4. IMMERSIONS AND SUBMERSIONS

We begin with a formulation of the Inverse Function Theorem which works for manifolds. This amounts mostly to reinterpreting what we already knew in terms of our new definitions.

Let $M \subseteq \mathbb{R}^p$ and $N \subseteq \mathbb{R}^q$ be manifolds. Suppose $f : M \rightarrow N$ is smooth, and $x \in M$. Write $y = f(x)$. Let $\phi_1 : U_1 \rightarrow V_1$ and $\phi_2 : U_2 \rightarrow V_2$ be charts in $M$ and $N$ respectively, with $x \in U_1$ and $y \in U_2$. After replacing $U_1$ with $U_1 \cap f^{-1}U_2$ and restricting $\phi_1$ to the new domain, we can assume that $fU_1 \subseteq U_2$. We now have a map $\psi = \phi_2 \circ f \circ \phi_1^{-1} : V_1 \rightarrow V_2$.

Let $a = \phi_1(x)$. By the chain rule, we get $da\psi = P \circ df \circ Q^{-1}$, where $P = dx\phi_1$ and $Q = dy\phi_2$ are both linear isomorphisms. In other words, $da\psi$ and $df$ are conjugate, so in particular have the same rank.

Suppose, for the moment, that $\dim M = \dim N = n$: that is, it is invertible. The same is true of $da\psi$, and so by usual Inverse Function Theorem (as given in Section 1), there are open sets $V'_1 \subseteq V_1$ and $V'_2 \subseteq V_2$ with $a \in V'_1$, such that $\psi|V'_1$ is a diffeomorphism to $V'_2$. Now let $U'_1 = \phi_1^{-1}V'_1$ and $U'_2 = \phi_2^{-1}V'_2$. We see that $\phi_1|U'_1 : U'_1 \rightarrow U'_2$ is a diffeomorphism.

Replacing $U_1$ and $U_2$ by these smaller open sets, we have now shown:

**Theorem 4.1.** Suppose $M, N$ are $n$-manifolds, and that $f : M \rightarrow N$ is smooth. Suppose $x \in M$ and that $df$ is invertible. Then there are open sets $U_1 \subseteq M$ and $U_2 \subseteq N$ with $x \in U_1$ such that $f|U_1$ is a diffeomorphism to $U_2$.

Note that we can assume that $U_1$ and $U_2$ are the domains of charts $\phi_1 : U_1 \rightarrow V_1$ and $\phi_2 : U_2 \rightarrow V_2$. We have seen that $\psi : V_1 \rightarrow V_2$ is a diffeomorphism, and it follows that $\psi \circ \phi_1 : U_1 \rightarrow V_2$ is also a chart. Replacing $\phi_1$ by this new chart, we get the following addendum to Theorem 4.1.
Theorem 4.2. With the same hypotheses as Theorem 4.1, we can find charts \( \phi_1 : U_1 \rightarrow V \) and \( \phi_2 : U_2 \rightarrow V \) around \( x \) and \( f(x) \) respectively, such that \( \phi_1 = \phi_2 \circ f \).

In other words, with respect to suitable local coordinates, \( f \) corresponds to the identity on \( \mathbb{R}^n \).

Example. The map \([t \mapsto (\cos t, \sin t)] : \mathbb{R} \rightarrow S^1\), or indeed any covering space (for people who know about covering spaces).

We want to generalise the above to the case where \( M \) and \( N \) have different dimensions.

First we need some linear algebra.

Let \( V, W \) be vector spaces, and set \( m = \text{dim} V, n = \text{dim} W \).

Definition. We say that a linear map, \( L : V \rightarrow W \), has maximal rank if \( \text{rank} L = \min\{m, n\} \).

There are two cases. If \( m \leq n \), then \( L \) is injective, and if \( m \geq n \), then \( L \) is surjective. (Of course, these cases overlap if \( n = m \) and \( L \) is invertible.)

The simplest examples are respectively the standard immersion, \( \iota \), of \( \mathbb{R}^m \) in \( \mathbb{R}^n \), given by

\[
\iota(x_1, \ldots, x_m) = (x_1, \ldots, x_n, 0, \ldots, 0),
\]

and the standard submersion, \( \sigma \), of \( \mathbb{R}^m \) to \( \mathbb{R}^n \) given by

\[
\sigma(x_1, \ldots, x_m) = (x_1, \ldots, x_n).
\]

(We will explain the terms “immersion” and “submersion” later.) While these two cases are qualitatively quite different, we will deal with them in parallel, since many of the results have similar formulations.

The following is a simple exercise in linear algebra:

Lemma 4.3. Let \( L : V \rightarrow W \) be a linear map of maximal rank. Then there are invertible linear maps \( P : V \rightarrow \mathbb{R}^m \) and \( Q : W \rightarrow \mathbb{R}^n \) such that \( Q \circ L \circ P^{-1} \) is a standard immersion or submersion.

One can prove this using bases. Alternatively, it can be reinterpreted as a statement about matrices, which can be proven using either column or row operations. From this point of view, the two cases can be viewed as the same result — swapping rows and columns.

We want a variation on this for smooth maps. We state it first for \( \mathbb{R}^m \) and \( \mathbb{R}^n \), which can be viewed as an elaboration on the Inverse Function Theorem. We will use \( 0^p \) to denote the origin in \( \mathbb{R}^p \).
Proposition 4.4. Suppose that \( U \subseteq \mathbb{R}^m \) is open and that \( f : U \rightarrow \mathbb{R}^n \) is a smooth map. Suppose that \( c \in U \) and that \( d_c f \) has maximal rank (that is, \( \min\{m, n\} \)). Then there are open sets \( U_1, V_1 \subseteq \mathbb{R}^n \) and \( U_2, V_2 \subseteq \mathbb{R}^2 \), with \( c \in U_1 \subseteq U \), and \( f(U_1) \subseteq U_2 \), together with diffeomorphisms, \( \theta_1 : U_1 \rightarrow V_1 \) and \( \theta_2 : U_2 \rightarrow V_2 \), such that \( \theta_1 c = 0^n \), and \( \theta_2 \circ f \circ \theta_1^{-1} : V_1 \rightarrow V_2 \) is the restriction of the standard immersion or submersion of \( \mathbb{R}^m \) to \( \mathbb{R}^n \).

Of course, this is really two results depending on whether \( m \leq n \) or \( m \geq n \). They intersect in the case where \( m = n \), where the Proposition is really just a formulation the Inverse Function Theorem. We will deal with the two cases separately. (In fact, we will see that \( \theta_1 \) or \( \theta_2 \) can be taken to be a linear map in the respective cases.)

To simplify the argument, we note that there is no loss in assuming that \( d_c f \) is just the standard immersion or submersion. This follows since by Lemma 4.3, we can find linear isomorphisms, \( P, Q \) of \( \mathbb{R}^n \) and \( \mathbb{R}^n \) respectively, so that \( P \circ d_c f \circ Q^{-1} \) is standard. Using the chain rule, we can then just replace \( f \) by \( P \circ f \circ Q^{-1} \). After postcomposing by a translation of \( \mathbb{R}^m \), we can also assume that \( c = 0^m \) and \( f(c) = 0^n \).

Proof. We split into the two cases:

1. \( m \leq n \). Identify \( \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m} \). Define a map \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) by \( F(x) = f(x_1, \ldots, x_m) + (0, \ldots, 0, x_{m+1}, \ldots, x_n) \). Note that \( d_{0^n} F \) is now the identity on \( \mathbb{R}^m \). Therefore, by the Inverse Function Theorem, there are open sets, \( U_2, V_2 \subseteq \mathbb{R}^m \) both containing the origin, and a diffeomorphism \( \theta_2 : U_2 \rightarrow V_2 \), which is the inverse of \( F|U_1 \). Let \( U_1 = U \cap V_2 \). Now if \( y \in U_1 \), we have \( \theta_2(f(y)) = \theta_2(f(y) + 0^n) = \theta_2(F(y, 0^{n-m})) = (y, 0^{n-m}) \). The result now follows by setting \( \theta_1 \) to be the identity restricted to \( U_1 \).

2. \( m \geq n \). Identify \( \mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^{m-n} \). Define a map \( G : U \rightarrow \mathbb{R}^m \) by \( G(x) = (f(x), \pi(x)) \), where \( \pi(x_1, \ldots, x_m) = (x_{m+1}, \ldots, x_n) \). Note that \( d_0 G \) is the identity on \( \mathbb{R}^m \). By the Inverse Function Theorem, we have open sets \( U_1, V_1 \subseteq \mathbb{R}^m \), both containing 0, such that \( G|U_1 : U_1 \rightarrow U_2 \) is a diffeomorphism. We now set \( \theta_1 = G|U_1, U_2 = fU_1 \), and \( \theta_2 \) to be the identity on \( U_2 \). \( \square \)

We can translate this back into a statement about manifolds:

Theorem 4.5. Suppose that \( M, N \) are manifolds, and that \( f : M \rightarrow N \) is smooth. Suppose that \( c \in M \), and that \( d_c f \) has maximal rank. Then there are charts, \( \phi_1 : U_1 \rightarrow V_1 \) and \( \phi_2 : U_2 \rightarrow V_2 \) around \( c \) and \( f(c) \) respectively, with \( \phi_1 x = 0 \), such that \( \phi_2 \circ f \circ \phi_1^{-1} : V_1 \rightarrow V_2 \) is the restriction of a standard immersion or submersion.
This uses Proposition 4.4 in the same way that Theorem 4.2 used the Inverse Function Theorem. The argument is essentially the same, so we leave it as an exercise.

**Definition.** We say that a map \( f : M \to N \) is an immersion if for all \( x \in M \), \( d_x f \) is injective. We say that it is a submersion if for all \( x \in M \), \( d_x f \) is surjective.

Clearly, these imply, respectively, that \( \dim M \leq \dim N \), and \( \dim M \geq \dim N \). Note that the composition of immersions is an immersion and a composition of submersions is a submersion.

Note that an immediate consequence of Theorem 4.5 is that an immersion is locally injective: that is, for all \( x \in M \), there is an open set, \( U \), containing \( x \) such that \( f|U \) is injective. Similarly, a submersion is open, that is \( f|U \) is open for any open \( U \subseteq M \).

**Examples.** Lots of familiar curves in the plane, such as the lemniscate (“figure of eight”), are examples of immersions. Provided they don’t have cusps — such as the cuspidal cubic.

The familiar picture of the Klein bottle drawn in 3-space is an example of a 2-manifold immersed in \( \mathbb{R}^3 \). The “Boy surface” in an immersion of the projective plane into \( \mathbb{R}^3 \).

**Definition.** A smooth map, \( f : M \to N \) between manifolds is an embedding if it is a diffeomorphism onto its range. We refer to the range, \( f(M) \), of such a map as a submanifold of \( N \).

Clearly any embedding is an injective immersion, thought the converse need not be true. A counterexample is the injective map of \([0, 1)\) to the plane whose image is a “figure of six”.

Note that if \( M \subseteq \mathbb{R}^p \) is a manifold in \( \mathbb{R}^p \) (according to our original definition of such), then \( M \) is a submanifold of \( \mathbb{R}^p \), according to the definition we have just given.

Recall that a map (between topological spaces) is proper if the preimage of any compact set is compact.

**Lemma 4.6.** Any proper injective immersion is an embedding.

**Proof.** This is an immediate consequence of the fact (mentioned in Section 1, that a continuous injective map between locally compact spaces is a homeomorphism onto its range. \( \square \)

We refer to the image of a proper embedding as a proper submanifold.

Note this is the same as being closed in \( \mathbb{R}^p \) (Exercise).

**Exercise:** Suppose that \( N, N' \) are manifolds, and that \( M \subseteq M' \) and \( M' \subseteq N' \) are submanifolds. Suppose that \( f : N \to N' \) is smooth, with
\( f(M) \subseteq M' \). Then \( f|_M : M \rightarrow M' \) is smooth (with respect to the intrinsic smooth structures). If \( f \) is a diffeomorphism, and \( f(M) = M' \), then \( f|_M \) is a diffeomorphism from \( M \) to \( M' \).

**Definition.** Let \( f : M \rightarrow N \) be a smooth map between manifolds. We say that \( x \in M \) is a regular point if \( d_x f \) is surjective, and a critical point otherwise. A point \( y \in N \) is a regular value if each point of \( f^{-1}(y) \) is a regular point, otherwise it is a critical value.

In other words, a critical value is the image of a critical point. Note that any point of \( N \setminus f(M) \) has empty preimage, and is therefore a regular value.

**Theorem 4.7.** If \( f : M \rightarrow N \) is smooth, and \( y \in f(M) \subseteq N \) is a regular value, then \( f^{-1}(y) \) is a proper submanifold of \( N \) of dimension \( m - n \).

Since \( f \) is continuous, \( f^{-1}(y) \), is closed, so it is a proper submanifold. The fact that we insist that manifolds are non-empty, explains why we disallow \( y \in M \setminus f(N) \), in this statement. Note that we can assume that \( \dim M \leq \dim N \) (otherwise, there is no regular point, and the statement becomes vacuous).

**Proof.** Let \( x \in f^{-1}(y) \). Choose charts as given by Theorem 4.5. Then \( f^{-1}(y) \cap U_1 \) is the set of points in \( U_1 \) with first \( m \) coordinates all equal to 0. Let \( \pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m-n} \) be the projection to the final \( m - n \) coordinates. Let \( U = f^{-1}(y) \cap U_1 \), and set \( \phi = \pi \circ \phi_1 : U \rightarrow \mathbb{R}^{m-n} \). The \( x \in U \), and \( \phi : U \rightarrow \mathbb{R}^{m-n} \) is a chart for \( M \). \( \square \)

We also note:

**Lemma 4.8.** With the hypotheses of Theorem 4.7, we have \( T_x(f^{-1}(y)) = \ker d_x f \).

**Proof.** By Lemma 3.2 any tangent vector in \( T_x(f^{-1}(y)) \) has the form \( \gamma : I \rightarrow f^{-1}(y) \), where \( \gamma(0) = x \). Now, \( f \circ \gamma \) is constant, so \( d_x f(\gamma'(0)) = (f \circ \gamma)'(0) = 0 \), so \( \gamma'(0) \in \ker d_x f \). This shows that \( T_x(f^{-1}(y)) \subseteq \ker d_x f \). But \( \dim T_x(f^{-1}(y)) = m - n = \dim(\ker d_x f) \), so \( T_x(f^{-1}(y)) = \ker d_x f \). \( \square \)

**Examples**

1. Define \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \) by \( f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 \). The only critical point is \( (0, 0, 0) \), so the only critical value is 0. Also, \( f^{-1}(t) \) is non-empty precisely if \( t \geq 0 \). Thus, \( f^{-1}(t) \) is a manifold if \( t > 0 \) (the
sphere of radius \(\sqrt{t}\)).

(2) Similarly, Define \(f : \mathbb{R}^3 \rightarrow \mathbb{R}\) by \(f(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2\). Again, the only critical point is \((0, 0, 0)\), so the only critical value is 0. In this case, \(f^{-1}(t)\) is always non-empty. Thus, \(f^{-1}(t)\) is a manifold if \(t \neq 0\): it is a hyperboloid. Note that \(f^{-1}(0)\) is a cone — not a manifold.

(3) Let \(T_{a,b} \subseteq \mathbb{R}^3\) be the torus, described in Example (6). Define \(f : T_{a,b} \rightarrow \mathbb{R}\) by \(f(x_1, x_2, x_3) = x_1\). In this case, \(f^{-1}(t)\) is a manifold for \(t \in (-a - b, -a + b) \cup (-a + b, a - b) \cup (a - b, a + b)\).

(4) We noted in Section 3 that \(GL(n, \mathbb{R})\) is an \(n^2\)-manifold, with the group operations smooth. From this, we can derive other similar examples. For example, consider the map \(\Delta : GL(n, \mathbb{R}) \rightarrow \mathbb{R}\), given by \(\Delta(A) = \det A\). Considered as a map from \(\mathbb{R}^{n^2}\) to \(\mathbb{R}\), it is smooth (in fact, polynomial). We claim that if \(t \neq 0\), then \(t\) is a regular value. For suppose \(\Delta(A) = t \neq 0\). Consider the path \(\gamma : \mathbb{R} \rightarrow GL(n, \mathbb{R})\) given by \(\gamma(u) = (1 + u)A\). Now \(\Delta \circ \gamma(u) = (1 + u)^n t\), so we get \(d_{\gamma(0)}(\gamma'(0)) = (\Delta \circ \gamma)'(0) = nt \neq 0\), and so \(dA\Delta\) has rank 1. It follows that \(\Delta^{-1}(t)\) is a manifold. Let \(SL(n, \mathbb{R}) = \Delta^{-1}(1) = \{A \in GL(n, \mathbb{R}) \mid \det A = 1\}\). This is a group. Moreover (using an earlier exercise) the group operations are smooth (since they are restrictions of the smooth group operations on \(GL(n, \mathbb{R})\)).

A similar argument shows that \(O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid A^T A = I\}\) is a manifold, with smooth group operations.

(5) Some “real world” examples arise from mechanical linkages. These have been used by engineers since ancient times. One can think of a linkage as a collection of rods connected at pivots, where they are allowed to flex. In mathematical terms, one can think of the pivots as a set of \(n\) points in the plane (for a planar linkage), and so described by a point in \(\mathbb{R}^{2n}\). The rods determine a number of constraints, such as the fact that certain pivots lie on a straight line, or that the distances between them is fixed. There are given by smooth equations, that is a map from \(\mathbb{R}^{2n}\) to some \(\mathbb{R}^d\). The configuration of the linkage is then constrained to lie in the preimage of a point. If this is a regular point (as one would expect generically, or as one might hope, if the linkage is to function smoothly), then this is a manifold. Examples?

(6) Analogous situations arise in many contexts in physics, where one has a number of invariants in a system. For example, we may have a
set of particles, or planets or whatever, whizzing around in space. The state of the system at a given moment may describe a finite number of coordinates of position and velocity, say. The energy of the system is constant, so they are constrained to live in some subset of the coordinate space — typically a manifold.

5. TANGENTS, NORMALS, ORIENTATIONS

Let $M \subseteq \mathbb{R}^n$ be an $m$-manifold.

**Definition.** The tangent bundle, $TM$, to $M$ is defined by:

$$TM = \{(x, v) \in M \times \mathbb{R}^n \mid v \in T_xM\}.$$ 

We write $p : TM \longrightarrow M$ for the projection, $p(x, v) = x$. Thus, $p^{-1}x = T_xM$.

**Proposition 5.1.** $TM$ is a manifold of dimension $2m$, and $p : TM \longrightarrow M$ is a submersion.

**Proof.** Let $\phi : U \longrightarrow V$ be a chart. Define a map $\psi : p^{-1}U \longrightarrow V \times \mathbb{R}^m$ by $\psi(x, v) = (x, d_x\phi(v))$. This is smooth, and has smooth inverse defined by $[(y, w) \mapsto (\phi^{-1}(y), d_y\phi^{-1}(w))]$. We see that the collection of such maps form an atlas for $TM$.

Now $p = \phi^{-1} \circ \sigma \circ \psi$, where $\sigma : V \times \mathbb{R} \longrightarrow V$ is projection to the first coordinate. Thus, $p$ is a composition of submersions, hence a submersion. \qed

**Examples** $T S^1$ is a cylinder. The map $[(x_1, x_2, t) \mapsto (x_1, x_2, -tx_2, tx_1)]$ gives a diffeomorphism from $S^1 \times \mathbb{R} \subseteq \mathbb{R}^3$ to $T S^1 \subseteq \mathbb{R}^2 \times \mathbb{R}^2 \equiv \mathbb{R}^4$.

Suppose $f : M \longrightarrow N$ is a smooth map between manifolds. We get an induced map, $f_* : TM \longrightarrow TN$ defined by $f_*(v) = df(v) \in T_xM$ where $x = pv$.

**Exercise :** $f_* : TM \longrightarrow TN$ is smooth.

The following is a special case of a more general definition we give later, in Section 7.

**Definition.** A section of $TM$ is a smooth map, $s : M \longrightarrow TM$ such that $p \circ s$ is the identity on $M$.

Note that a section, $s$, is necessarily an immersion. In fact, a proper embedding of $M$ in $TM$. (More about this in Section ???)
In other words, we are assigning to each $x \in M$ a tangent vector, $s(x) \in T_x M$, in a smooth manner. In more familiar terms, $s$ is just a smooth “vector field”, on $M$.

**Definition.** A frame field on $M$ is a family, of $m$ smooth vector fields, $v_1, \ldots, v_m$, such that $v_1(x), \ldots, v_m(x)$ forms a basis for $T_x M$ for all $x \in T_x M$.

Note that a frame field need not always exist on all of $M$. (Will see an example in Section 7.)

However, one can always find frame field locally. In fact, suppose $\phi : U \rightarrow \mathbb{R}^n$ is a chart for $M$. Let $e_1, \ldots, e_m$ be the standard basis for $\mathbb{R}^m$ (or indeed any fixed basis). If $x \in U$, set $v_i(x) = (d_\phi)^{-1} e_i$. Then $v_1, \ldots, v_n$ is a frame field on $U$ (which we can think of an open submanifold of $M$).

**Notation :** Again, the notation $\frac{\partial}{\partial x_i}$ is often used informally for $v_i(x)$ defined locally. In this context, we can think of $\frac{\partial}{\partial x_i}$ as denoting a vector field on $U \subseteq M$. If $\lambda_i : U \rightarrow \mathbb{R}$ are smooth functions, then $[x \mapsto \sum_{i=1}^m \lambda_i(x)v_i(x)]$ is also a vector field, often informally denoted as $\sum \lambda_i \frac{\partial}{\partial x_i}$.

We say that a smooth frame field is orthonormal if $(v_i(x))_i$ is an orthonormal basis with respect to the dot product induced by the embedding of $T_x M$ in $\mathbb{R}^n$. That is, $v_i(x).v_j(x) = \delta_{ij}$ where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.

We note:

**Lemma 5.2.** If $M$ admits a (global) frame field, then it admits a global orthonormal frame field.

**Proof.** Recall the Gram-Schmidt process for producing an orthonormal basis from a given basis. Start with any frame field $v_i(x)$. Set $v'_i(x) = v_1(x)$. Then set

$$v'_2(x) = v_2(x) - \frac{v_1(x).v_2(x)}{v_1(x).v_1(x)}v_1(x).$$

Then set

$$v'_3(x) = v_3(x) - \frac{v_1(x).v_3(x)}{v_1(x).v_1(x)}v_1(x) - \frac{v_2(x).v_3(x)}{v_2(x).v_2(x)}v_2(x).$$

e tc. Finally, set $w_i(x) = v'_i(x)/||v'_i(x)||$. Then $(w_i(x))_i$ is orthonormal.
It suffices to observe that all the above operations are smooth (they are given by nice simple formulae). Therefore the resulting maps \([x \mapsto w_i(x)]\) are smooth. \(\square\)

In particular, we see that there is always a locally defined orthonormal frame field for a manifold embedded in euclidean space.

We now move on to consider normal vectors.

Recall that \((T_x M)^{\perp}\) is the orthogonal complement to \(T_x M\) in \(\mathbb{R}^n\); that is the space of “normal vectors” to \(M\) at \(x\), with respect to the dot product on \(\mathbb{R}^n\).

**Definition.** The normal bundle to \(M\) in \(\mathbb{R}^n\) is defined by:

\[ \nu(M, \mathbb{R}^n) = \{(x, v) \in M \times \mathbb{R}^n \mid v \in (T_x M)^{\perp}\}. \]

We write \(p : \nu(M, \mathbb{R}^n) \rightarrow M\) for the projection map.

**Proposition 5.3.** \(\nu(M, \mathbb{R}^n)\) is an \(n\)-manifold, and \(p : \nu(M, \mathbb{R}^n) \rightarrow M\) is a submersion.

(Here, we should think of the dimension, \(n\), as \(n = m + (n - m)\). Note that \(m\) of the coordinates come from \(M\), and the remaining \(n - m\) from the normal space.)

**Proof.** Let \(x_0 \in M\). Let \(\phi : U \rightarrow \mathbb{R}^m\) be a chart, with \(x_0 \in U\), and let \(v_1(x), \ldots, v_m(x)\), be a frame field defined on \(U\) (as discussed above). In particular, \(v_1(x_0), \ldots, v_m(x_0)\) is a basis for \(T_{x_0}(M) \subseteq \mathbb{R}^m\), and we extend this arbitrarily to a basis \(v_1(x_0), \ldots, v_m(x_0), v_{m+1}, \ldots, v_n\) of \(\mathbb{R}^n\). By continuity, we see easily that \(v_1(x), \ldots, v_m(x), v_{m+1}, \ldots, v_n\) is a basis of \(\mathbb{R}^n\) for all \(x\) in some open neighbourhood, say \(U'\), of \(x\) in \(U \subseteq M\). Now for \(x \in U'\) set \(v_i(x) = v_i\) for \(i \in \{m + 1, \ldots, n\}\). We apply the Gram-Schmidt process to this to give us an orthonormal frame, \(w_1(x), \ldots, w_n(x)\), for \(\mathbb{R}^n\), which varies smoothly in \(x\). (This is just an extension of the construction used in proving Lemma 5.2.)

Now \(w_i(x) \in T_x M\) for \(i \leq m\), and \(w_i(x) \in (T_x M)^{\perp}\) for \(i > m\). Set \(e_i(x) = w_{m+i}(x)\) for \(i > m\). Then \(e_1(x), \ldots, e_{n-m}(x)\) is an orthonormal basis for \((T_x M)^{\perp}\). (We will no longer need \(w_1(x), \ldots, w_m(x)\).)

We now define a map \(\psi : p^{-1}U' \rightarrow V \times \mathbb{R}^{n-m}\) by setting \(\psi(x, v) = (\phi, \lambda)\), where \(\lambda = (v.e_1(x), \ldots, v.e_{n-m}(x)) \in \mathbb{R}^{n-m}\). This is a smooth map, and has smooth inverse given by \((y, \lambda) \mapsto (\phi^{-1}y, \sum_{i=1}^{n-m} \lambda_i e_i(x))\), where \(\lambda = (\lambda_1, \ldots, \lambda_{n-m})\).

We finally note that \(p\) is a submersion for a similar reason as for the tangent bundle. \(\square\)
Examples. \( \nu(S^m, \mathbb{R}^{m+1}) \) is diffeomorphic to \( S^n \times \mathbb{R} \).

We can define a section of \( \nu(M, \mathbb{R}^m) \) as a smooth map, \( \kappa : M \rightarrow \nu(M, \mathbb{R}^m) \) with \( p \circ \kappa \) the identity on \( M \). Such a section is commonly called a normal field.

We now consider orientations on a manifold. We begin with some general linear algebra.

Let \( V \) be a vector space of dimension \( n > 0 \). Let \( I(V) \) be the set of linear isomorphisms from \( V \) to \( \mathbb{R}^n \). Given \( \rho, \sigma \in I(V) \), we get a linear automorphism, \( \sigma \circ \rho^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n \). Note that \( \det(\sigma \circ \rho^{-1}) \neq 0 \).

We write \( \sigma \sim \rho \) if \( \det(\sigma \circ \rho^{-1}) > 0 \). This is easily seen to define an equivalence relation on \( I(V) \), and we write \( \text{Or}(V) = I(V)/\sim \). Thus \( |\text{Or}(V)| = 2 \). In the case where \( \dim V = 0 \), we set \( \text{Or}(V) = \{-1, +1\} \).

Definition. An orientation on \( V \) is an element of \( \text{Or}(V) \). An oriented vector space is a (finite dimensional) vector space equipped with an orientation.

This can be thought of, perhaps more intuitively, in terms of bases. Let \( V \) be an oriented vector space of positive dimension. Note that a basis, \( v_1, \ldots, v_m \), of \( V \) determines an element of \( I(V) \) sending the basis to the standard basis of \( \mathbb{R}^m \). We refer to \( (v_i)_i \) as positively oriented if this map lies in the class of the orientation, and negatively oriented otherwise.

Note that \( \mathbb{R}^m \) itself comes with a natural “standard” orientation, namely the class of the identity map. In other words, the standard basis is deemed to be positively oriented.

If \( V, W \) are oriented vector spaces of the same dimension, we say that a linear isomorphism \( L : V \rightarrow W \), is orientation preserving if it sends some (hence any) positively oriented basis to a positively oriented basis. Otherwise, it is orientation reversing. (Of course, this can also be expressed directly in terms of the orientation classes of isomorphism to \( \mathbb{R}^m \).)

Note that an automorphism, \( L \), of \( \mathbb{R}^m \) with the standard orientation is orientation preserving if \( \det L > 0 \), and orientation reversing if \( \det L < 0 \).

Suppose that \( V, W \) are vector spaces and \( E = V \oplus W \). Then orientations on \( V \) and \( W \) determine an orientation on \( E \). For example, choose positively oriented bases for \( V \) and \( W \). Their union is a basis for \( E \), which we deem to be positively oriented in \( E \). Exercise: Check this is well defined, independently of the bases we choose. Conversely, an orientation on \( V \) and an orientation on \( E \) determine an orientation on \( W \).
We can now define an orientation on an $m$-manifold, $M$, in $\mathbb{R}^n$ when $m > 0$.

**Definition.** An orientation on $M$ is an assignment of an orientation to each tangent space $T_x M$ such that there is an atlas of charts, $\phi_\alpha : U_\alpha \to \mathbb{R}^m$, indexed by some set $\mathcal{A}$, such that for all $\alpha \in \mathcal{A}$ and all $x \in U_\alpha$, the map $d_x \phi_\alpha : T_x M \to \mathbb{R}^m$ orientation preserving (i.e. in the orientation class of the orientation of $T_x M$).

We refer to $M$ as an oriented manifold.

**Definition.** We say that $M$ is orientable if it admits an orientation.

Given an orientation on $M$ we have an opposite orientation obtained by reversing the orientation on each tangent space. To see that this is indeed orientation, take the atlas, and postcompose every chart with an orientation reversing linear automorphism of $\mathbb{R}^m$ (e.g. $[ (x_1, \ldots, x_m) \mapsto (-x_1, \ldots, x_m)]$), to give another smooth atlas with all orientations reversed.

Not every manifold is orientable (as we will see). However, every manifold is “locally orientable” in the sense that every point is contained in an open set, $U$, which is orientable. In fact, if $\phi : U \to V$ is chart, then we just use the maps $d_x \phi : T_x M \to \mathbb{R}^m$ to define the orientation on the tangent space $T_x U = T_x M$ at $x$.

It is easily seen that orientability is invariant under diffeomorphism.

**Examples :**

(1): Clearly $\mathbb{R}^n$ is orientable.

(2): $S^n \subseteq \mathbb{R}^{n+1}$ is orientable. If we take the atlas given by stereographic projections from the north and south poles, then the transition function reverses orientation. (Recall that it is inversion in an $(n-1)$-sphere.) To fix this, we just postcompose one of the charts with any orientation reversing diffeomorphism of $\mathbb{R}^n$, so as to give an oriented atlas.

(3): The direct product of any two orientable manifolds is orientable. (So for example, the torus is orientable. So is the cylinder $S^1 \times \mathbb{R}$.)

(4) The Möbius band is not orientable. (In particular, it is not diffeomorphic to the cylinder.)

We have noted that if $M$ is orientable, then it has at least two orientations. It might have many, since orientations on different connected
components of $M$ are independent of each other. However, if $M$ is connected, it has precisely two:

**Lemma 5.4.** A connected orientable manifold has precisely two orientations.

*Proof.* Choose any $x_0 \in M$. After reversing one of the orientations if necessary, we can assume that they both agree on $T_{x_0}M$. We want to show that they are the same everywhere.

Let $x \in M$. By Lemma 3.5, there is a continuous path, $\gamma : [0, 1] \rightarrow M$ with $\gamma(0) = x_0$ and $\gamma(1) = x$. Suppose that $t \in [0, 1]$ and that the two orientations agree on $T_{\gamma(u)}M$ for all $u < t$. We claim they must agree on $T_{\gamma(v)}M$ for all $v$ in some neighbourhood of $t$.

To see this, let $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^m$ and $\phi_\beta : U_\beta \rightarrow \mathbb{R}^m$ be charts for the respective orientations, with $\gamma(t) \in U_\alpha \cap U_\beta$. For each $y \in U_\alpha \cap U_\beta$, we have a linear automorphism $(d_y \phi_\beta \circ (d_y \phi_\alpha)^{-1} : \mathbb{R}^m \rightarrow \mathbb{R}^m$. Let $\Delta(y) = \det((d_y \phi_\beta \circ (d_y \phi_\alpha)^{-1})$. Then $\Delta(y) \neq 0$, and $\Delta : U_\alpha \cap U_\beta \rightarrow \mathbb{R}$ is smooth, so in particular continuous. Now for $u < t$, $\Delta(y) > 0$, so by the Intermediate Value Theorem, we must have $\Delta(\gamma(v)) > 0$ for all $v$ in a neighbourhood of $t$ in $[0, 1]$, and the statement follows.

It now follows easily, by considering the supremum of such $t$, that the orientations agree on $T_{\gamma(t)}M$ for all $t \in [0, 1]$, in particular, they agree on $T_xM$.

Suppose that $M \subseteq \mathbb{R}^n$ is a smooth manifold. Given $x \in M$, then $\mathbb{R}^n \cong T_x\mathbb{R}^n = T_xM \oplus (T_xM)^\perp$. Now $\mathbb{R}^n$ comes with a standard orientation. Thus, as discussed before, an orientation on $T_xM$ determines an orientation on $(T_xM)^\perp$ and conversely.

Consider the case when $n = m + 1$. Then $(T_xM)^\perp \cong \mathbb{R}$, and so an orientation on $T_xM$, hence on $(T_xM)^\perp$ gives us a unique $\kappa(x) \in (T_xM)^\perp$ with $||\kappa(x)|| = 1$ and with $\{\kappa(x)\}$ a positive basis for $(T_xM)^\perp$. We can think of $\kappa(x)$ as the “outward” unit normal vector. If $M$ is oriented, then $\kappa(x)$ is defined everywhere. Also, the map $[x \mapsto \kappa(x)]$ is smooth (Exercise: see the proof of Proposition 5.3.) Thus, $\kappa(x)$ is a normal field.

Conversely, if $\kappa$ is a (global) nowhere vanishing normal field on $M$, then $\kappa$ gives rise to an orientation on $T_xM$ for all $x$. From this, it is not hard to construct an orientable atlas.

From this one can deduce:

**Theorem 5.5.** Let $M \subseteq \mathbb{R}^{m+1}$ be an $m$-manifold. The following are equivalent:

1. $M$ is orientable.
(2) $M$ admits a nowhere-vanishing normal field.
(3) $M$ admits a unit normal field.

**Definition.** Suppose that $M$ and $N$ are oriented manifolds, and that $f : M \rightarrow N$ is a diffeomorphism. We say that $f$ preserves orientation if $d_x f : T_x M \rightarrow T_{f(x)} N$ respects the given orientations for all $x \in M$. We say that it reverses orientation if it reverses orientation for all $x \in M$.

**Exercises:**

(1) If $M, N$ are connected and oriented, then every diffeomorphism either preserves or reverses orientation.

(2) Let $f : S^n \rightarrow S^n$ be the antipodal map on the $n$-sphere $S^n \subseteq \mathbb{R}^{n+1}$. (That is, $f(x) = -x$.) Show that $f$ is orientation preserving if $n$ is odd, and orientation reversing if $n$ is even.