F_\tau) \) for some face \( F_\tau \) of \( B_x \).

Let \( \mathcal{C}_\tau \subset H_1(M) \) be the cone generated by closed positive transversals to \( \tau^{(2)} \).
Let \( \mathcal{C}_\tau^\vee \subset H_2(M) \) be the cone of classes intersecting everything in \( \mathcal{C}_\tau \) nonnegatively.
Let $\mathcal{C}_\tau \subset H_1(M)$ be the cone generated by closed positive transversals to $\tau^{(2)}$.

Let $\mathcal{C}_\tau^\vee \subset H_2(M)$ be the cone of classes intersecting everything in $\mathcal{C}_\tau$ nonnegatively.

**Theorem (L):** $\mathcal{C}_\tau^\vee = \text{cone}(F_\tau)$

In words: the veering triangulation linear-algebraically determines the cone over a face of the norm ball, and computes $x$ over that face.
Recall $\tau^{(2)}$ is not quite a branched surface in $M$. Say $S \hookrightarrow M$ is **carried** by $\tau^{(2)}$ if

- $S \cap \hat{M}$ is carried by $\tau^{(2)}$ in the normal sense
- Each component of $S - \hat{M}$ is $\pi_1$-injective annulus or meridional disk in a filling torus
Theorem (L): Let $S$ be an embedded surface in $M$. Then: $S$ is carried by $\tau^{(2)}$ up to isotopy iff $S$ is taut and $[S] \in \text{cone}(F_\tau)$.

i.e. $\tau^{(2)}$ sees $\text{cone}(F_\tau)$ at the level of isotopy, not just homology.

technical point: $F_\tau$ could be "the empty face." This happens when $\tau^{(2)}$ doesn't carry anything. Equivalently: $\mathcal{C}_\tau = H_1(M)$. 
For which $M$ are hypotheses of the theorem satisfied?

- when $M$ fibers with pseudo-Anosov monodromy (Agol)
- when $M$ admits a pseudo-Anosov flow with no perfect fits (Agol-Guéritaud)
- when $M$ admits a taut $\mathbb{R}$-covered foliation, or more generally a taut foliation with one-sided branching (Calegari, Fenley)

Unresolved: given a taut surface in $M$, is it carried by the 2-skeleton of a veering triangulation?
Another perspective: given $\tau$, suppose you are interested in $\mathring{M}$ (the unfilled manifold). In this case there is again a natural Euler class $e_\tau \in H_1(M)$, giving a face $F_\tau$ of $B_\chi(M)$.

Might want to know: is $\tau$ layered? non-measured? dimension of $F_\tau$? is $F_\tau$ fibered face?

**Theorem (LMT):** Let $\tau$ be a veering triangulation of $\mathring{M}$. Then:

- $\mathcal{C}_\tau = \text{cone}(F_\tau)$.
- The codimension of cone($F_\tau$) is the dimension of the largest linear subspace contained in $\mathcal{C}_\tau$.
- Moreover, the following are equivalent:
  1. the union of all closed transversals to $\tau^{(2)}$ lies in an open half-space in $H_1(\mathring{M})$
  2. $\tau$ is layered
  3. $F_\tau$ is fibered
Veering polynomial

Let $G = H_1(M; \mathbb{Z})/\text{torsion}$

**Theorem (LMT).** Given $\tau$, there is an element $V_\tau \in \mathbb{Z}[G]$ called the *veering polynomial* that recovers the Teichmüller polynomial $\Theta_{F_\tau}$ when $\tau$ is layered.

More explicitly, $V_\tau$ factors as

$$V_\tau = V^{AB} \cdot \Theta_\tau,$$

where $\Theta_\tau$ equals the Teichmüller polynomial up to a unit and $V^{AB}$ has a simple formula.
Remark 1. The veering polynomial behaves well under Dehn filling and recovers the Teichmüller polynomial in general. Combined with the earlier results about filled manifolds, it is a generalization of the Teichmüller polynomial to "veering faces" of the Thurston norm ball.

Remark 2. It can be constructed 2 ways:
A. as the determinant of a presentation matrix for a \( \mathbb{Z}[G] \)-module, or
B. as the Perron polynomial of a directed graph.

Remark 3. Anna Parlak wrote a computer program that computes these things and is writing a couple of papers about it.

[Anna Parlak, Computation of the taut, the veering and the Teichmüller polynomials, in preparation]
[Anna Parlak, The taut polynomial and the Alexander polynomial]
Thank you