Symplectic twist maps without conjugate points

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Abstract. For sequences of symplectic twist maps without conjugate points, an invariant Lagrangian subbundle is constructed. This allows one to deduce that absence of conjugate points is a rare property in some classes of map.

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1 Introduction and results

In this paper we construct an analogue of L Green's invariant subbundles for the case of discrete variational principles related to the dynamics of sequence of symplectic twist maps of $T^*\mathbb{T}^d$. Such a construction was first performed by L Green [9] for Riemannian geodesic flows but has turned out to be much more general. For example, it can be extended to optical Hamiltonian flows [7]. The construction of invariant subbundles is very useful in many examples of the so-called Hopf-type rigidity.

In particular, we apply L Green's construction to the so-called Frenkel-Kontorova variational problem which is related to a sequence of generalized standard maps. We prove a result which can be seen as an analogue of a rigidity result of Knauf and Croke-Fathi which was proved for conformally flat Riemannian metrics [11, 6].

In the discrete time case Hopf rigidity was established first for convex plane billiards [1, 17].

There are still very many problems related to the rigidity and integrability of twist maps and we hope that our results will be useful for their solutions.

Let us introduce the setting (see also the recent book by Chris Gole [8] for a detailed exposition).

For each $n \in \mathbb{Z}$, let $S_n : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^2-smooth function satisfying the following:

$$\begin{aligned} &1. \quad S_n \quad \text{is} \quad \mathbb{Z}^d - \text{periodic}: \quad S_n(q+e, Q+e) = S_n(q,Q) \\ &\text{for any } (q,Q) \in \mathbb{R}^d \times \mathbb{R}^d \text{ and } e \in \mathbb{Z}^d. \end{aligned} \tag{1.1}$$

2. S_n satisfies the uniform twist condition : for any $\xi \in \mathbb{R}^d$ the quadratic form

$$\sum_{i,j} \frac{\partial^2 S_n(q,Q)}{\partial q_i \partial Q_j} \xi_i \xi_j \le -K||\xi||^2 \text{ for a positive constant } K. \tag{1.2}$$

Such a function defines two closely related objects.

The first is the variational functional defined on the sequences $\{q_n\}, n \in \mathbb{Z}$,

$$F(\{q_n\}) = \sum_{n=-\infty}^{+\infty} S_n(q_n, q_{n+1}). \tag{1.3}$$

The functional is a formal sum but the extremals are well defined and satisfy the equations

$$\partial_2 S_{n-1}(q_{n-1}, q_n) + \partial_1 S_n(q_n, q_{n+1}) = 0 \text{ for all } n \in \mathbb{Z}. \tag{1.4}$$

The second object is the symplectic diffeomorphism T_n of $T^*\mathbb{T}^d$ generated by the function S_n. In the standard coordinates (p,q) it is given by the following implicit formula

$$T_n(p,q) = (P,Q) \text{ if } P = +\partial_2 S_n(q,Q), p = -\partial_1 S_n(q,Q). \tag{1.5}$$

Here and throughout the paper ∂_1, ∂_2 stand for the derivatives with respect to the q_i, Q_j variables respectively.

We refer the reader to [8], [10] and [13] for general theory of symplectic twist maps — note that in eq.(1.2) we follow [13]'s choice of twist condition rather than either of those of [10].

The basic example for us will be

Example 1 Let $S_n = \frac{1}{2}||Q - q||^2 + V_n(q)$ where V_n is a $\mathbb{Z}^d$-periodic smooth function (called the potential). In this case we shall call F a Frenkel-Kontorova functional. In what follows we will assume that the sequence of the potential functions V_n depends either periodically on n, or V_n vanishes for all but finitely many values of n. The corresponding map T_n is a generalized standard map of $T^*\mathbb{T}^d$:

$$T_n : (p, q) \mapsto (p + \nabla V_n(q), p + q + \nabla V_n(q)).$$

It is important to notice that in this case for any n, T_n can be considered as acting on $\mathbb{T}^{2d}$ and not just on $T^*\mathbb{T}^d$; this follows from the fact that for any $e \in \mathbb{Z}^d$,

$$T_n(p + e, q) = (P + e, Q + e).$$

The correspondence between the extremals of the functional F and the orbits of the sequence T_n is the following. Let a sequence $\{q_n\}$ be an extremal for F. Let $p_n = -\partial_1 S(q_n, q_{n+1})$ and form the sequence $\{x_n = (p_n, q_n)\}$. Then $\{x_n\}$ is an orbit of the evolution, i.e. $T_n(x_n) = x_{n+1}$. Conversely, if $\{x_n = (p_n, q_n)\}$ is an orbit then the corresponding sequence $\{q_n\}$ is extremal for the variational principle written above.

Similarly, invariant fields along the orbits of $\{T_n\}$ correspond to the so-called Jacobi fields along the extremals. For an orbit $\{x_n\}$, let $\zeta_n \in T_{x_n}T^*\mathbb{T}^d$ be a tangent vector at $x_n = (p_n, q_n)$; then the field $\{\zeta_n\}$ is invariant under the derivative T_*, i.e. $(T_n)_*(\zeta_n) = \zeta_{n+1}$, if and only if the vectors $\xi_n = \pi_*(\zeta_n)$ satisfy the Jacobi equation (here $\pi : (p, q) \mapsto q$ is the canonical projection):

$$b^T_{n-1}\xi_{n-1} + a_n\xi_n + b_n\xi_{n+1} = 0 \tag{1.6}$$

with the matrices

$$b_n = \partial_{12}S_n(q_n, q_{n+1}), a_n = \partial_{11}S_n(q_n, q_{n+1}) + \partial_{22}S_{n-1}(q_{n-1}, q_n)$$

(the symbols $\partial_{11}S, \partial_{12}S, \partial_{22}S$ denote the matrices of second derivatives of S).

We will use the following definition first introduced for the discrete case in [1].

Definition. Two points of the extremal configuration $\{q_n\}$ are called *conjugate* if there exists a non-trivial Jacobi field ξ_n vanishing at these two points.

Denote by R^n_m the evolution transformation, i.e.

$R^n_m = T_{n-1} \circ \ldots \circ T_m$, for $n > m$, $R^m_m = Id$ and $R^n_m = (R^m_n)^{-1}$, for $n < m$.

With the above correspondence one can interpret the definition geometrically by saying that q_m and q_n, for $m < n$, are conjugate if

$$(R^{n-1}_m)_* (\mathcal{V}(x_m)) \bigcap \mathcal{V}(x_n) \neq \{0\}$$

where $\mathcal{V}(x)$ denotes the vertical subspace at x and $x_n = R^n_0(x_0)$ is the orbit corresponding to $\{q_n\}$.

Theorem 1 *If none of the extremals of the functional F have conjugate points then for every n there exists a field $\mathcal{W}_n$ of Lagrangian subspaces $\mathcal{W}_n(x) \subseteq T_x T^*\mathbb{T}^d$ depending measurably on x and such that*

1. *Invariance: $(T_n)_*\mathcal{W}_n(x) = \mathcal{W}_{n+1}(T_n x)$*
2. *At every point $x, \mathcal{W}_n(x)$ is transversal to the vertical subspace $\mathcal{V}(x)$.*

We shall use a partial order $\leq$ on the subset of Lagrangian subspaces which are transversal to the vertical one, defined as follows. To every such subspace $\mathcal{L}(x)$ corresponds a symmetric matrix L, by $\mathcal{L}(x) = \{\xi : dp(\xi) = L\ dq(\xi)\}$. Given two subspaces $\mathcal{L}_{1,2}$ we say $\mathcal{L}_1 \leq \mathcal{L}_2$ if $L_1 \leq L_2$, i.e. $L_2 - L_1$ is non-negative.

Theorem 2 *If none of the extremals of the functional F have conjugate points then for the fields $\mathcal{W}_n(x)$ the following holds*

1. $(T_{n+1}^{-1})_*\mathcal{V}(T_n x) \leq \mathcal{W}_n(x) \leq (T_{n-1})_*\left(\mathcal{V}(T_n^{-1}x)\right)$,
   or in terms of the matrices this reads
   $-\partial_{11}S_n(q, q_+) \leq W_n(x) \leq \partial_{22}S_{n-1}(q_-, q)$,
   for all x where $q = \pi(x), q_- = \pi(T_{n-1}^{-1}x), q_+ = \pi(T_n x)$.
2. *The following inequality holds true for all x*
   $W_{n+1}(T_n x) - W_n(x) \leq \partial_{11}S_n(q, q_+) + \partial_{22}S_n(q, q_+) + \partial_{12}S_n(q, q_+) + \partial_{21}S_n(q, q_+)$
   with equality in only the case when
   $\partial_{12}S_n(q, q_+) = \partial_{21}S_n(q, q_+)$ *and* $W_{n+1}(T_n x) = \partial_{22}S_n(q, q_+) + \partial_{21}S_n(q, q_+)$
   and $W_n(x) = -\partial_{11}S_n(q, q_+) - \partial_{12}S_n(q, q_+)$.

As an application of this to Frenkel-Kontorova functionals we obtain

Theorem 3 *Consider the Frenkel-Kontorova functional with a sequence of potential functions V_n which is either periodic in n or has all but finitely many of the V_ns constant functions. Then either there exist extremals with conjugate points or all the potential functions are constants.*

The next section contains necessary preliminaries about Jacobi fields in the discrete case. We prove the theorems in section 3. Discussion and open questions conclude the paper.

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2 Nonsingular Jacobi fields

In this section we prove first that the assumption that no extremal has conjugate points implies that each extremal is in fact a strict local minimum configuration. As a consequence of this we construct a special non-singular solution of the *matrix* Jacobi equation. The first fact is stated as

Lemma 1 *If all the extremals of F have no conjugate points then each is a strict local minimum between any two of its points.*

Proof of Lemma 1

Let $\{q_n\}, n \in \mathbb{Z}$, be an extremal. For $M \leq N$, denote

$$F_{MN}(u_M, \ldots, u_N) = S_{M-1}(q_{M-1}, u_M) + \sum_{n=M}^{N-1} S_n(u_n, u_{n+1}) + S_N(u_N, q_{N+1}).$$

We claim that the matrix $\delta^2 F_{M,N}$ of second variation of F_{MN} is positive definite. To prove this, note that by a simple calculation it has the following block matrix form:

$$\begin{pmatrix} a_M & b_M & & 0 \\ b_M^T & a_{M+1} & \ddots & \\ & \ddots & \ddots & b_{N-1} \\ 0 & & b_{N-1}^T & a_N \end{pmatrix} \tag{2.1}$$

with the matrices a_i, b_i introduced in eq.(1.6). It follows that the kernel of this matrix consists exactly of the Jacobi fields vanishing at q_{M-1} and q_{N+1}. Thus by the non conjugacy assumption, the matrix is non-degenerate. But then it has to be positive definite by the fact that it depends continuously on the configuration (and so its signature is constant) and there always exist segments which minimize the functional (a consequence of (1.1),(1.2)) and so have positive definite second variation (see for example [8] for the proof). This completes the proof of the lemma. $\square$

Note that as a consequence, every orbit is a global minimum between any two of its points, though we do not need this fact.

Let us consider a minimal configuration $\{q_n\}, n \in \mathbb{Z}$. For given $k \in \mathbb{Z}$, define a *matrix* solution of the Jacobi eq.(1.6) $\xi_n^{(k)}$ such that $\xi_k^{(k)} = 0$ and $\xi_{k+1}^{(k)}$ is invertible, by iteration from this pair. Then by the no conjugate points assumption, all $\xi_n^{(k)}$ are invertible ($n \neq k$) and hence

$$A_n^{(k)} = -b_n \xi_{n+1}^{(k)} \left[\xi_n^{(k)}\right]^{-1} (n > k) \tag{2.2}$$

are defined and do not depend on the choice of $\xi_{k+1}^{(k)}$. Moreover one can easily see that

$$A_{k+1}^{(k)} = a_{k+1}, \text{ and for } n > k, \quad A_{n+1}^{(k)} = a_{n+1} - b_n^T \left[A_n^{(k)}\right]^{-1} b_n. \tag{2.3}$$

In particular all the $A_n^{(k)}$ are symmetric. A crucial observation for us is that all these matrices are in fact positive definite. Indeed, if on the contrary, for some $m > k$, $A_m^{(k)}$ is not positive definite then for some vector $\eta \neq 0$, $< A_m^{(k)}\eta, \eta > \leq 0$. Then define the segment of Jacobi field

$$\eta_n = \xi_n^{(k)} \left[\xi_m^{(k)}\right]^{-1} \eta, \quad k \leq n \leq m;$$

for $n = k$ and $n = m$ we have $\eta_k = 0$, $\eta_m = \eta$. One can easily compute the value of the quadratic form $\delta^2 F_{k+1,m}$ on the variation $(\eta_{k+1}, \ldots, \eta_m)$. Using eq.(2.1) one has

$$\delta^2 F_{k+1,m}\left(\eta_{k+1}, \ldots, \eta_m\right) = < -b_m \eta_{m+1}, \eta_m > = < A_m^{(k)}\eta, \eta >,$$

which contradicts the positivity of $\delta^2 F$.

We claim that the limit

$$\lim_{k \to -\infty} A_n^{(k)} = A_n \tag{2.4}$$

exists and A_n is a positive definite matrix sequence with the recursion rule

$$A_{n+1} = a_{n+1} - b_n^T A_n^{-1} b_n. \tag{2.5}$$

Indeed, it is easy to see by induction that $A_n^{(k)}$ is monotone in k: $A_n^{(k)} > A_n^{(k-1)}$, for all $n > k$. The initial step $A_{k+1}^{(k)} > A_{k+1}^{(k-1)}$ follows from

$$A_{k+1}^{(k)} = a_{k+1} \text{ and } A_{k+1}^{(k-1)} = a_{k+1} - b_k^T \left[A_k^{(k-1)}\right]^{-1} b_k, \text{ so } A_{k+1}^{(k)} - A_{k+1}^{(k-1)} = b_k^T \left[A_k^{(k-1)}\right]^{-1} b_k.$$

The induction step is also simple: if

$$A_n^{(k)} > A_n^{(k-1)}$$

then

$$A_{n+1}^{(k)} - A_{n+1}^{(k-1)} = -b_n^T \left([A_n^{(k)}]^{-1} - [A_n^{(k-1)}]^{-1}\right) b_n.$$

Thus the limit (2.4) exists and is a non-negative definite matrix. Moreover A_n is positive definite since it is necessarily non-degenerate (together with $A_n^{(k)}$, the limit A_n has to satisfy the recurrence relation (2.5) which can be written without the inverses of A_n). The claim is justified. We summarize the result in the following

Theorem 4 *For any strict local minimal configuration $\{q_n\}$ there exists a non-singular solution ξ of the matrix Jacobi equation such that the matrices $A_n = -b_n \xi_{n+1} \xi_n^{-1}$ are symmetric positive definite and satisfy*

$$A_{n+1} = a_{n+1} - b_n^T A_n^{-1} b_n. \tag{2.6}$$

3 Proofs of the main theorems

In this section we use the construction of the previous section to prove Theorems 1 and 2, and then apply them to prove Theorem 3.

Proof of Theorem 1

Consider the evolution transformations R^n_m defined above and the orbit of the point $x, x_n = R^n_0 x$, and consider the corresponding extremal $q_n = \pi x_n$.

Define $\mathcal{W}_n(x) = \lim_{k\to-\infty} \mathcal{W}^{(k)}_n(x)$, where $\mathcal{W}^{(k)}_n(x) = (R^n_k)_*(\mathcal{V}(R^k_n x))$. Note, that by the assumption of no conjugate points the Lagrangian subspaces $\mathcal{W}^{(k)}_n(x)$ are transversal to the vertical subspaces $\mathcal{V}(x)$. Moreover, one can easily check that the corresponding matrices $W^{(k)}_n$ satisfy:

$$W^{(k)}_n(x) = -\partial_{11} S_n(\pi(x), \pi(T_n(x)) + A^{(k)}_n. \tag{3.1}$$

Therefore, by the properties of $A^{(k)}_n$ of the previous section, the matrices W_n are well defined and satisfy the equation:

$$\lim_{k\to-\infty} W^{(k)}_n(x) = W_n(x) = -\partial_{11} S_n(\pi(x), \pi(T_n(x)) + A_n. \tag{3.2}$$

Notice that $W_n(x)$ depends measurably on x, since for every n, k, $\mathcal{W}^{(k)}_n(x)$ is a smooth field of Lagrangian subspaces. The invariance property of the fields $\mathcal{W}_n$ follows immediately from the transformation rule

$$\mathcal{W}^{(k)}_{n+1}(x) = (T_n)_* \mathcal{W}^{(k)}_n(T^{-1}_n x)$$

for $\mathcal{W}^{(k)}_n$ which is immediate from the definition. This yields the proof of theorem 1.□

Proof of Theorem 2 As in the proof of Theorem 1 consider the orbit of the point x. In order to prove the inequalities 1 and 2 of Theorem 2, we shall use strongly that all the matrices A_n are positive definite. Then (3.1), (3.2) imply

$$-\partial_{11} S_n(\pi(x), \pi(T_n(x)) \leq W_n(x). \tag{3.3}$$

And therefore

$$-\partial_{11} S_n(q, q_+) \leq W_n(x).$$

Also, using the relation (2.5), we have

$$\begin{aligned} W_{n+1}(T_n x) &= -\partial_{11} S_{n+1}(\pi(T_n x), \pi(T_{n+1} \circ T_n x)) + A_{n+1} = \\ &= -\partial_{11} S_{n+1}(\pi(T_n x), \pi(T_{n+1} \circ T_n x)) + a_{n+1} - b^T_n A^{-1}_n b_n = \\ &\partial_{22} S_n(\pi x, \pi(T_n x)) - b^T_n A^{-1}_n b_n \leq \partial_{22} S_n(\pi x, \pi(T_n x)). \end{aligned} \tag{3.4}$$

Thus we have

$$W_n(x) \leq \partial_{22} S_{n-1}(\pi(T^{-1}_{n-1} x), \pi(x)) = \partial_{22} S_{n-1}(q_-, q). \tag{3.5}$$

Notice that the inequalities (3.3) and (3.5) can be expressed geometrically by

$$(T_{n+1}^{-1})_* \mathcal{V}(T_n x) \le \mathcal{W}_n(x) \le (T_{n-1})_* \Big(\mathcal{V}(T_n^{-1} x)\Big)$$

This proves the first part of Theorem 2.

In order to prove the second part we subtract the two expressions (3.4) and (3.2) for W. We have

$$W_{n+1}(T_n x) - W_n(x) = \partial_{22} S_n(\pi x, \pi(T_n x)) + \partial_{11} S_n(\pi x, \pi(T_n x)) - A_n - b_n^T A_n^{-1} b_n \tag{3.6}$$

This can be rewritten as

$$\begin{aligned} W_{n+1}(T_n x) - W_n(x) = \partial_{11} S_n(q, q_+) + \partial_{22} S_n(q, q_+) - \\ - \left(A_n^{\frac{1}{2}} + b_n^T A_n^{-\frac{1}{2}}\right)\left(A_n^{\frac{1}{2}} + A_n^{-\frac{1}{2}} b_n\right) + b_n + b_n^T. \end{aligned} \tag{3.7}$$

Notice that the first matrix in brackets of (3.7) is the transpose of the second one and thus

$$\begin{aligned} W_{n+1}(T_n x) - W_n(x) \le \partial_{11} S_n(q, q_+) + \partial_{22} S_n(q, q_+) + b_n + b_n^T = \\ = \partial_{11} S_n(q, q_+) + \partial_{22} S_n(q, q_+) + \partial_{12} S_n(q, q_+) + \partial_{21} S_n(q, q_+). \end{aligned} \tag{3.8}$$

Moreover the inequality (3.8) is strict except when

$$A_n = -b_n = -b_n^T. \tag{3.9}$$

In the last case the expressions for $W_{n+1}(T_n x)$ and $W_n(x)$ are

$$\begin{aligned} W_{n+1}(T_n x) &= \partial_{22} S_n(q, q_+) + \partial_{12} S_n(q, q_+) \\ W_n(x) &= -\partial_{11} S_n(q, q_+) - \partial_{21} S(q, q_+). \end{aligned} \tag{3.10}$$

This finishes the proof of Theorem 2. $\square$

Proof of Theorem 3

In the case of a Frenkel-Kontorova functional we have

$$S_n(q, Q) = \frac{1}{2}(Q - q)^2 + V_n(q),$$

where V_n is periodic in q. In this case the partial derivatives of S_n are

$$\begin{aligned} \partial_{22} S_n = -\partial_{12} S_n = -\partial_{21} S_n = I \\ \partial_{11} S_n = I + Hess(V_n). \end{aligned} \tag{3.11}$$

Suppose that all the extremals of the Frenkel-Kontorova functional are without conjugate points. Then construct the fields of Lagrangian subspaces $\mathcal{W}_n$ and the corresponding matrix functions W_n as in Theorems 1,2. Let us define

$$w_n(x) = tr W_n(x)$$

then w_n is a bounded measurable function satisfying the following inequality (a consequence of Theorem 2)

$$w_{n+1}(T_n x) - w_n(x) \leq tr\left(\partial_{11} S_n(q, q_+) + \partial_{22} S_n(q, q_+) + 2\partial_{12} S_n(q, q_+)\right).$$

In other words we get the following

$$w_{n+1}(T_n x) - w_n(x) \leq \Delta V_n(q). \tag{3.12}$$

We shall see below that if all the extremals of the Frenkel-Kontorova functional have no conjugate points then for almost all x there is equality in (3.12). Therefore by Theorem 2, (3.10) holds, i.e. by the formulae (3.11)

$$W_n = -HessV_n \text{ and } W_{n+1}(T_n x) = 0.$$

In other words

$$W_n \equiv -Hess(V_n) \equiv 0$$

for all n. But then all the functions V_n are constant. This will finish the proof of Theorem 3.

In order to establish equality in (3.12) we shall consider two cases. In the first case the sequence V_n is periodic, i.e. $V_{n+p} \equiv V_n$ for some positive integer p and for all n. In the second case the sequence V_n is of compact support, i.e. $V_n \equiv const$ for $|n| > N$ for some N. Consider first the periodic case. In this case obviously $\mathcal{W}_{n+p} \equiv \mathcal{W}_n$ and thus $w_n \equiv w_{n+p}$. Now we apply (3.12) p times to obtain

$$\begin{aligned} w_{n+p}(T_{n+p-1} \circ \cdots \circ T_n x) - w_n(x) \leq \Delta V_n(\pi x) + \\ + \Delta V_{n+1}(\pi(T_n x)) + \ldots + \Delta V_{n+p-1}(\pi(T_{n+p-2} \circ \cdots \circ T_n x)). \end{aligned} \tag{3.13}$$

Let us recall the additional property of the standard maps T_n that the phase space is effectively compact (see remark in Example 1). This implies immediately that each field $\mathcal{W}_n(x) = \mathcal{W}_n(p, q)$ depends periodically on p as well as on q. Thus the function w_n is a periodic bounded function on $\mathbb{T}^{2d}$. Now we can finish the argument by the following reasoning. If there is strict inequality in (3.12) for some n on a set of positive measure then one has strict inequality in (3.13) also on a set of positive measure. But then the strict inequality remains after the integration of (3.13) over the whole phase space $\mathbb{T}^{2d}$. But this is a contradiction, because since all the transformations T_n are symplectic (and hence measure preserving) then one can easily see that the integrals of both sides of (3.13) over $\mathbb{T}^{2d}$ vanish. This finishes the proof of the claim in the periodic case.

In the second case the idea is similar. The important ingredient in its realization is the following claim. The limit

$$\lim_{n \to \pm\infty} w_n(x) = 0$$

exists and the convergence is uniform in x. In fact for those n which lie to the left of the support of V_n it easily follows from the construction that $W_n(x) = 0$ and then $w_n(x) = 0$ for all x. For large positive n we have $V_n \equiv 0$, thus the recursion rule for the matrices W_n, A_n from (2.6) and (3.2) is:

$$A_{n+1} = 2I - A_n^{-1} \text{ and } W_n = -I + A_n. \tag{3.14}$$

Then for the eigenvalues of A_n the same recursion rule holds

$$\lambda_{n+1} = 2 - \frac{1}{\lambda_n}.$$

Recall that all the matrices A_n are positive definite. Therefore all λ_n are positive and then one can easily see that the sequence λ_n is monotonically decreasing and converges to 1. Moreover, it is clear from the formula that λ_{n+1} is less than 2. Therefore, A_n converges (uniformly for all orbits) to I and thus W_n to 0. This proves the claim. In order to finish the proof of the Theorem one proceeds exactly as in the previous case. One takes N sufficiently large and sums up the inequality (3.12) from $-N$ to N. This completes the proof of theorem 4. □

4 Discussion and some open questions

1. The variational principle (1.3) can be considered on other configuration manifolds different from tori, for example on hyperbolic manifolds. It would be interesting to understand the consequences of the no conjugate points condition for these cases. Another very interesting direction would be to study, along the lines of this paper, variational principles of the form (1.3) on configurations $\{q_n\}$ for n lying on some lattice $\mathbb{Z}^k$ (see also [12]). Some results in this direction were obtained in [3] for multi-continuous-time systems.

2. An important problem is to understand to what extent the smoothness of W is required. An example of not smooth enough W would give a qualitatively new system without conjugate points.

3. The integration trick used in the proof of Theorem 3 worked well due to compactness of the phase space for the standard map. In many interesting cases, however, the phase space is not compact. Then new integral-geometric approaches are required. For example it is not clear yet how to apply this to the so-called outer billiard problem [16]. It would be reasonable to conjecture that the only outer billiards without conjugate points on the affine plane are the elliptic ones. In some cases the lack of compactness can be overcome [4, 2].

4. It was proved by J Moser [14] for area-preserving twist maps that every such map can be seen as the time-one map of an optical Hamiltonian function. This result was generalized in [5] to higher dimensions for those twist maps with symmetric matrix $\partial_{12}S$ (see [8] for the proof and discussion). It is not clear what can be said about the interpolation problem for symplectic twist maps without conjugate points. Is it true that they can be interpolated by flows without conjugate points?

5. One might prefer an extended notion of conjugate points for symplectic twist maps, which mimics more closely the properties of maps arising from optical Hamiltonian flows, by allowing a conjugate point to occur in between two integer times. To formalise this, we say that an orbit of Lagrange planes *crosses the vertical* between times n and $n+1$ if the signature of the associated quadratic form changes. Then for $m < n$ we can say time m is conjugate to $(n, n+1)$ along orbit (x_i) if the orbit of the plane which is vertical at time m crosses the vertical between times n and $n+1$. Similarly for $m > n+1$ by using the backwards dynamics. Also we can say $(m, m+1)$ is conjugate to $(n, n+1)$ if the orbit of the vertical plane at time m crosses the vertical between times $m, m+1$ and between times $n, n+1$. The definition of this paper is incorporated by saying times m and n are conjugate if the orbit of the vertical at time m has non-zero intersection with the vertical at time n. Of course, if all orbits have no conjugate points in this extended sense then they have no conjugate points in the restricted sense and hence the conclusions of the paper still follow. Advantages of the extended definition are that possession of conjugate points becomes stable and that for discretisations of an orbit of an optical Hamiltonian system its conjugate points are inherited.

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