Anosov parameter values for the triple linkage and a physical system with a uniformly chaotic attractor

TJ Hunt
Open University, Walton Hall,
Milton Keynes MK7 6AA, UK

RS MacKay
Mathematics Institute, University of Warwick,
Coventry CV4 7AL, UK

30 Sept 2002, revised May 8, 2003

Abstract

Firstly, the free frictionless dynamics of the triple linkage of Thurston and Weeks is proved to be Anosov on each positive energy level for a non-empty open set of parameter values. Secondly, in the presence of weak friction and any smooth potential, an open set of physically implementable feedback control laws are found for which the system possesses a uniformly hyperbolic attracting submanifold.

PACS: XXX, **AMS**: XXX, **keywords**: chaos, uniform hyperbolicity, Anosov, geodesic flow, linkage

1 Introduction

There is a well worked out mathematical theory of uniformly hyperbolic sets (e.g. [KH]). Their definition and principal properties are recalled in Appendix A. The main point is that, if not just a union of hyperbolic periodic orbits, a locally maximal uniformly hyperbolic set has well characterised chaotic behaviour. As a convenient shorthand, we call any locally maximal uniformly hyperbolic set other than a union of hyperbolic periodic orbits *uniformly chaotic*.

Although the theory is usually presented in discrete time, continuous time is much more relevant to physics, so the latter is the context used in the present paper. Mathematical examples of uniformly chaotic sets in continuous time include: (i) suspensions of discrete-time examples like a horseshoe, Plykin attractor, solenoid map, DA attractor and hyperbolic toral automorphism, and (ii) the unit tangent bundle for a geodesic flow on a compact Riemannian surface with everywhere negative Gaussian curvature or higher dimensional manifold with everywhere negative sectional curvature.

The principal types of chaos of relevance to physics are those which can be observed without special choice of initial conditions. Thus physical interest focusses on (i) chaotic attractors, and (ii) systems with a foliation into compact invariant submanifolds on each of which the whole dynamics is chaotic. Physicists have been disappointed, however, that the mathematical theory of uniformly hyperbolic sets never seems to apply to any physical examples of these two types of chaotic dynamics, and that the mathematical examples can not be realised physically (see Appendix B for a discussion). Suspensions of horseshoes occur in many physical systems, of course, but are unstable. Thus two long standing questions have been whether there are (i) physical examples of a Hamiltonian system with Anosov energy levels (an *energy level* is the set of points where the Hamiltonian takes a given value), and (ii) physical examples of a system with a uniformly chaotic attractor. Both questions are given positive answers in this paper. Of course, "physical" does not have a definition, but we hope that most physicists will agree that our examples qualify.

Our examples are based on the *triple linkage* of [TW]. For its free frictionless motion we prove existence of an open set of parameter values for which each positive energy level is Anosov, thus answering question (i). To answer question (ii), we allow weak friction and construct feedback control laws such that the system has an attracting uniformly chaotic invariant submanifold near one of the energy levels.

The triple linkage consists of three disks in a plane, free to rotate about pivots fixed in a triangle, but constrained by three rods connecting one point $p_i, i \in \{1, 2, 3\}$ of each disk to a floating pivot x (see Figure 1). Of course, it is not really necessary that all lie in one plane; it is enough that the projections to a plane satisfy the constraints. The free motion at given positive energy is (up to time-scale) the geodesic flow of the kinetic energy metric. This is just the metric that assigns length $ds = \sqrt{2K(v)}\ dt$

to an infinitesimal displacement $v\ dt$, where $K(v)$ is the kinetic energy for a velocity vector v. Thurston and Weeks presented the triple linkage (with the disks replaced by rods, but this makes no difference) as an example of a system whose configuration space has interesting topology: for parameters in the regime of interest to us, it is a surface of genus three (Section 2); but they did not address the dynamics. Surfaces of genus greater than one can support Riemannian metrics of negative Gaussian curvature, and their geodesic flows are Anosov. So we asked ourselves if for some parameter values the curvature of the kinetic energy metric for the triple linkage might be everywhere negative.

We began by a numerical investigation of the curvature. To reduce our search we restricted attention to the D_6-symmetric case of disks of equal moments of inertia I pivoted on the vertices of an equilateral triangle of radius 1, connected by equal rods of length l_2 and mass m, attached to points of the disks at equal radius l_1 from their centres. The rods are not required to have uniform line density, nor to be straight, nor to end at the pivots, nor even to be one-dimensional, but the D_6 symmetry forces them to have centre of mass on the line through its two pivots. Then the kinetic energy can be written

$$K(\dot{x},\dot{p}) = \frac{1}{2}(\mu_1 \dot{x}^2 + \mu_2 \dot{x}.\sum_i \dot{p}_i + \mu_3 \sum_i \dot{p}_i^2) \tag{1}$$

(the values of the mass parameters μ_j are related to physical parameters in Section 2). After an extensive search [H], parameter values $l_1 = 7/40, l_2 = 41/40, \mu_1 = 11/50, \mu_2 = 3/100, \mu_3 = 23/100$ were found for which the curvature was computed to be negative everywhere (Section 3). To make a proof, however, would probably require computer-assisted estimates.

To obtain a rigorous result we took a cue from the fact that these parameter values have l_1 small and l_2 very close to 1, and that calculations are much easier if $\mu_1, \mu_2 = 0$, to consider a limiting metric for which we proved the curvature to be negative except at 8 points (Section 4). That is sufficient for the flow to be Anosov.

Using structural stability of Anosov flows, we deduced a parameter regime where the free motion of the triple linkage is Anosov on each energy level and other properties (Section 5):

Theorem: Write $l_2 = 1 + bl_1$. Then there exist $l_0, m_0, b_0 > 0$ such that for $l_1 < l_0, m < m_0, b < b_0$ the free motion of the triple linkage on each positive energy level is Anosov, mixing and the correlations of Hölder continuous functions decay exponentially.

These properties are preserved even if the D_6-symmetry is broken slightly, and for all high enough energy levels if a potential is added. Furthermore, they imply diffusion in any Abelian cover.

Even for parameter values where the free motion is not Anosov, a potential can be chosen to make a given energy level (and all nearby ones) Anosov, provided the configuration space has genus greater than one (Section 6).

To obtain a uniformly chaotic attractor when friction is taken into account, we add torques chosen to contract strongly onto or near a particular energy level (Section 7). By the theory of normal hyperbolicity, this gives a normally hyperbolic attracting submanifold close to the energy level. The motion on it is close to the free motion on that energy level and hence Anosov and mixing.

2 The triple linkage

In this section, we examine the topology of the configuration space of the triple linkage as a function of its length parameters, and determine the kinetic energy in terms of the length and mass parameters.

Given l_1, l_2, the region accessible to the central pivot x is the intersection of three annuli, of inner and outer radii $|l_2 - l_1|$ and $l_1 + l_2$, centred on the pivots for the three disks. The accessible region may have more than one component. When the position x is given in this region, there are generically two possibilities for the position of each p_i. We label the possibilities by $\sigma_i \in \{\pm\}$ using $+$ for the choice that is clockwise relative to x and $-$ for the anticlockwise choice. The possibilities for each p_i merge into one, denoted $\pm$, on certain pieces of the boundary of the accessible region, and the 8 patches corresponding to the choices of σ glue together along these edges to form a union of compact orientable surfaces. The genus g of each can be computed using Euler's formula $F - E + V = 2 - 2g$ for the numbers of faces, edges and vertices in this decomposition into patches.

We computed all the possibilities for the configuration space of the triple linkage and the results are shown in Figure 2. The topology of the configuration spaces of many other linkages was studied by [Ni].

A necessary condition for a geodesic flow on a surface to be Anosov is that it have genus greater than 1 [Kl], so only those cases where the configuration space has a component of genus greater than 1 are worth considering. A sufficient condition is that the curvature of the metric be negative everywhere, though one can relax it to be zero at a finite number of points (cf. [Kn]).

Morse proved that for a surface of genus greater than 1, the geodesic flow of any metric has an invariant subset with semiconjugacy to the negative curvature case (the proof was streamlined and generalised in [DM]). The subset could be small, however, and the semiconjugacy could map most points to a few periodic orbits, so it would be much better to know that the whole flow is Anosov.

We decided to concentrate on the regions I, G, M where the configuration space is a surface of genus 3. Figure 3 shows the case $l_1 = 19/180, l_2 = 181/180$, represented as a subset of $\mathbb{T}^3$ using the angles θ_i of the three disks.

The mass parameters for the kinetic energy are:

$$\begin{aligned} \mu_1 &= \frac{3}{l_2^2}(I_c + m|p-c|^2), \\ \mu_2 &= -\frac{2}{l_2^2}(I_c + m(x-c).(p-c)), \\ \mu_3 &= \frac{I}{l_1^2} + \frac{I_c + m|x-c|^2}{l_2^2}, \end{aligned} \tag{2}$$

where I_c is the moment of inertia of a rod about its centre of mass c. The masses of the pivots can be incorporated into the mass parameters too, any part fixed to a disk being incorporated into its inertia and any part fixed to a rod being incorporated into its inertia. Real pivots, however, often contain parts that are fixed to neither, e.g. ball races. Then the exact result depends on their construction. One could calculate their effect for any given construction, but for simplicity of exposition, we ignore that.

The mass parameters μ satisfy some restrictions due to positivity of m, I, I_c, $|p-c|^2$ and $|x-c|^2$ and the relation $|(x-c)-(p-c)| = l_2$. The complete set of restrictions is $\mu_1, \mu_3 > 0$ and $\mu_2^2 < \frac{4}{3}\mu_1\mu_3$ (see Appendix C). Note that although μ_2 may be negative if the rods extend far enough beyond the pivots, it is positive for a uniform rod joining the pivots (and for all nearby mass distributions).

3 Numerics on the curvature

On eliminating the p_i in favour of x and the σ_i, the kinetic energy of the triple linkage can be written entirely in terms of $\dot{x}$ and the σ_i. We chose the origin for x to be at the centre of the triangle of fixed pivots, and oriented the triangle to have a vertex at $(1, 0)$. By D_6-symmetry, it is enough to look at the $+++$ and $-++$ patches. The kinetic energy metric depends on 5 parameters: l_1, l_2 and the mass parameters μ_i, but an overall scale factor on the masses just scales the kinetic energy, so we can reduce parameter space to 4 dimensions (in our numerics we took $\mu_1 + 3\mu_2 + 3\mu_3 = 1$).

The curvature of the metric can be computed by Brioschi's formula (e.g. [DC, G]), though the formulae are immense. Although still long, the calculation simplifies at the symmetry points, i.e. the fixed points for (non-identity) elements of the D_6 action. So we evaluated the curvature there to eliminate large regions of the 4-dimensional parameter space. To make the curvature negative at the "type I" fixed points ($(x, \sigma) = (1 - l_1 - l_2, 0, \pm, +, -)$, $(1 - l_1 - l_2, 0, \pm, -, +)$, and their images under three-fold rotation) required most of the mass to be in μ_3 and the lengths to be near $l_2 = 1, l_1 = 0$. The "type II" fixed points ($(1 \pm (l_2 - l_1), 0, \pm, +, -)$, $(1 \pm (l_2 - l_1), 0, \pm, -, +)$ and their three-fold rotations) turned out to pose no further restriction. The "type O" points, however, ($x = 0$, $\sigma = +++$ or $---$) required further restriction to a neighbourhood of the hyperbola $l_2^2 = 1 + l_1^2$.

Having hit on a promising parameter set $l_1 = 7/40, l_2 = 41/40, \mu_1 = 11/50, \mu_2 = 3/100, \mu_3 = 23/100$, we computed the curvature on the whole of the $+++$ and $-++$ patches. This was done with intelligent use of the chain rule in Mathematica. The result is shown in Figure 4. Indeed the curvature appears to be negative everywhere.

4 The limit system

A priori, the regime $\mu = (0, 0, 1/3)$ does not look promising because at the type O symmetry points the principal curvatures are equal so the Gaussian curvature is a square, thus non-negative. It has the advantage, however, that the kinetic energy metric is just the restriction of a standard Euclidean metric on $\mathbb{T}^3$, which facilitates the calculations.

For $i = 1, 2, 3$, let θ_i be the angle of the i^{th} disk from the position where the point of attachment of the connecting rod is closest to the centroid of the equilateral triangle. The condition that the attachment points be on a circle of radius l_2 can be written

$$l_2^2(a+b+c)(a+b-c)(b+c-a)(c+a-b) = a^2b^2c^2, \tag{3}$$

where a, b, c are the distances between them. Writing $l_2 = 1 + bl_1$, we find the configuration space to first order in l_1 has the equation

$$b = -(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)/3. \tag{4}$$

This is a (non-empty real analytic) submanifold of $\mathbb{T}^3$ for all values of $b \in [-1, 1]$ except the critical values of the right hand side which are $-1, -1/3, 1/3, 1$. Solid-state physicists will recognise it as the Fermi surface for independent electrons in a tight binding model of a cubic lattice with only nearest neighbour hopping [AM]. Riemannian geometers will recognise the case $b = 0$ as the Schwarz P-surface, an example of a minimal surface (which was the basis for constructions of [BD]).

If the rods have zero mass ($m = 0$) and the moments of inertia of the disks are scaled to $I = 1$, then the kinetic energy metric is simply the restriction of the Euclidean metric $ds = \sqrt{\sum_i d\theta_i^2}$ on $\mathbb{T}^3$ to the configuration surface. The unit normal vector of this first order approximation is $(\sin\theta_1, \sin\theta_2, \sin\theta_3)/Z^{1/2}$, where $Z = \sum_i \sin^2\theta_i$. Note that $Z \neq 0$ for b not a critical value. In any open set which is a graph over (θ_1, θ_2) the projection of its derivative is

$$Z^{-3/2}\begin{bmatrix} \cos\theta_2\sin^2\theta_3 + \cos\theta_3\sin^2\theta_2 & -\sin\theta_1\sin\theta_2\cos\theta_3 \\ -\sin\theta_1\sin\theta_2\cos\theta_3 & \cos\theta_1\sin^2\theta_3 + \cos\theta_3\sin^2\theta_1 \end{bmatrix}. \tag{5}$$

The curvature (the determinant of the above, multiplied by the appropriate projection factor $Z/\sin^2\theta_3$) is

$$Z^{-2}(\cos\theta_1\cos\theta_2\sin^2\theta_3 + \cos\theta_2\cos\theta_3\sin^2\theta_1 + \cos\theta_3\cos\theta_1\sin^2\theta_2). \tag{6}$$

This can be written as

$$(9b^2/2 + 3bp - s/2)Z^{-2}, \tag{7}$$

where $p = \prod_i \cos\theta_i$ and $s = \sum_i \cos^2\theta_i = 3 - Z$. The same result is obtained for open sets requiring other pairs of coordinates, so it holds everywhere.

If $b = 0$ then the curvature is $-\frac{s}{2Z^2}$, which is non-positive (this follows also from minimality). It is equal to zero only if all $\cos\theta_i$ are zero, i.e. at the eight points $\theta_i \in \{\pm\pi/2\}$. This is sufficient for its geodesic flow to be Anosov (cf. [Kn]).

5 Perturbation of the limit system

Now we add the higher order corrections in l_1 and allow the rods to have small positive mass. By structural stability of Anosov flows, the geodesic flow remains Anosov. It is also mixing because geodesic flows are "contact" flows and contact Anosov flows on connected manifolds are mixing [KH] (both topologically and for every Gibbs measure, which includes Liouville measure for Anosov geodesic flows). The correlations of Hölder continuous functions decay exponentially by [Do]. This proves our Theorem.

One can break the D_6-symmetry weakly too (which can introduce "rattleback" terms in the kinetic energy), and the theorem still holds. One can even allow slight out-of-plane effects, e.g. if the axes of rotation of the disks are not quite parallel, provided no new degrees of freedom are introduced.

One consequence of the exponential decay of correlations is that there is diffusion in any Abelian cover. An obvious Abelian cover is obtained by lifting the configuration space from $\mathbb{T}^3$ to $\mathbb{R}^3$ (as is easy to visualise from Figure 3). Thus if one were to measure the total angles $\theta_j, j = 1, 2, 3$, turned in time t, one should see the vector $\theta(t)$ perform a random walk in $\mathbb{R}^3$, when observed on a scale large compared with 2π. Actually, the first Betti number of the unit tangent bundle to X_3 is 7, so this can be extended to a 7-dimensional random walk.

6 Adding a potential

If the linkage is subjected to a potential energy V (but still no friction), then the motion is that of the Lagrangian $L(q,\dot{q}) = K(\dot{q}) - V(q)$. The energy $H = K + V$ is conserved. For large energies, the perturbation of the vector field due to V is relatively small in C^1, so structural stability implies that all high enough energy levels remain Anosov. They also remain mixing, because nonintegrability of the invariant foliations of a transitive Anosov flow is preserved under C^1-small perturbation [Ch]. It follows by [Do] that exponential decay of correlations is also preserved.

An alternative strategy for obtaining Anosov energy levels for the triple linkage is to start from any parameter value such that the configuration space has a component which is a surface of genus greater than 1 and then add a carefully designed potential, as follows. For E greater than the maximum of V, the motion on energy level $H^{-1}(E)$ is up to time-reparametrisation the geodesic flow of the Maupertuis (or Jacobi) metric $2\sqrt{(E - V(q))K(dq)}$. This is conformally equivalent to the kinetic energy metric $\sqrt{2K(dq)}$. The conformal factor $\sqrt{2(E - V(q))}$ can be chosen to make the Maupertuis metric have constant negative curvature [Au]: the equation $\Delta\phi - e^{2\phi} = \kappa$, where Δ and κ are the Laplacian and the curvature of the kinetic energy metric, has a unique solution ϕ on the configuration space and then one chooses $V = E - \frac{1}{2}e^{2\phi}$. It follows that the chosen energy level is Anosov (by constant negative curvature), and so are all the nearby ones (by structural stability). It is not guaranteed that the flow remains mixing, however, because the time-reparametrisation might remove the joint nonintegrability of the invariant foliations.

7 Including friction

If there is friction (meaning additional forces whose inner product with the velocity is negative definite) then by LaSalle's principle and compactness of the configuration space, the energy tends to zero as time goes to infinity (or in the case with a potential, to a critical value of the potential), so there is no chaos.

If the friction depends linearly on velocity and there is no potential, then the flow induces one on the unit tangent bundle of the configuration space, given by rescaling the rate of time to obtain speed 1. If the friction is weak then the resulting flow is close to the geodesic flow, hence still Anosov and mixing. This is a somewhat artificial fix, however, because the rescaled time is bounded and the result is not robust to the addition of a potential or nonlinearity in the friction. So we seek alternatives.

The effect of friction can be compensated by suitable driving forces. To make a physically realisable system, the compensating forces must lie in an open set of suitable ones. So it is not enough just to design external forces which exactly annihilate the friction forces. Instead, we suppose the friction is weak and apply feedback forces which contract strongly onto or close to a particular energy level $H^{-1}(E)$. We choose E sufficiently above the maximum of the potential so that H is dominated by the kinetic energy K. Then for m small the desired effect can be achieved by torques $\gamma(K - E)\dot{\theta}_i$ on the three disks, with γ large (γ can be allowed to be state-dependent). This is because with no friction nor potential, and $m = 0$, such external torques would make the kinetic energy $K = \frac{I}{2}\sum_i \dot{\theta}_i^2$ evolve by $\dot{K} = 2\gamma K(K - E)/I$, so K would converge rapidly to E from all initial conditions except the rest states, and on $K^{-1}(E)$ the torques are zero so there is no change to the motion there. The linearised contraction rate onto $K^{-1}(E)$ is $2\gamma E/I$. If this is chosen larger than the maximum contraction rate for the flow tangent to $K^{-1}(E)$ (in the limit of Section 4, this is $\sqrt{2E/I}$, because it is the product of the speed and the square root of minus the most negative curvature of the metric, which is easily shown to be -1 and to occur at $\theta = (0, \pi, \pi/2)$ and its images under symmetry, so it suffices to choose γ a little larger than $\sqrt{I/2E}$), then $K^{-1}(E)$ is normally hyperbolic and hence persists to a nearby attracting invariant submanifold under all small perturbations [F], such as the addition of weak friction, a potential small compared to E, small m, and small deviations from the above feedback law (provided they are autonomous). Because the unperturbed motion on $K^{-1}(E)$ is Anosov and Anosov flows are structurally stable, the flow on the resulting submanifold is Anosov. It is also mixing, because nonintegrability of the invariant foliations is preserved by C^1-small perturbation.

This feedback control could be implemented by detectors to measure the rotation rate of each disk (e.g. car speedometers), a device to compute the torques (which could perhaps be mechanical, though a microcomputer would be easier), and actuators to apply the torques to the disks (e.g. induction motors).

Acknowledgements

TJH was supported by an EPSRC studentship.

Appendix A: Properties of uniformly hyperbolic sets

Definition: An invariant subset Λ for a C^1 vector field $\dot{x} = v(x)$ on a manifold M (equipped with a norm $|.|$ on its tangent spaces), with flow denoted $\phi : \mathbb{R} \times M \to M$, is said to be *uniformly hyperbolic* if there are $C, A, \lambda > 0$ and a splitting $E_x^+ \oplus E_x^- \oplus \mathbb{R}v(x)$ of the tangent space to M at each point $x \in \Lambda$, such that (i) $\xi \in E_x^\pm, t \in \mathbb{R}$ implies $D\phi_t\xi \in E^\pm_{\phi_t(x)}$ and

$$|D\phi_t\xi| \leq Ce^{-\lambda|t|} \text{ for } t \in \mathbb{R}_\pm,$$

and (ii) $\xi^\pm \in E_x^\pm, \alpha \in \mathbb{R}$ imply $|\xi^+ + \xi^- + \alpha v(x)| \geq A \max(|\xi^\pm|, |\alpha||v(x)|)$ (the splitting is automatically continuous and the second condition is satisfied automatically if Λ is compact). The case where the whole manifold M is uniformly hyperbolic is called *Anosov*.

Any invariant set that is contained in a small enough neighbourhood of a uniformly hyperbolic set is also uniformly hyperbolic, so without loss of generality one can restrict attention to *locally maximal* uniformly hyperbolic sets.

Locally maximal uniformly hyperbolic invariant sets Λ have the following nice properties (e.g. [KH]):

1. Λ is closed.

2. Each chain transitive component of Λ is locally maximal and topologically transitive (for a volume-preserving system such as a Hamiltonian system on an energy level, this implies that each connected component of Λ is topologically transitive).

3. If compact, Λ has a finite Markov partition, i.e. equivalence of the flow on Λ (up to a "negligible" subset) to a flow on a finite directed graph ("symbolic dynamics").

4. Λ persists under C^1-small perturbation of v, and the dynamics on its continuation remain topologically conjugate to the unperturbed dynamics ("structural stability").

5. If a small time-dependent perturbation is added to v, each orbit of the perturbed system that remains in a small enough neighbourhood of Λ is "shadowed" by some orbit of the unperturbed system in Λ.

6. The sets of points whose orbits converge together with the orbit of a given point $x \in \Lambda$ in forwards (respectively backwards) time form injectively immersed submanifolds, tangent to $E_x^\pm$ and as smooth as the system, which we call the forwards (respectively backwards) contracting submanifolds of that point. If Λ contains all its backwards contracting submanifolds then it attracts a neighbourhood and they foliate the basin of attraction. In the Anosov case, the whole manifold is foliated by both the forwards and backwards contracting submanifolds. The foliations are invariant and Hölder continuous; in general they are not smoother, but for Anosov geodesic flows of Riemannian metrics on surfaces they are C^1.

7. If Λ is chain transitive and attracting, it supports a unique invariant probability measure μ (called "SRB" after Sinai, Ruelle and Bowen) describing the asymptotic distribution of almost all orbits in its basin of attraction. For volume-preserving flows the SRB measure is the volume (up to normalisation). There are formulae relating the SRB measure to properties of its periodic orbits.

8. There is a simple formula for how the SRB measure changes under perturbation of the vector field [R].

9. In some circumstances it can be proved that the SRB measure is mixing, and explicit bounds on the decay of correlation functions can be obtained. For example, Anosov geodesic flows on a compact connected manifold are mixing, and if the flow is $C^{2+\varepsilon}$ and the invariant foliations are C^1 (as is true for Anosov geodesic flows in two dimensions) then the correlations of Hölder continuous functions decay at least exponentially [Do].

10. In particular, if the velocity correlation functions $C_{ij}(t) = \int h_i(v(\phi_t(x)))h_j(v(x))\ d\mu(x)$ for a basis $(h_i)_{i=1\ldots I}$ of any subspace L of first de Rham cohomology $H^1(M)$ are integrable, one obtains "diffusion" in the Abelian cover of M corresponding to L. The simplest sense in which there is diffusion is that $\int \Delta_i(x)\Delta_j(x)\ d\mu(x) \sim 2D_{ij}t$ as $t \to \infty$, where $\Delta_i(x) = \int_0^t h_i(v(\phi_t(x)))\ dt$ and $D_{ij} = \int_0^\infty C_{ij}(t)\ dt$, but diffusion holds in stronger senses too [DP] (cf. [BS] for a periodic Lorentz gas, though this is not a uniformly hyperbolic system).

Appendix B: Comments on some standard examples of chaos

The chaotic attractors of most dynamical systems of physical origin seem not to be uniformly chaotic. For example, the Lorenz system fails even if one takes a parameter regime with no near-tangencies of the invariant manifolds of the saddle, because the attractor contains the saddle so the splitting in the definition of uniformly hyperbolic set can not be continuous there, whereas continuity of the splitting is an automatic consequence of the definition of uniform hyperbolicity; this may look just a technical point but it causes properties 3,4 and 5 to fail. The Sinai billiard fails because the vector field is discontinuous at bounces and even worse the billiard map is discontinuous at grazing incidence. The periodically forced van der Pol oscillator (which can be included in our autonomous context by adding the equation $\dot{t} = 1$) in chaotic parameter regimes appears to have near-tangencies of the invariant manifolds of a periodic orbit, and its Poincaré map does not appear to have the minimum of three "holes" that are necessary to obtain a uniformly hyperbolic attractor [P].

Also the mathematical examples of uniform chaos do not seem to be physically realisable. Compact manifolds of negative curvature can not be embedded in $\mathbb{R}^3$ so their geodesic flows have not been regarded as physical. Similarly the suspensions of the solenoid map and toral automorphisms live on 3-manifolds that are difficult to imagine, so they are regarded as artificial. The suspension of a Plykin attractor is realisable in a system of three ODEs (e.g. [H]), but explicit realisations so far look very contrived. Ruelle and Takens (refined with Newhouse [NRT]) proved that arbitrarily close to a constant vector field on a torus of dimension $N \geq 3$ (which could arise by weakly coupling N limit cycle oscillators or by bifurcations from a periodic orbit) are vector fields containing uniformly chaotic attractors, but we are not aware of any positive identifications of this possibility in physical systems.

Appendix C: Restrictions on the mass parameters

The restrictions $\mu_1 > 0, \mu_3 > 0$ and $\mu_2^2 < \frac{4}{3}\mu_1\mu_3$ on the mass parameters are derived as follows.

The necessity of the first two conditions is clear from the formulae (2). To show the third is necessary, let s be the position of the centre of mass of a rod along the line through p and x, scaled so that $s = 0$ puts it at p and $s = 1$ puts it at x. Put $\alpha = I/l_1^2$ and $\beta = I_c/l_2^2$. Then

$$(\frac{\mu_2}{2} + \beta)^2 = (ms(1-s))^2 = (\frac{\mu_1}{3} - \beta)(\mu_3 - \alpha - \beta). \tag{8}$$

So

$$\frac{\mu_2^2}{4} - \frac{\mu_1\mu_3}{3} = -\beta(\frac{\mu_1}{3} + \mu_2 + \mu_3) - \alpha(\frac{\mu_1}{3} - \beta). \tag{9}$$

Now $\frac{\mu_1}{3} + \mu_2 + \mu_3 = m + \alpha > 0$ and $\frac{\mu_1}{3} - \beta = ms^2 \geq 0$, so $\mu_2^2 - \frac{4}{3}\mu_1\mu_3 < 0$.

To show sufficiency of the conditions, given μ satisfying the constraints, choose any $\alpha > 0$ such that $\mu_1, \mu_2, \mu_3' = \mu_3 - \alpha$ still satisfy them. Then let $m = \frac{\mu_1}{3} + \mu_2 + \mu_3'$, $\beta = \frac{1}{m}(\frac{\mu_1\mu_3'}{3} - \frac{\mu_2^2}{4})$ and $s = \frac{1}{m}(\frac{\mu_1}{3} + \frac{\mu_2}{2})$. This combination of parameters realises μ.

References

[AM] Ashcroft NW, Mermin ND, Solid state physics (Saunders, 1976)

[Au] Aubin T, Some nonlinear problems in Riemannian geometry (Springer, 1998)

[BS] Bunimovich LA, Sinai YaG, Statistical properties of Lorentz gas with periodic configuration of scatterers, Commun Math Phys 78 (1981) 479–497.

[BD] Burns K, Donnay VJ, Embedded surfaces with ergodic geodesic flows, Int J Bif Chaos 7 (1997) 1509–27.

[Ch] Chernov NI, Markov approximations and decay of correlations for Anosov flows, Ann Math 147 (1998) 269–324.

[DP] Denker M, Phillips W, Approximation by Brownian motion for Gibbs measures and flows under a function, Ergod Th Dyn Sys 4 (1984) 541–552.

[DM] Denvir J, MacKay RS, Consequences of contractible geodesics on surfaces, Trans Am Math Soc 350 (1998) 4553–68.

[DC] Do Carmo MP, Differential geometry of curves and surfaces (Prentice Hall, 1976)

[Do] Dolgopyat D, On decay of correlations in Anosov flows, Ann Math 147 (1998) 357–390.

[F] Fenichel N, Persistence and smoothness of invariant manifolds for flows, Indiana U Math J 211 (1971) 193–226.

[G] Gray A, Modern differential geometry of curves and surfaces (CRC, 1993)

[H] Hunt TJ, Low dimensional dynamics: bifurcations of cantori and realisations of uniform hyperbolicity, PhD thesis (Cambridge, 2000)

[KH] Katok A, Hasselblatt B, Introduction to the modern theory of dynamical systems (Cambridge, 1995).

[Kl] Klingenberg W, Riemannian manifolds with geodesic flow of Anosov type, Ann Math 99 (1974) 1–13.

[Kn] Knauf A, Ergodic and topological properties of Coulombic periodic potentials, Commun Math Phys 110 (1987) 89–112.

[NRT] Newhouse S, Ruelle D, Takens F, Occurrence of strange Axiom A attractors near quasi periodic flows on $\mathbb{T}^m, m \geq 3$, Commun Math Phys 64 (1978) 35–40.

[Ni] Niemann SH, Geometry and mechanics (PhD thesis, Mathematics, Oxford, 1977).

[P] Plykin RV, Sources and sinks of A-diffeomorphisms of surfaces, Mat USSR Sbornik 23:2 (1974) 233–253.

[R] Ruelle D, Differentiation of SRB states, Commun Math Phys 187 (1997) 227–241.

[TW] Thurston WP, Weeks JR, The mathematics of three-dimensional manifolds, Sci Am 251 (July 1984) 94–106.

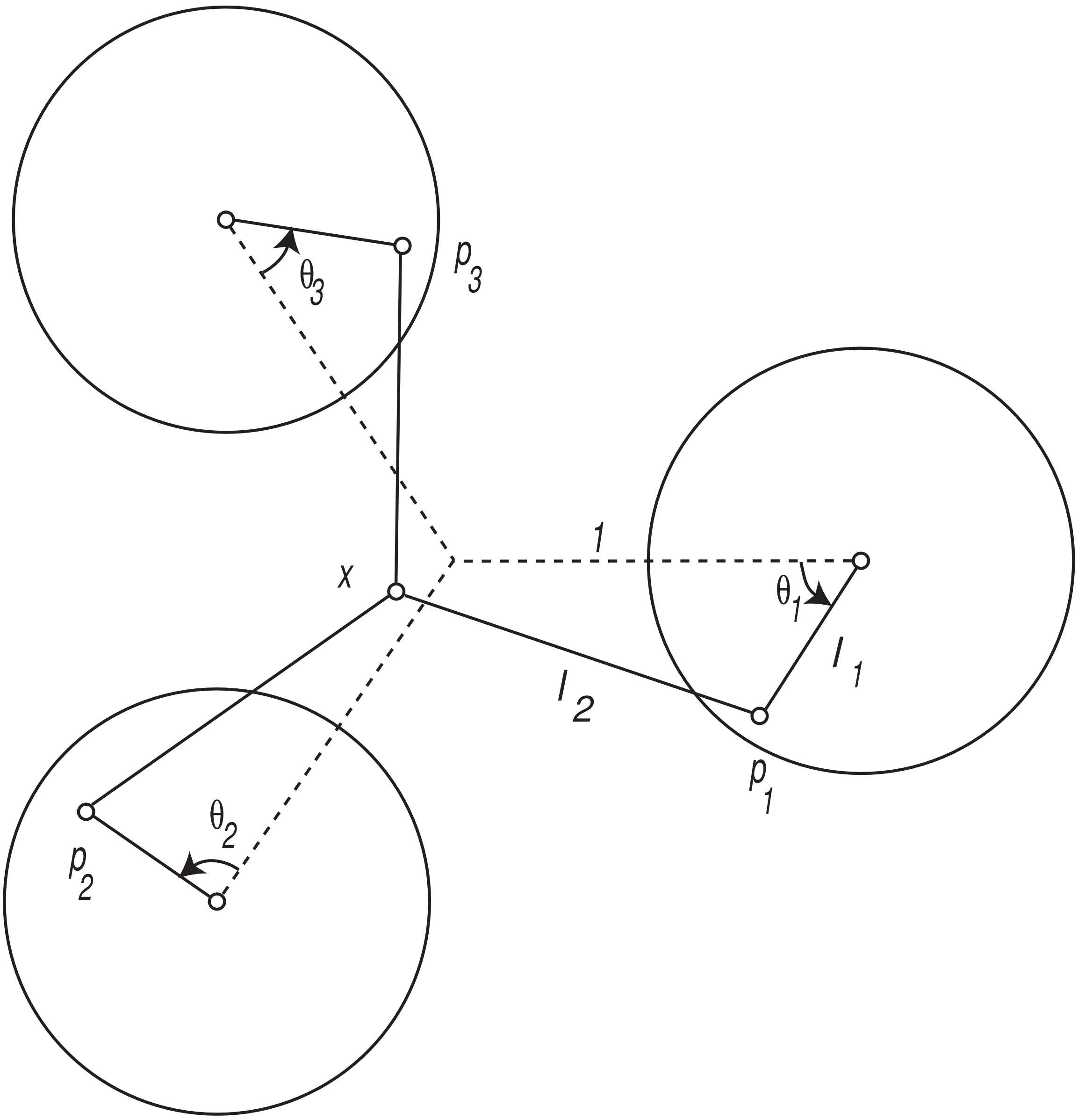

Figure 1: The triple linkage

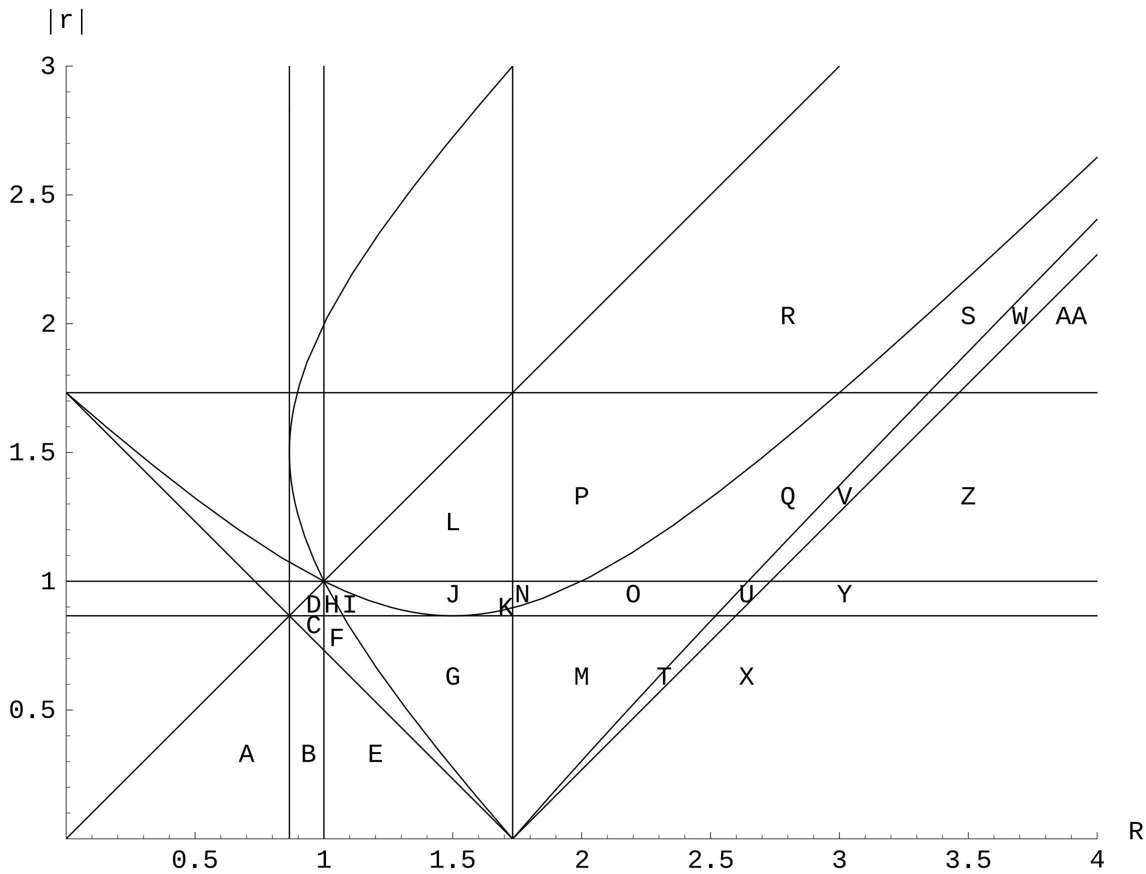

Figure 2: Partition of the plane of parameters $R = l_1 + l_2, r = l_1 - l_2$ into regions where the configuration space of the triple linkage has a given diffeomorphism type. $I, G, M : X_3$; $H, F, E, J, N : X_0$; $X : X_{12}$; $Z, AA : X_7$; $Q, V, S : 3X_0$; $Y : X_7 \cup X_0$; $W : 6X_0$; $K, O : 4X_0$; $U : 7X_0$; $T : X_3 \cup 3X_0$. X_g denotes the oriented surface of genus g. The regions A, B, C, D, L, P, R and all those above the diagonal are unattainable.

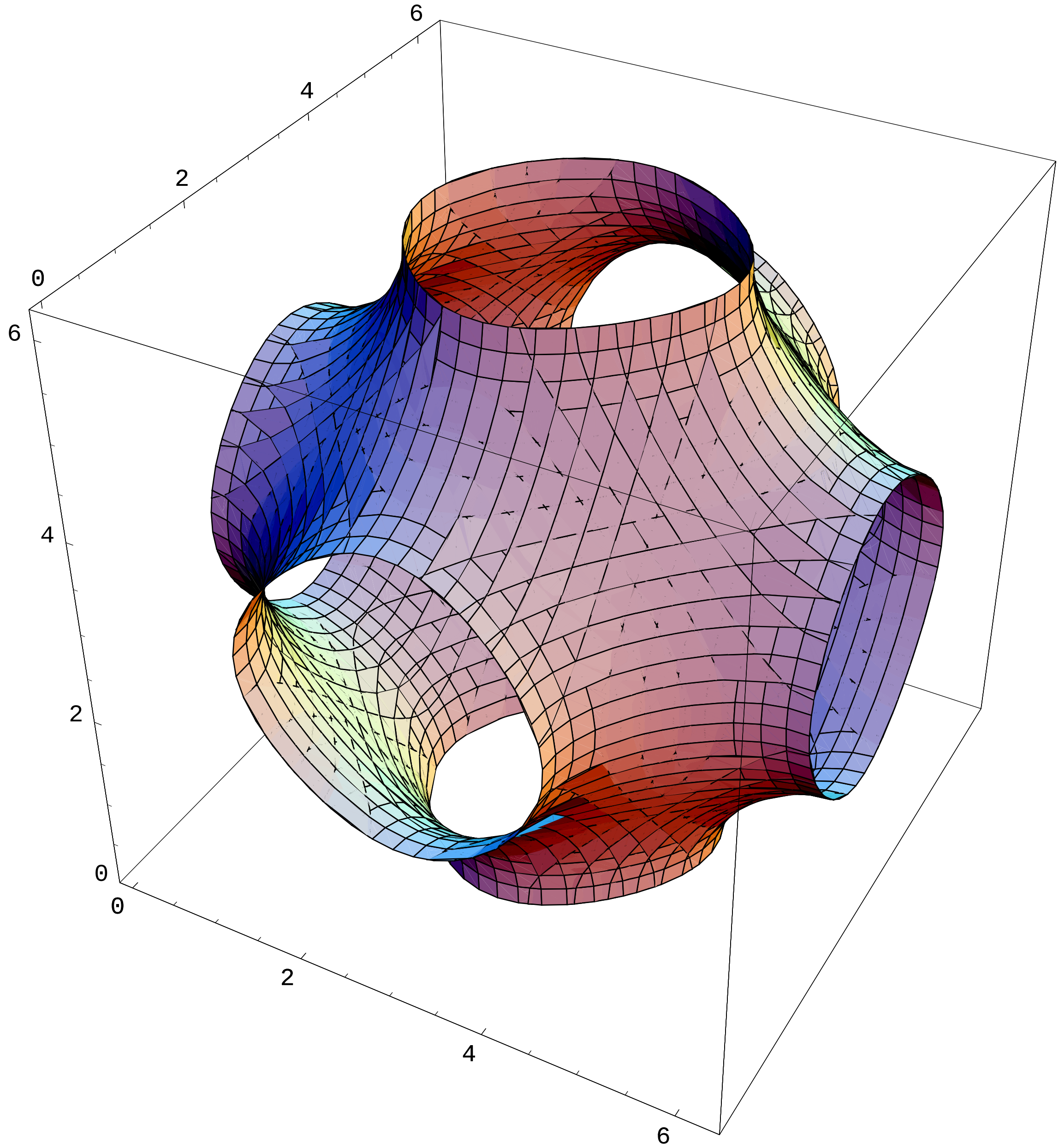

Figure 3: The configuration space as a subset of $\mathbb{T}^3$ for $l_1 = 19/180$, $l_2 = 181/180$.

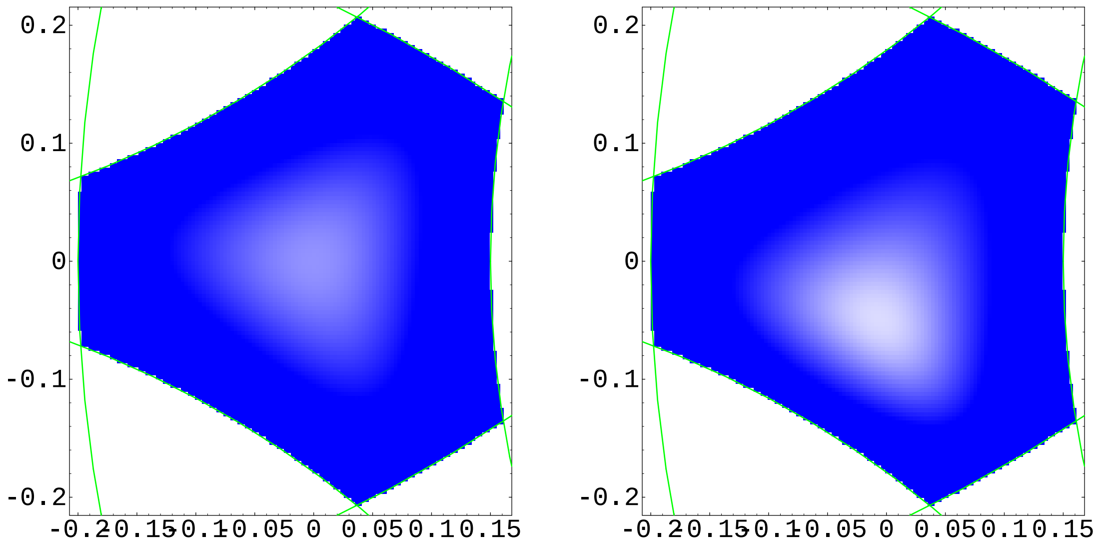

Figure 4: The curvature on the $+++$ and $-++$ patches of the configuration space for the triple linkage at parameter values $\mu_1 = 11/50, \mu_2 = 3/100, \mu_3 = 23/100, l_1 = 7/40, l_2 = 41/40$. The darkness indicates the degree of negativity of the curvature.