Optimal Scheduling in a Periodic Environment

R.S.MacKay[1], S.Slijepčević[1,2], J.Stark[3]

July 1997

revised 11 October 1999

[1] The Nonlinear Centre, DAMTP, Silver Street, Cambridge, CB3 9EW, UK.

[2] Department of Mathematics, Bijenička 30, 10000 Zagreb, Croatia.

[3] Centre for Nonlinear Dynamics and its Applications, UCL, Gower Street, London, WC1E 6BT, UK.

Abstract

We consider the problem of scheduling a sequence of actions when the benefit obtained from an action depends on only the current time modulo a period and the time since the previous action. A simple example is a model of a shuttle bus service where the profit from running a bus depends on the time of day and the time since the previous bus. Mathematically, such problems can be formulated as maximising $\sum_{n=1}^{N} V(t_{n-1}, t_n)$ *over schedules* $(t_n)_{n\geq 0}$ *as* t_N *goes to infinity, for functions* $V(t,t')$ *satisfying the periodicity condition* $V(t+T, t'+T) = V(t,t')$*, for some* $T > 0$*. The problem is related to Aubry-Mather theory in the dynamics of area-preserving maps. We extend this theory in order to prove the existence of optimising schedules for each initial condition* t_0*, to characterise the properties of such schedules and to analyse their dependence on* t_0 *and on the parameters of the model.*

MSC codes: 49J99, 65K10, 90A16, 37A40

Keywords: dynamic optimisation, Aubry-Mather theory

1 Introduction

Many scheduling problems arise in a time-periodic environment, where the periodicity is for instance due to seasonal or daily factors. For example:

- Suppose you are in charge of a 24-hour shuttle bus service and the revenue you gain from running a bus depends on the time of day and the time since the previous one. How do you schedule the service to maximise your total profit?
- Or, suppose you can grow and harvest a crop any time of the year, but the yield from a harvest depends on the time of year and the time since the previous harvest. Then what is the harvesting strategy which maximises the yield over the long term?
- Similarly, suppose you can take holidays (vacations) any time of year, but the benefit you obtain from a holiday depends on the time of year and the time since your previous holiday. What is your optimal holiday schedule?

Such dynamic optimisation problems can all be expressed in the following mathematical form. A sequence of actions is to be scheduled, with the n^{th} action occurring at time t_n. The benefit or profit obtained from the n^{th} action is given by $V(t_{n-1}, t_n)$, where V is some continuous real-valued function, which we call the *profit function*. In general, the domain of V is not the whole of $\mathbb{R}^2$ and some conditions need to be placed on the set A of allowed (t, t') on which V is defined. The function V is assumed to be periodic, in the sense that for some period $T > 0$,

$$V(t+T, t'+T) = V(t, t') \tag{1}$$

for all $(t, t') \in A$. Given t_0, we wish to optimise

$$\sum_{n=1}^{N(P)} V(t_{n-1}, t_n) \tag{2}$$

as $P \to \infty$, where $N(P)$ is the largest N such that $t_N \leq P$. We will make precise below how "as $P \to \infty$" should be interpreted.

There is a well developed theory of dynamic optimisation (e.g. [CHL, KS, SL, W, YK]). The place where one might hope to find a solution to our problem is called "undiscounted infinite horizon dynamic optimisation", but there is a big difference between our problem and the one usually treated there. They wish to optimise $\sum_{n=1}^{N} V(t_{n-1}, t_n)$ as $N \to \infty$, in particular mean profit per action, whereas we wish to optimise (2) as $P \to \infty$, in particular mean profit per unit of time. The theory of chapter 5 of [CHL], for example, does not apply.

Another comment is that much of the theory (though [CHL,YK] are exceptions) is in the context of concave V. Concavity of V is incompatible with (non-trivial) periodicity (1). Under a weakened notion of concavity, however, we show that the problem (2) is a variation on a well understood problem in the theory of dynamical systems, namely to find an "Aubry-Mather set" for an "area-preserving twist map" (see [BAN] for a survey).

Aubry-Mather theory was developed in two contexts. Aubry [ALD] was interested primarily in minimum energy configurations of chains of atoms in a one-dimensional periodic potential. The relevance of this to our problem becomes readily apparent if we draw a correspondence between the position of the n^{th} atom and the time of the n^{th} action (and replace minimising by maximising). Although concentrating on configurations with minimal energy per atom, Aubry also considered those with minimal "grand potential per unit length" [A], which corresponds to our mean profit per unit time. Mather [MAT], on the other hand, was mainly concerned with the dynamics of area-preserving twist maps and existence of quasi-periodic orbits. Given that such maps are the discrete-time analogues of 1 degree of freedom Hamiltonian systems, and the close connection between Hamiltonian dynamics and the calculus of variations, it is again reasonable to envisage some relationship to our scheduling problem (see section 2.3).

We exploit and extend Aubry-Mather theory to derive some general results for the solutions of (2). In particular, under suitable hypotheses, we shall show the existence of optimising schedules for all initial conditions t_0, characterise their properties, and discuss their robustness to perturbations.

Although our primary interest is in sequences (t_n) indexed by non-negative integers n, we will also study finite sequences and doubly infinite sequences. It is necessary to choose some terminology to distinguish between these cases, and also a name to allow us to treat them simultaneously when possible.

Definition 1.1 *A* **schedule** *on a set S of consecutive integers is a function $\mathbf{t} : S \to \mathbb{R}$.*

A **sequence** *is a schedule on the whole of $\mathbb{Z}$.*

A **semi-sequence** *is a schedule on the set $\mathbb{Z}_+ = \{0, 1, 2, ...\}$.*

A **segment** *is a schedule on a finite set of consecutive integers.*

Since in general the infinite sum $\sum_{n=1}^{\infty} V(t_{n-1}, t_n)$ will not converge, the first problem is to decide what we mean by "optimising (2) as $P \to \infty$". A natural choice is to say that a schedule is optimising if every finite sub-schedule maximises the sum (2) subject to appropriate end conditions. This is formalised by the following hierarchy of definitions; our notion of "optimising (2) as $P \to \infty$" is the last of these.

First, for a segment $(t_m, ..., t_p)$ we define

$$V(t_m, ..., t_p) = \sum_{n=m+1}^{p} V(t_{n-1}, t_n),$$

or $-\infty$ if $(t_{n-1}, t_n) \notin A$ for some $n \in \{m+1, ..., p\}$.

Definition 1.2 *A schedule* $\mathbf{t}$ *is* **feasible** *if for all n such that $n-1, n \in S$, then $(t_{n-1}, t_n) \in A$.*

Definition 1.3 *A schedule* $\mathbf{t}$ *is* **maximising** *if it is feasible and for all $m, p \in S$ with $m < p$, and all segments* $\mathbf{u}$ *with $u_m = t_m$, $u_p = t_p$, then $V(t_m, ..., t_p) \geq V(u_m, ..., u_p)$. We denote the set of all maximising sequences by $\mathcal{M}$.*

Definition 1.4 *A schedule* $\mathbf{t}$ *is* **optimising** *if it is feasible and for all $m, p \in S$ with $m < p$, all $k > 0$ and all segments* $\mathbf{u}$ *on $[0, k]$ with $u_0 = t_m$, $u_k = t_p$, then $V(t_m, ..., t_p) \geq V(u_0, ..., u_k)$.*

The difference between maximising and optimising is that the latter allows us to compare a sub-segment with sub-segments containing a different number of actions, whilst the former requires the number of actions to be the same. Maximising sequences are the prime subject of interest in Aubry-Mather theory which has developed a comprehensive understanding of their properties and of the structure of the sets of maximising sequences of a given mean spacing. In the present case, however, the notion of an optimising sequence is more appropriate: if one can make more profit in a given time by taking a larger number of actions, one will do so (note that V represents the profit for a single action, and hence is assumed to incorporate the additional costs involved in taking such an action). We thus optimise (2) as $P \to \infty$ by finding an optimising semi-sequence starting from t_0.

Whilst the theory of maximising sequences for periodic profit functions is well developed, optimising sequences and semi-sequences appear to have received little attention in this context. The principal mathematical results of this paper therefore consist in extending results about maximising sequences from standard Aubry-Mather theory to optimising semi-sequences. In particular, since every optimising sequence or semi-sequence is maximising, we study how to decide which of the maximising ones are also optimising. We do this by defining the "average profit rate" of a sequence; this is closely related to the physical notion of "mean action" arising in Aubry-Mather theory, though not identical to it (see section 5.2). It turns out to be the same for all maximising sequences and semisequences with the same mean spacing. The set of optimising (semi) sequences is contained in the set of maximising ones whose mean spacing maximises the average profit rate. In the irrational case, we prove they coincide; but in the rational case, one has to examine their "relative profits" to decide whether the "advancing" or "retreating" maximising semi-sequences, or both, are optimising.

Under hypotheses (D1-4), (H1-4) of section 2, and (H5),(H6) of section 5, such characterisations then allow us to deduce the following main results for optimising semi-sequences:

1. For each $t_0 \in \mathbb{R}$ there exists an optimising semi-sequence $\mathbf{t}$.

2. Every optimising semi-sequence $\mathbf{t}$ has a "mean spacing" $\alpha \in \mathbb{R}$ (measured in units of T) such that $\frac{t_n - t_0}{nT} \to \alpha$ as $n \to \infty$, and furthermore for all $m, n \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, if $(n-m)\alpha > k$ then $t_n - t_m > kT$, and similarly if $(n-m)\alpha < k$ then $t_n - t_m < kT$.

3. All optimising semi-sequences have the same mean spacing α_M.

4. If α_M is irrational, then the set of optimising semi-sequences is closed and *totally ordered*, i.e. if $t_0 > u_0$, then for all $n > 0$, $t_n > u_n$. If one identifies semi-sequences which differ only by adding the same integer multiple of T to each t_n, then the set of *recurrent* optimising semi-sequences (i.e. semi-sequences for which there exists an $n > 0$ and a k with $t_n - kT$ arbitrarily close to t_0) is either the whole set of optimising semi-sequences (which is topologically a circle) or a Cantor set. Each optimising semi-sequence is either recurrent or asymptotic to a recurrent one, and if it is recurrent, then it is the unique optimising semi-sequence for that t_0. There are at most countably many t_0 for which the optimising semi-sequence is non-unique.

5. If α_M is rational, equal to p/q in lowest terms, then every optimising semi-sequence is periodic of type (p, q): $t_{n+q} = t_n + p$, or asymptotic to such, with $t_{n+q} > t_n + p$ ("advancing") or $t_{n+q} < t_n + p$ ("retreating"). The union of the periodic and advancing optimising semi-sequences form a closed totally ordered set, as does the union of the periodic and retreating optimising semi-sequences. If there is a periodic optimising semi-sequence for a given t_0 then it is the unique optimising semi-sequence from t_0. For the other t_0, there is a function of "relative profits" which decides whether there is an advancing or retreating optimising semi-sequence, or both. There are at most countably many t_0 for which there are more than two optimising semi-sequences.

6. If a family of profit functions V_μ depends continuously on a parameter μ then the optimal mean spacing α_M is continuous in μ.

7. If the strength of the periodic variation in V_μ is weak and the family is "transversal", then there is a set of nearly full measure of parameter values μ for which α_M is irrational and the optimising semi-sequences schedule actions densely throughout the period. If on the other hand the variation is sufficiently strong, certain intervals are excluded. In our example above, the former corresponds to a shuttle service which sends buses at a dense set of times of the day and the latter to one which shuts down during quiet periods (e.g. during the night).

In the case that (H6) is not satisfied, the average profit rate is always negative, optimal semi-sequences do not exist and the best strategy is to change business (see the comment at the end of section 5.4).

Finally note that in many works on dynamic programming problems, the future is discounted by some factor $0 \leq \lambda < 1$. This corresponds to optimising

$$\sum_{n=1}^{N(P)} \lambda^{t_n} V(t_{n-1}, t_n)$$

instead of (2). It leads to a problem about orbits of a dissipative rather than area-preserving map. Very little has been done in the context of Aubry-Mather theory for problems of this type (though M. Casdagli worked on it during his thesis), and hence it is not clear how useful the techniques developed in this paper would be in this case.

The paper is organised as follows. In section 2, we give hypotheses on the profit function and its domain A, discuss feasible schedules, introduce the "rotation interval" of the problem, and discuss stationary schedules. In section 3, we discuss maximising sequences; this is an adaptation of standard Aubry-Mather theory to the case of more general domains A. In section 4, we discuss maximising semi-sequences; this is an extension of ideas of Katok. In section 5, we introduce the average profit rate function, and the relative profit for asymptotically periodic semi-sequences, and obtain our main results on optimising semi-sequences. Robustness of optimising schedules is discussed in a brief section 6. Some illustrations are given in section 7, and the paper concludes with a discussion in section 8.

2 Hypotheses, feasible schedules and stationary schedules

2.1 The domain

We require the domain $A \subset \mathbb{R}^2$ of the function V to satisfy the following conditions:

(D1) Periodicity: There exists $T > 0$ such that $(t, t') \in A$ if and only if $(t + T, t' + T) \in A$.

(D2) Extendibility: Given any $t \in \mathbb{R}$, there exist $\underline{t}$ and $\overline{t}$ such that $(\underline{t}, t) \in A$ and $(t, \overline{t}) \in A$.

(D3) Rectangles: If $(\underline{u}, \overline{v}) \in A$ and $(\overline{u}, \underline{v}) \in A$, $\underline{u} < \overline{u}$, $\underline{v} < \overline{v}$, then $(\underline{u}, \underline{v}) \in A$ and $(\overline{u}, \overline{v}) \in A$.

(D4) Openness: A is an open subset of $\mathbb{R}^2$.

In (D1) we require A to be compatible with periodicity of V. Condition (D2) allows us to extend any feasible schedule both forwards and backwards. The rectangles condition (D3) is required for the concavity condition on V in the next subsection to be well defined. Openness of A may be stronger than necessary, but some such condition is required to obtain a non-trivial set of feasible schedules. We emphasise that our results depend crucially on the feasible schedules forming a continuum rather than a discrete space. We make the standing assumption that the domain A satisfies (D1) - (D4). However, we will sometimes emphasise which assumption is necessary to obtain particular properties of A, for the purpose of possible generalisations.

The assumptions (D1)-(D3) sound rather technical. Their most important purpose is to establish existence of periodic feasible sequences (Proposition 2.8) and therefore existence of maximising periodic sequences (a variant of the Poincaré-Birkhoff theorem). A reader not interested in technical details can assume that the domain looks like $A = \{(t, t') : t' - t \in (\rho^-, \rho^+)\}$ (which will be the most natural assumption in applications) and skip the rest of this subsection.

Definition 2.1 *Given a domain A, let*

$$\beta^-(t) = \inf\{t' : (t, t') \in A\}$$
$$\beta^+(t) = \sup\{t' : (t, t') \in A\}$$

The functions β^+ and β^- can take the values ∞ and $-\infty$ respectively, though as the next lemma shows, if this occurs at one t, then it occurs at all t.

Lemma 2.1 *The functions $\beta^\pm$ satisfy $\beta^\pm(t + T) = \beta^\pm(t) + T$ and are non-decreasing, i.e. if $t < t'$ then $\beta^\pm(t) \le \beta^\pm(t')$. If $\beta^+(t) = \infty$ for some t, then $\beta^+(t) = \infty$ for all t, and similarly if $\beta^-(t) = -\infty$ for some t, then $\beta^-(t) = -\infty$ for all t.*

Proof: The formula $\beta^\pm(t+T) = \beta^\pm(t)+T$ follows immediately from the fact that $(t, t') \in A$ if and only if $(t + T, t' + T) \in A$. To show the monotonicity of β^-, suppose that for some $t < t'$ we had $\beta^-(t) > \beta^-(t')$. Then there exist u and u' such that $\beta^-(t') < u' < \beta^-(t) < u$ and $(t, u) \in A$, $(t', u') \in A$. Hence by (D3), $(t, u') \in A$. But $u' < \beta^-(t)$, which contradicts the definition of $\beta^-(t)$. Hence $\beta^-(t) \le \beta^-(t')$ for all $t < t'$, as required. Similarly suppose that we had $\beta^+(t) > \beta^+(t')$ for some $t < t'$. Then choose u and u' such that $(t, u) \in A$, $(t', u') \in A$, but now with $\beta^+(t) > u > \beta^+(t') > u'$. Then by (D3), $(t', u) \in A$, with $u > \beta^-(t')$ which contradicts the definition of $\beta^+(t)$. The same argument shows that if $\beta^+(t) = \infty$ for some t, then $\beta^+(t') = \infty$ for all $t' > t$. Hence $\beta^+(t' - nT) = \infty$ for all $t' > t$, $n \in \mathbb{Z}$, in other words $\beta^+(t) = \infty$ for all $t \in \mathbb{R}$. A similar argument applies to β^-.

Lemma 2.2 *If A satisfies (D2) and (D3), then $(t, t') \in A$ for all t' such that $\beta^-(t) < t' < \beta^+(t)$.*

Proof: By (D2), there exists a u such that $(u, t') \in A$. If $u < t$, choose a v such that $\beta^-(t) < v < t'$ and $(t, v) \in A$. Then by (D3), $(t, t') \in A$ as required. Similarly if $u > t$, choose v so that $t' < v < \beta^+(t)$ and $(t, v) \in A$. Then again $(t, t') \in A$ by (D3).

Lemma 2.3 *If A satisfies (D2),(D3),(D4), then A is path-connected (and hence connected).*

Proof: Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the map $\pi(t, t') = t$. Let A_0 be a path-connected component of A. Suppose that there exists a $t \in \mathbb{R}$ such that $t \notin \pi(A_0)$. Let $u = \sup\{t' \in \pi(A_0) : t' < t\}$. By (D2) the image of π is the whole of $\mathbb{R}$ and hence there exists u' such that $(u, u') \in A$. Since A is open, there exists an $\epsilon > 0$ such that $(v, u') \in A$ for all $v \in (u - \epsilon, u + \epsilon)$, and furthermore all such (v, u') are in the same component of A as (u, u'). Choose $v \in (u - \epsilon, u)$ such that $v \in \pi(A_0)$, with say $(v, v') \in A_0$. Then by Lemma 2.2, (v, v') and (v, u') are in the same component of A. Hence $(v, u') \in A_0$, and hence $(v'', u') \in A_0$ for all $v'' \in (u - \epsilon, u + \epsilon)$. This contradicts the definition of u, and hence we must have $\pi(A_0) = \mathbb{R}$. Thus for any $(t, t') \in A$ there exists a u such that $(t, u) \in A_0$. Then again by Lemma (2.2), (t, t') and (t, u) are in the same component of A, i.e. $(t, t') \in A_0$. Thus $A \subset A_0$ and hence $A = A_0$, as required.

Lemma 2.4 *If A satisfies (D2), (D3), (D4) then β^- is right continuous and β^+ is left continuous.*

Proof: Assume that β^- is finite. Given an $\epsilon > 0$, we can find a t' such that $\beta^-(t) < t' < \beta^-(t) + \epsilon$ and $(t, t') \in A$. Since A is open, there exists a $\delta > 0$ such that $(u, t') \in A$ for all $u \in (t - \delta, t + \delta)$. Hence in particular $\beta^-(u) < t' < \beta^-(t) + \epsilon$ for all $u \in [t, t + \delta)$. But β^- is non-decreasing, so $\beta^-(t) \leq \beta^-(u) < \beta^-(t) + \epsilon$ for all $u \in [t, t + \delta)$, as required.

Similarly, if β^+ is finite, we choose t' such that $\beta^+(t) - \epsilon < t' < \beta^+(t)$ and $(t, t') \in A$. Then again there exists a $\delta > 0$ such that $(u, t') \in A$ for all $u \in (t - \delta, t + \delta)$, and hence $\beta^+(t) - \epsilon < t' \leq \beta^+(u)$ for all $u \in (t - \delta, t]$, and so $\beta^+(t) - \epsilon < \beta^+(u) \leq \beta^+(t)$ for all $u \in (t - \delta, t]$.

Corollary 2.5 *If A satisfies (D2), (D3), (D4), then $\beta^+(t) - \beta^-(t)$ is lower semi-continuous.*

Proof: By the above, given an $\epsilon > 0$, we can find $\delta > 0$ such that $\beta^-(t') < \beta^-(t) + \epsilon$ for all $t' \in [t, t + \delta)$ and $\beta^+(t') > \beta^-(t) - \epsilon$ for all $t' \in (t - \delta, t]$. But $\beta^\pm$ are non-decreasing and hence $\beta^+(t') \geq \beta^-(t)$ for all $t' \in [t, t + \delta)$ and $\beta^-(t') \leq \beta^-(t)$ for all $t' \in (t - \delta, t]$. Hence for all $t' \in [t, t + \delta)$ we have $\beta^+(t') - \beta^-(t') \geq \beta^-(t) - (\beta^-(t) + \epsilon) = -\epsilon$. Similarly, for $t' \in (t - \delta, t]$ we have $\beta^+(t') - \beta^-(t') \geq \beta^-(t) - \epsilon - \beta^-(t) = -\epsilon$.

Corollary 2.6 *The function $\beta^+(t) - \beta^-(t)$ is bounded away from 0, i.e. there exists a constant $c > 0$ such that $\beta^+(t) - \beta^-(t) > c$ for all $t \in \mathbb{R}$.*

Proof: A lower semi-continuous function attains its lower bound on a compact domain. Thus since $\beta^+(t) - \beta^-(t) > 0$ for all $t \in [0,T]$, there must be a minimum $2c > 0$ such that $\beta^+(t) - \beta^-(t) \geq 2c > c$ for each $t \in [0,T]$. But $\beta^+(t+T) - \beta^-(t+T) = \beta^+(t) - \beta^-(t) > c$ for all $t \in \mathbb{R}$.

We now begin the study of feasible schedules. Given $t_0 \in \mathbb{R}$, denote $t_n^\pm = (\beta^\pm)^n(t_0)$. Note that $t_{n+1}^\pm = \beta^\pm(t_n^\pm)$.

Lemma 2.7 *Given any $t_0 \in \mathbb{R}$, suppose that $t_n \in (t_n^-, t_n^+)$. Then there exists a sequence $t_0, ..., t_n$ such that $(t_{i-1}, t_i) \in A$ for $i = 1, ..., n$.*

Proof: By induction. The statement obviously holds for $n = 1$. Suppose it holds for some $n \geq 1$, and choose any $t_{n+1} \in (t_{n+1}^-, t_{n+1}^+)$.

(i) First suppose that $t_{n+1} \in (t_{n+1}^-, \beta^+(t_n^-))$. Then $(t_n^-, t_{n+1}) \in A$, and since A is open, $(t_n^- + \epsilon, t_{n+1}) \in A$ for sufficiently small $\epsilon > 0$. Let $t_n = t_n^- + \epsilon$ with ϵ sufficiently small so that $t_n < t_n^+$. Then by our induction hypothesis there exists a sequence $t_0, ...t_n$ such that $(t_{i-1}, t_i) \in A$ for $i = 1, ..., n$. But $(t_n, t_{n+1}) \in A$, thus completing the inductive step.

(ii) Similarly if $t_{n+1} \in (\beta^-(t_n^+), t_{n+1}^+)$ then $(t_n^+, t_{n+1}) \in A$, and since A is open, $(t_n^+ - \epsilon, t_{n+1}) \in A$ for sufficiently small $\epsilon > 0$. Let $t_n = t_n^+ + \epsilon$ and proceed as before.

(iii) This leaves the case $t_{n+1} \in [\beta^+(t_n^-), \beta^-(t_n^+)]$, (which is only non-trivial if $\beta^+(t_n^-) \leq \beta^-(t_n^+)$. Choose any $u^- \in (t_{n+1}^-, \beta^+(t^-(n))$ and $u^+ \in (\beta^-(t^+(n), t_{n+1}^+)$. Then $(t_n^-, u^-), (t_n^+, u^+) \in A$ and since A is path connected, there is a path in A joining (t_n^-, u^-) to (t_n^+, u^+). Since $u^- < \beta^+(t_n^-) \leq t_{n+1} \leq \beta^-(t_n^+) < u^+$, this path must intersect the horizontal $\mathbb{R} \times t_{n+1}$. Let (t_n, t_{n+1}) be any point of intersection; note that $(t_n, t_{n+1}) \in A$. Then we must have $t_n^- < t_n < t_n^+$ since $\beta^+(t) < t_{n+1}$ for all $t \leq t_n^-$ and $\beta^-(t) > t_{n+1}$ for all $t \geq t_n^+$. Hence by our induction hypothesis there exists a sequence $t_0, ..., t_n$ such that $(t_{i-1}, t_i) \in A$ for $i = 1, ..., n$. But we have already shown that $(t_n, t_{n+1}) \in A$, completing the inductive step.

Recall that if β is the lift of a homeomorphism of the circle (that means a homeomorphism of $\mathbb{R}$ which satisfies $\beta(t+T) = \beta(t) + T$), then it has a unique rotation number, independent of t_0, defined by

$$\rho = \lim_{n \to \infty} \frac{t_n - t_0}{nT},$$

where $t_n = \beta^n(t_0)$ is the n^{th} iterate of t_0 (e.g. [KH]). This still holds if β is merely non-decreasing, but not necessarily continuous [RT1]. So in particular $\beta^\pm$ have well defined rotation numbers $\rho^\pm$. Note that despite the fact that $\beta^+ > C + \beta^-$, we can have $\rho^+ = \rho^-$. However, if either ρ^- or ρ^+ is irrational then $\rho^- < \rho^+$ [RT2].

Definition 2.2 *The open interval (ρ^-, ρ^+), or single point $\rho^\pm$ if equal, is called the* **rotation interval**.

We now construct periodic feasible sequences.

Definition 2.3 *A sequence* $\mathbf{t}$ *is* **periodic** *with period $(p, q) \in \mathbb{N}^2$ if for each $n \in \mathbb{Z}$, $t_{n+q} = t_n + pT$.*

Corollary 2.8 *If $\rho^- < \rho^+$ and $\frac{p}{q}$ is such that $\rho^- < \frac{p}{q} < \rho^+$, then for any $t_0 \in \mathbb{R}$ there exists a sequence $t_0, ..., t_q$ with $t_q = t_0 + pT$ and $(t_{i-1}, t_i) \in A$ for $i = 1, ...q$.*

Proof: If $\rho^- < \frac{p}{q} < \rho^+$, then $t_q^- < t_0 + pT < t_q^+$ for all $t_0 \in \mathbb{R}$ [RT2], so apply Lemma 2.7.

Lemma 2.9 *If $\rho^+ = \rho^- = \frac{p}{q}$, then there exists t_0 such that $t_q^- < t_0 + pT < t_q^+$ and hence there exists a sequence $t_0, ..., t_q$ with $t_q = t_0 + pT$ and $(t_{i-1}, t_i) \in A$ for $i = 1, ..., q$.*

Proof: If $\rho^+ = \rho^- = \frac{p}{q}$, then $t_q^- \leq t_0 + pT \leq t_q^+$ [RT2]. By Corollary 2.6, $\beta^+(t) > \beta^-(t) + c$ for all $t \in \mathbb{R}$. Suppose that $t_n^+ > t_n^-$ for some $n \in \mathbb{N}$. Then $t_{n+1}^+ = \beta^+(t_n^+) > \beta^-(t_n^+) + c \geq \beta^-(t_n^-) + c = t_{n+1}^- + c$. Since $t_1^+ > t_1^- + c$, we have $t_n^+ \geq t_n^- + c$ for all $n \in \mathbb{N}$ and $t_0 \in \mathbb{R}$.

We aim to show that there exists a t_0 such that $t_0 + pT - c < t_q^- \leq t_0 + pT - \frac{c}{2}$, since then $t_q^+ > t_q^- + c > t_0 + pT$, as required. So suppose not. Then for all $t_0 \in \mathbb{R}$ we have either $t_q^- > t_0 + pT - \frac{c}{2}$ or $t_q^- \leq t_0 + pT - c$. Both of these must occur for some t_0, for suppose we had $t_q^- \geq t_0 + pT - \frac{c}{2}$ for all $t_0 \in \mathbb{R}$. Then $t_q^+ > t_q^- + c \geq t_0 + pT + \frac{c}{2}$, or in other words $(\beta^+)^q(t_0) > t_0 + pT + \frac{c}{2}$ for all $t_0 \in \mathbb{R}$. Hence for any $n \geq 1$ we have $(\beta^+)^{nq}(t_0) > (\beta^+)^{(n-1)q}(t_0) + pT + \frac{c}{2}$, and thus by induction $(\beta^+)^{nq}(t_0) > t_0 + npT + n\frac{c}{2}$. Thus $\rho^+ > \frac{p}{q} + \frac{c}{2}$, which is a contradiction. Similarly we cannot have $t_q^- \leq t_0 + pT - c$ for all $t_0 \in \mathbb{R}$, as this would imply $\rho^- \leq \frac{p}{q} - c$. We can thus find $u, v \in \mathbb{R}$ such that $(\beta^-)^q(u) > u + pT - \frac{c}{2}$ and $(\beta^-)^q(v) \leq v + pT - c$. Since $(\beta^-)^q(t + nT) = (\beta^-)^q(t) + nT$ for all $n \in \mathbb{Z}$, we may assume that $u < v < u + T$. Let $u' = \sup\{t \in [u, v] : (\beta^-)^q(t) > t + pT - \frac{c}{2}\}$. Then $(\beta^-)^q(u') \leq u' + pT - \frac{c}{2}$, since if $(\beta^-)^q(u') > u' + pT - \frac{c}{2}$, then for any $\epsilon > 0$ we have $(\beta^-)^q(u' + \epsilon) \geq (\beta^-)^q(u') > u' + pT - \frac{c}{2}$ and hence for all sufficiently small $\epsilon > 0$ we have $(\beta^-)^q(u' + \epsilon) > u' + \epsilon + pT - \frac{c}{2}$ which contradicts the definition of u'. Similarly if $(\beta^-)^q(u') \leq u' + pT - c$, then $(\beta^-)^q(u' - \epsilon) \leq (\beta^-)^q(u') \leq u' + pT - c$, and hence for all sufficiently small $\epsilon > 0$ we have $(\beta^-)^q(u' - \epsilon) < u' + pT - \frac{c}{2}$, which is again a contradiction. Hence $u' + pT - c < (\beta^-)^q(u') \leq u' + pT - \frac{c}{2}$, as required.

Let Ω be the space of feasible sequences:

$$\Omega = \{\mathbf{t} \in \mathbb{R}^{\mathbb{Z}} : (t_{n-1}, t_n) \in A,\ n \in \mathbb{Z}\}$$

By the above results, Ω is non-empty and always contains at least one periodic sequence $\mathbf{t}$.

Lemma 2.10 *If $\mathbf{t} \in \Omega$, then*

$$\rho^- \leq \liminf_{|n|\to\infty} \frac{t_n - t_0}{nT} \leq \limsup_{|n|\to\infty} \frac{t_n - t_0}{nT} \leq \rho^+. \tag{3}$$

In particular, if $\rho^+ = \rho^- = \frac{p}{q}$, then every feasible sequence has mean spacing $\frac{p}{q}$.

Proof: Since $(t_{i-1}, t_i) \in A$ for each i, we have $\beta^-(t_{i-1}) < t_i < \beta^+(t_{i-1})$. Applying induction and monotonicity of $\beta^\pm$ we get $(\beta^-)^i(t_0) < t_i < (\beta^+)^i(t_0)$ for each $i > 0$. Hence:

$$\frac{(\beta^-)^i(t_0) - t_0}{nT} < \frac{t_n - t_0}{nT} < \frac{(\beta^+)^i(t_0) - t_0}{nT}. \tag{4}$$

Applying lim inf to the left, and lim sup to the right inequality, we prove the claim. The proof for negative i is analogous.

2.2 Hypotheses on V

We assume that the profit function V is defined $V : A \to \mathbb{R}$, and continuous. Furthermore, we assume the following four conditions.

(H1) Periodicity: For all $(t,t') \in A$: $V(t+T,t'+T) = V(t,t')$.

(H2) Boundary: V converges to $-\infty$ as (t,t') tends to the boundary of A. If the set $\{u = t-t' : (t,t') \in A\}$ is unbounded (i.e. β^+ or β^-, or both, are not finite), we require more: $\lim_{|u|\to\infty} V(t,t+u) = -\infty$ uniformly in t.

(H3) Concavity: If $(\underline{u},\overline{v}) \in A$ and $(\overline{u},\underline{v}) \in A$, $\underline{u} < \overline{u}$, $\underline{v} < \overline{v}$, then $V(\underline{u},\overline{v}) + V(\overline{u},\underline{v}) < V(\underline{u},\underline{v}) + V(\overline{u},\overline{v})$.

(H4) Transversality: If t maximises both $V(u,t)+V(t,v)$ and $V(u',t)+V(t,v')$, where either $u \neq u'$ or $v \neq v'$, then $(u-u')(v-v') < 0$.

Note that the right hand side in (H3) is well defined because of property (D2). In the case $A = \mathbb{R}^2$ the conditions above are the same as those in [BAN]. These are reasonable assumptions in the context of the three examples in the Introduction. For example, (H3) says that if you have two schedules $\mathbf{t}$ and $\mathbf{s}$ (possibly with different initial conditions) for which one overtakes the other at some time, i.e. $t_n < s_m$ and $s_{m+1} < t_{n+1}$ for some m, n, then the sum of the values of the two schedules is increased by switching them at that time.

In the standard Aubry-Mather theory, if V is C^2, it is sufficient to assume that

$$\frac{\partial^2 V}{\partial t \partial t'} > 0. \tag{5}$$

Proposition 2.11 *If the map V is C^2 and satisfies the twist condition (5), then it satisfies conditions (H3) and (H4).*

Proof: (i) By direct calculation we obtain

$$V(\underline{u},\underline{v}) + V(\overline{u},\overline{v}) - V(\underline{u},\overline{v}) - V(\overline{u},\underline{v}) = \int_{\underline{u}}^{\overline{u}} \int_{\underline{v}}^{\overline{v}} V_{12}(t,t')dtdt' > 0.$$

(ii) Assume that (u,t,v) and (u',t,v') are maximising. Then we have $V_2(u,t) + V_1(t,v) = 0$ and $V_2(u',t) + V_1(t,v') = 0$. The twist condition implies that functions $V_2(\cdot,t)$ and $V_1(t,\cdot)$ are strictly increasing. Hence $V_2(u,t) < V_2(u',t)$ is equivalent to $u < u'$, $V_1(t,v) > V_1(t,v')$ and $v > v'$.

Readers familiar with the theory of billiards may be concerned that the profit function for that problem (chord length) does not satisfy our hypotheses. The domain is the closed strip $0 \leq t' - t \leq 1$ and $V = 0$ on the boundaries. But it can be included in our framework by extending V to a larger domain and then restricting attention to rotation numbers in [0,1].

2.3 Area-preserving maps

When V is C^1 we may differentiate (2) with respect to t_n and we obtain the *Euler equation*

$$V_2(t_{n-1}, t_n) + V_1(t_n, t_{n+1}) = 0 \tag{6}$$

where subscripts 1 and 2 refer to partial derivatives with respect to the first and second argument, respectively. In the C^1 case this is a necessary condition for $\mathbf{t}$ to be an optimum in any sense at all, otherwise a small change in t_n can be made to increase the payoff for the period t_{n-1} to t_{n+1} (A is assumed open).

Definition 2.4 *A schedule* $\mathbf{t}$ *is* **stationary** *if for all* n *such that* $n-1, n, n+1 \in S$*, the derivative of (2) with respect to* t_n *exists and is zero, i.e.* $V_2(t_{n-1}, t_n) + V_1(t_n, t_{n+1}) = 0$.

Under the condition (5) with $V \in C^2$, the map $V_1(x, \cdot)$ is strictly monotone, so for fixed t_{n-1} and t_n, equation (6) has at most one solution. Therefore it defines a mapping:

$$f : (t_{n-1}, t_n) \mapsto (t_n, t_{n+1})$$

provided $-V_2(t_{n-1}, t_n)$ is in the range of $V_1(t_n, \cdot)$. It preserves the area $V_{12}(t_{n-1}, t_n) dt_{n-1} \wedge dt_n$ and commutes with the translation $(t_{n-1}, t_n) \mapsto (t_{n-1} + T, t_n + T)$. It is conventional to introduce a "momentum" coordinate $p_n = V_2(t_{n-1}, t_n)$; then the induced mapping $(t_n, p_n) \mapsto (t_{n+1}, p_{n+1})$ preserves the standard area $dp \wedge dt$.

The optimisation problem becomes one of selecting a particular orbit of this area-preserving map. Although this viewpoint is not essential (indeed the optimisation theory we will give does not require $V \in C^1$), it can be helpful in smooth examples. Apart from the possibility that the domain and the range of the map f need not be the same, this is the standard setting for Aubry-Mather theory. It is easy to construct examples with a non-invariant domain.

Example: Take the profit function $V(x, y) = \log(y - x) - (y - x) + 2C\pi \sin(2\pi x)$, on the domain $A = \{(x, y) : y > x\}$. Then V and A satisfy the axioms (see Section 6 for a detailed discussion of similar families of functions). Choose $C > 1$. Then equation (6) gives:

$$\frac{1}{x_2 - x_1} - 1 = \frac{1}{x_3 - x_2} - 1 + C \cos 2\pi x_2. \tag{7}$$

Taking $(x_1, x_2) = (1, 2) \in A$, we obtain $x_3 - x_2 = \frac{1}{1-C} < 0$, so $(x_2, x_3) \notin A$.

3 Maximising sequences

The concavity assumptions (H3), (H4) and the periodicity condition (H1) are just the ingredients required for Aubry-Mather theory. This is normally presented in the case of a smooth V defined on the whole of $\mathbb{R}^2$, but the results still hold under the weaker assumptions above. For a thorough presentation of Aubry-Mather theory see [BAN] (or earlier unpublished notes [MS] or more recent unpublished notes [BAE]).

Again, Ω will be the space of all feasible sequences $\Omega = \{\mathbf{t} \in \mathbb{R}^{\mathbb{Z}} : (t_{n-1}, t_n) \in A, \forall n \in \mathbb{Z}\}$. Recall that by Corollary 2.8, Ω is not empty and always contains at least one periodic sequence $\mathbf{t}$ (such that $\exists p \in \mathbb{Z}, q \in \mathbb{N}$ with $t_{n+q} = t_n + pT$).

We will extend the function V to $V : \mathbb{R}^2 \to [-\infty, \infty)$, defining $V(t,t') = -\infty$ if $(t,t') \notin A$. We assume the standard topology on $[-\infty, \infty)$ (generated by the open intervals of $\mathbb{R}$ and the sets of the form $[-\infty, x)$, $x \in \mathbb{R}$) and ordering $-\infty < x$ for each $x \in \mathbb{R}$. V then remains continuous. As in the introduction, we will use the following notation:

$$V(t_m, t_{m+1}, ..., t_{n-1}, t_n) = \sum_{i=m+1}^{n} V(t_{i-1}, t_i).$$

We define $V(t_m, t_{m+1}, ..., t_{n-1}, t_n) = -\infty$ if the segment $(t_m, t_{m+1}, ..., t_{n-1}, t_n)$ is not feasible (if for some $m \leq k < n$, $(t_k, t_{k+1}) \notin A$).

We will need the following definitions:

Definition 3.1 *The translation $T_{(m,n)} : \Omega \to \Omega$ is defined by:*

$$T_{(m,n)}(\mathbf{t}) = \mathbf{t}^*, \;\; t_i^* = t_{i-m} + nT.$$

The sequence $\mathbf{t}$ *is* **rotationally ordered** *(or* **monotone***) if: $t_i < t_j + pT$ if and only if $t_{i+1} < t_{j+1} + pT$.*

Translations map feasible sequences to feasible sequences because of the periodicity condition. Note that the sequence $\mathbf{t}$ is rotationally ordered if and only if for all $a, b \in \mathbb{Z}^2$, $\mathbf{t}$ and $\mathbf{t}^* = T_{(a,b)}(\mathbf{t})$ are comparable in the partial order defined by the standard order on every component (comparable means that either $\mathbf{t} < \mathbf{t}^*$ or $\mathbf{t} = \mathbf{t}^*$ or $\mathbf{t} > \mathbf{t}^*$).

Definition 3.2 *Sequences* $\mathbf{t}$ *and* $\mathbf{t}^*$ **cross**

(i) at $i \in \mathbb{Z}$ if $t_i = t_i^*$ and $(t_{i-1} - t_{i-1}^*)(t_{i+1} - t_{i+1}^*) < 0$.

(ii) between i and $i+1$ if $(t_i - t_i^*)(t_{i+1} - t_{i+1}^*) < 0$.

By the transversality condition (H4), two maximising sequences either cross or are comparable.

In the rest of the paper we ignore the degenerate case when the rotation interval is a single point $\rho^{\pm}$ (the main results are true in that case too, but some proofs require slight modification).

3.1 Periodic maximising sequences

Given $p \in \mathbb{Z}, q \in \mathbb{N}$ such that $p/q \in (\rho^-, \rho^+)$, let $\Omega_{(p,q)} = \{\mathbf{t} \in \Omega \; : \; t_{n+q} = t_n + pT, \, n \in \mathbb{Z}\}$. By Corollary (2.8), $\Omega_{(p,q)}$ is non-empty. Define $W_{(p,q)} : \Omega_{(p,q)} \to \mathbb{R}$ by:

$$W_{(p,q)}(\mathbf{t}) = \sum_{i=1}^{q} V(t_{i-1}, t_i). \tag{8}$$

Proposition 3.1 *If $\Omega_{(p,q)}$ is non-empty, there exists a $\mathbf{t} \in \Omega_{(p,q)}$ which maximises $W_{(p,q)}$.*

Proof: We can write $W_{(p,q)}(\mathbf{t}) = W_{(p,q)}(t_0, ..., t_{q-1})$, taking $t_q = t_0 + pT$. Because of periodicity, we can assume that $t_0 \in [0,T]$. Since $\Omega_{(p,q)}$ is non-empty, let $\mathbf{t}^* \in \Omega_{(p,q)}$ and $C = W_{(p,q)}(\mathbf{t}^*)$. The following three cases exhaust the possibilities:

(i) Assume that β^+ is finite. If for some $t \in [0,T]$ we have $(t,t') \in A$, then $t' < \beta^+(t) \leq \beta^+(T)$, and so $t - t' > -\beta^+(T)$. By the periodicity condition (D1), we have for each $(t,t') \in A$,

$$t - t' > -\beta^+(T). \tag{9}$$

Now if $\mathbf{t} \in \Omega_{(p,q)}$, we have $t_1 < t_0 + \beta^+(T) \leq T + \beta^+(T)$, $t_2 < t_1 + \beta^+(T) < T + 2\beta^+(T)$,..., $t_{q-1} < T + (q-1)\beta^+(T)$. Equivalently, we have $t_{q-1} > t_q - \beta^+(T) \geq pT - \beta^+(T)$,..., $t_1 > pT - (q-1)\beta^+(T)$. Define $\tilde{\Omega}_{(p,q)} = [0,T] \times [pT - (q-1)\beta^+(T), T + \beta^+(T)] \times ... \times [pT - \beta^+(T), T + (q-1)\beta^+(T)]$. Thus if $\mathbf{t} \in \Omega_{(p,q)}$, then $\mathbf{t} \in \tilde{\Omega}_{(p,q)}$, and if we define $W_{(p,q)}(\mathbf{t}) = -\infty$ for those $\mathbf{t} \in \tilde{\Omega}_{(p,q)}$ that are not feasible, then by condition (H2) $W_{(p,q)}$ is a continuous function from $\tilde{\Omega}_{(p,q)}$ to $[-\infty, \infty)$. Since $\tilde{\Omega}_{(p,q)}$ is compact, there exists a $\mathbf{t} \in \tilde{\Omega}_{(p,q)}$ which maximises $W_{(p,q)}$. Since $W_{(p,q)}(\mathbf{t}) \geq C$, $\mathbf{t}$ must be a feasible sequence and hence $\mathbf{t} \in \Omega_{(p,q)}$.

(ii) Assume that β^- is finite. If $t \in [0,T]$ and $(t,t') \in A$, then $t' > \beta^-(t) \geq \beta^-(0)$, and because of periodicity we have for each $(t,t') \in A$

$$t - t' < T - \beta^-(0). \tag{10}$$

In a similar fashion to case (i) we define $\tilde{\Omega}_{(p,q)} = [0,T] \times [-T - \beta^-(0), (p+2)T + (q-1)\beta^-(0)] \times ... \times [-(q-1)T - (q-1)\beta^-(0), (p+2)T + \beta^-(0)]$, and again we can extend $W_{(p,q)}$ to a continuous function $W_{(p,q)} : \tilde{\Omega}_{(p,q)} \to [-\infty, \infty)$. Since $\tilde{\Omega}_{(p,q)}$ is compact, there exists a $\mathbf{t} \in \tilde{\Omega}_{(p,q)}$, that is feasible and maximises $W_{(p,q)}$.

(iii) If both β^-, β^+ are infinite, $A = \mathbb{R}^2$. Condition (H2) (uniform convergence on the boundary) now implies that the set of sequences $\mathbf{t} \in \Omega_{(p,q)}$, $t_0 \in [0,T]$, and $W_{(p,q)}(\mathbf{t}) \geq C$ is compact (see [BAN]). Hence $W_{(p,q)}$ reaches a maximum value for some $\mathbf{t} \in \Omega_{(p,q)}$.

Proposition 3.2 *Suppose that $u_m, ..., u_n$ is a feasible segment. Then there exists a segment $t_m, ..., t_n$ with $t_m = u_m$, $t_n = u_n$ which maximises $V(t_m, ..., t_n)$ with respect to all variations fixing t_m and t_n.*

Proof: As in the proof of Proposition 3.1, we construct a compact $\tilde{\Omega} \subset \mathbb{R}^{n-m-1}$, containing $(u_{m+1}, ..., u_{n-1})$, where all interesting segments must lie. Since $\tilde{\Omega}$ is compact, $V(t_m, ..., t_n)$ reaches a maximum value for some segment $(u_m, t_{m+1}, ..., t_{n-1}, u_n)$.

We can now apply the standard techniques of Aubry-Mather theory. We will prove in full detail only the changes required to take account of the fact that A might not be the whole of $\mathbb{R}^2$. For the rest, the reader is referred to [BAN].

Proposition 3.3 Aubry's Fundamental Lemma*[ALD]: Let $(u_m, ..., u_n)$ and $(v_m, ..., v_n)$ be two maximising segments. Then they cross at most once between m and n.*

Proof: If $u_j < v_j$, $u_{j+1} > v_{j+1}$, $u_k > v_k$, $u_{k+1} < v_{k+1}$, then by the condition (D2), $(u_j, v_{j+1}) \in A$, $(v_j, u_{j+1}) \in A$, $(u_k, v_{k+1}) \in A$, $(u_k, v_{k+1}) \in A$, and the functions below are well defined. Using the concavity condition (H3) we have:

$$\begin{aligned} V(u_j, v_{j+1}, ..., v_k, v_{k+1}) + V(v_j, u_{j+1}, ..., u_k, v_{k+1}) &= V(u_j, v_{j+1}) + V(v_{j+1}, ..., v_k) + V(v_k, u_{k+1}) \\ &\quad + V(v_j, u_{j+1}) + V(u_{j+1}, ..., v_k) + V(u_k, v_{k+1}) \\ &> V(v_j, v_{j+1}, ...v_{k+1}) + V(u_j, u_{j+1}, ..., u_{k+1}). \end{aligned}$$

This contradicts the fact that both of the segments $(u_m, ..., u_n)$ and $(v_m, ...v_n)$ are maximising. The above argument also works if one crossing takes place at an integer. If both crossings take place at an integer, then use (H4) instead of (H3).

Corollary 3.4 *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are maximising sequences, they cross at most once.*

Recall that we denote the set of maximising sequences by $\mathcal{M}$.

Corollary 3.5 *If* $\mathbf{u} \in \mathcal{M}$ *and* $\mathbf{v} \in \mathcal{M}$ *are periodic with the same period* (p, q)*, then* $\mathbf{u}$ *and* $\mathbf{v}$ *do not cross. If* $\mathbf{u} \in \mathcal{M}$ *is periodic with minimal period* (p, q)*, then* q *and* p *are relatively prime.*

Proof: [BAN], Corollary (3.2).

Theorem 3.6 *If* $\mathbf{t} \in \Omega_{(p,q)}$*,* $\mathbf{t}^* \in \Omega_{(p,q)}$*, maximise* $W_{(p,q)}$*, they do not cross.*

If $\mathbf{t} \in \Omega_{(p,q)}$ *maximises* $W_{(p,q)}$*, then it is rotationally ordered (i.e. does not cross any of its translates* $T_{(a,b)}(\mathbf{t})$ *).*

There exists $\mathbf{t} \in \mathcal{M}$ *periodic with period* (p, q)*.*

Proof: Assume that $\mathbf{t} \in \Omega_{(p,q)}$, $\mathbf{t}^* \in \Omega_{(p,q)}$ which maximise $W_{(p,q)}$ cross. We define the periodic sequences $\mathbf{t}^- \in \mathbb{R}^{\mathbb{Z}}$, $\mathbf{t}^+ \in \mathbb{R}^{\mathbb{Z}}$ by $t_i^- = \min\{t_i, t_i^*\}$ and $t_i^+ = \max\{t_i, t_i^*\}$. As in the proof of Proposition 3.3, there are two cases. If $\mathbf{t}$ and $\mathbf{t}^*$ cross between some i and $i+1$, then $W_{(p,q)}(\mathbf{t}^+) + W_{(p,q)}(\mathbf{t}^-) < W_{(p,q)}(\mathbf{t}) + W_{(p,q)}(\mathbf{t}^*)$ (a consequence of (H3)), which is a contradiction. If $\mathbf{t}$ and $\mathbf{t}^*$ cross at $i \in \mathbb{Z}$, then $W_{(p,q)}(\tilde{\mathbf{t}}^+) + W_{(p,q)}(\tilde{\mathbf{t}}^-) < W_{(p,q)}(\mathbf{t}) + W_{(p,q)}(\mathbf{t}^*)$ (a consequence of (H4), $\tilde{\mathbf{t}}^+, \tilde{\mathbf{t}}^- \in \Omega_{(p,q)}$ coincide with $\mathbf{t}^-$, $\mathbf{t}^+$ except for indices $nq + i$, $n \in \mathbb{N}$), again a contradiction.

This implies that $\mathbf{t} \in \Omega_{(p,q)}$ which maximises $W_{(p,q)}$ does not cross any of its translates. This fact and Corollary (3.5) imply that every maximum $\mathbf{t}$ of $W_{(p,q)}$ is also a maximum of $W_{(p^*,q^*)}$ for every $(p^*, q^*) = n(p, q)$. It follows that a periodic $\mathbf{t}$ which maximises $\Omega_{(p,q)}$ is a maximising sequence. (Detailed proof: [BAN], Theorem (3.3)).

3.2 Mean spacing of maximising sequences

We continue applying the standard techniques of Aubry-Mather theory to prove that each maximising sequence $\mathbf{t}$ has a well defined *mean spacing*

$$\lim_{|n| \to \infty} \frac{t_n - t_0}{nT}. \tag{11}$$

The mean spacing is normalised to the period T. In the example from the introduction it is the reciprocal of the mean frequency of bus departures (normalised to the period T).

We will use the following notions of asymptoticity (equivalent to analogous notions for the corresponding area-preserving maps):

Definition 3.3 *Sequences* $\mathbf{t}$ *and* $\mathbf{t}^*$ *are:*

α-asymptotic if $\lim_{i\to-\infty}|t_i - t_i^*| = 0$;

ω-asymptotic if $\lim_{i\to\infty}|t_i - t_i^*| = 0$.

A sequence $\mathbf{t}$ *is ω-asymptotic to the boundary, if* $\lim_{n\to\infty} d((t_n, t_{n+1}), B) = 0$*, where B denotes the boundary of the domain A, and B is not empty.*

A sequence $\mathbf{t}$ *is α-asymptotic to the boundary if* $\lim_{n\to-\infty} d((t_n, t_{n+1}), B) = 0$.

A sequence is asymptotic to the boundary if it is α- or ω- asymptotic to the boundary.

Denote the set of maximising sequences (with product topology), not asymptotic to the boundary by $\mathcal{M}^*$.

Remark: Since the average profit rate (to be defined in Section 5) of the sequences asymptotic to the boundary will be $-\infty$, we will be able to exclude them from future considerations: they can never be optimising.

Lemma 3.7 *Suppose that* $\mathbf{t}$ *and* $\mathbf{t}^*$ *are in $\mathcal{M}$, α- or ω-asymptotic, not asymptotic to the boundary, and that $|t_{i+1} - t_i|$ is bounded for all $i \le 0$ (resp. $i \ge 0$). Then* $\mathbf{t}$ *and* $\mathbf{t}^*$ *do not cross.*

We will see later that for maximising sequences, $|t_{i+1} - t_i|$ is always bounded.

Proof: Assume that $\mathbf{t}$ and $\mathbf{t}^*$ are in $\mathcal{M}$, ω-asymptotic and that they cross between i and $i+1$. Because of (H3),

$$V(t_i, t_{i+1}^*) + V(t_i^*, t_{i+1}) > V(t_i, t_{i+1}) + V(t_i^*, t_{i+1}^*). \tag{12}$$

Since $\mathbf{t}$ and $\mathbf{t}^*$ are not ω-asymptotic to the boundary, there exists a sequence $n_j \to \infty$ and $\epsilon > 0$ such that $d((t_{n_j}, t_{n_j+1}), B) > 2\epsilon$. Then for each $d((t_{n_j}, t_{n_j+1}), (t_{n_j}^*, t_{n_j+1}^*)) < \epsilon$ both $(t_{n_j}^*, t_{n_j+1})$ and $(t_{n_j}, t_{n_j+1}^*)$ are in the domain, and are at least ϵ from the boundary. The periodicity condition implies:

$$|V(t_{n_j}^*, t_{n_j+1}) - V(t_{n_j}^*, t_{n_j+1}^*)| = |V(t_{n_j}^* - k_jT, t_{n_j+1} - k_jT) - V(t_{n_j}^* - k_jT, t_{n_j+1}^* - k_jT)| \tag{13}$$

for some integer k_j such that $0 \le t_{n_j} - k_jT < T$. By our hypothesis, $|t_{n_j+1} - k_jT|$ and $|t_{n_j+1}^* - k_jT|$ are bounded by some $K > 0$ for j large enough. Let $C = \{(t,t') \in A : |t| \le K, |t'| \le K, d((t,t'), B) \ge \epsilon\}$, where B denotes the boundary of the domain A. The set C is bounded and closed, so is compact. V is defined on C, so is uniformly continuous. Since $\mathbf{t}$ and $\mathbf{t}^*$ are ω-asymptotic, we have $\lim_{j\to\infty}|t_j - t_j^*| = 0$, and hence because of (13) and uniform continuity of V on C we obtain

$$\lim_{j\to\infty}|V(t_{n_j}^*, t_{n_j+1}) - V(t_{n_j}^*, t_{n_j+1}^*)| = 0, \tag{14}$$

and in the same way,

$$\lim_{j\to\infty}|V(t_{n_j}, t_{n_j+1}^*) - V(t_{n_j}, t_{n_j+1})| = 0. \tag{15}$$

Maximality of $(t_i, ..., t_{j+1})$ and $(t_i^*, ..., t_{j+1}^*)$ implies (for $j > N$ again large enough so that everything is well defined):

$$V(t_i, t_{i+1}^*) + V(t_{i+1}^*, ..., t_j^*) + V(t_j^*, t_{j+1}) \le V(t_i, t_{i+1}) + V(t_{i+1}, ..., t_j) + V(t_j, t_{j+1}), \tag{16}$$

and

$$V(t_i^*, t_{i+1}) + V(t_{i+1}, ...t_j) + V(t_j, t_{j+1}^*) \leq V(t_i^*, t_{i+1}^*) + V(t_{i+1}^*, ...t_j^*) + V(t_j^*, t_{j+1}^*). \tag{17}$$

Adding (16) and (17), we obtain a contradiction to (12), (14), (15) for $j \to \infty$. The other cases can be treated similarly.

Theorem 3.8 *Suppose* $\mathbf{t} \in \mathcal{M}$ *is not asymptotic to the boundary. Then* $\mathbf{t}$ *and* $T_{(a,b)}\mathbf{t}$ *do not cross for any* $(a, b) \in \mathbb{Z}^2$.

Proof: The statement is trivial for $a = 0$. So we assume that $a > 0$ (the case $a < 0$ is treated similarly). Suppose that $\mathbf{t}$ and $\mathbf{t}^* = T_{(a,b)}$ cross. Without loss of generality, they cross at 0 or between 0 and 1. According to Lemma (3.3) they do not cross again. We can assume that $t_j^* < t_j$ for $j < 0$ and $t_j^* > t_j$ for $j > 0$ (otherwise we interchange $\mathbf{t}$ and $\mathbf{t}^*$). These equations imply that for every $j \leq 0$ the sequence

$$t_{j-ka} + kbT, \;\; k \in \mathbb{N}$$

is decreasing and that for every $j > 0$ the sequence

$$t_{j+ka} - kbT, \;\; k \in \mathbb{N}$$

is decreasing.

We distinguish three cases:

(i) $\frac{b}{a} \notin [\rho^-, \rho^+]$. Since for $j \leq 0$

$$\frac{t_{j-ka} + kbT}{-kaT} > \frac{t_j}{-kaT},$$

we get that

$$\limsup_{n \to -\infty} \frac{t_n}{nT} \geq \frac{b}{a}. \tag{18}$$

For $j > 0$ we see:

$$\frac{t_{j+ka} - kbT}{kaT} < \frac{t_j}{kaT}.$$

We get

$$\liminf_{n \to \infty} \frac{t_n}{nT} \leq \frac{b}{a}. \tag{19}$$

Since $\frac{b}{a} \notin [\rho^-, \rho^+]$, either (18) or (19) is in contradiction with Lemma 2.10.

(ii) $\frac{b}{a} \in (\rho^-, \rho^+)$. There exists a periodic orbit $\mathbf{u}$ with period (a, b) such that $u_0 < t_0$. By Lemma (3.3) we have either $u_j < t_j$ for $j \leq 0$ or $u_j < t_j$ for $j \geq 0$. We will treat the case $u_j < t_j$ for $j \leq 0$ (the other case is completely analogous). For $j \leq 0$ the sequence $k \mapsto t_{j-ka} + kb$ is decreasing and bounded below by $u_{j-ka} + kb = u_j$. Hence it is convergent and

$$v_j := \lim_{k \to \infty} (t_{j-ka} + kb) = \lim_{k \to \infty} (T_{(ka,kb)} t)_j$$

exists for $j \leq 0$ and $v_{j-a} + b = v_j$. From the periodicity of $\mathbf{v}$ it follows that both $\mathbf{t}$ and $\mathbf{t}^*$ are α-asymptotic to $\mathbf{v}$ and that $|t_{i+1} - t_i|$ is bounded for $j \to -\infty$. This is in contradiction with Lemma 3.7 and our assumption that $\mathbf{t}$ and $\mathbf{t}^*$ cross.

(iii) Assume that $\frac{b}{a} = \rho^-$ (the case $\frac{b}{a} = \rho^+$ is analogous). Since ρ^- is the mean spacing of the non-decreasing function β^-, Corollary (4) from [RT1] implies that there exists a non-decreasing periodic function f which agrees with β^+ except on a finite set in every interval $[nT, (n+1)T]$ and has a periodic orbit $\mathbf{u}$ with period (a, b). Since β^- is left-continuous, $u_{n+1} \leq \beta^-(u_n)$. We can choose $\mathbf{u}$ such that $u_0 < t_0$, and then we get by induction

$$u_{n+1} \leq \beta^-(u_n) \leq \beta^-(t_n) < t_{n+1}.$$

Hence $u_j < t_j$ for each $j > 0$. We then construct a sequence v with period (a, b), α-asymptotic to $\mathbf{t}$ and $\mathbf{t}^*$ in the same way as in (ii). This is again a contradiction with Lemma 3.7 and our assumption that $\mathbf{t}$ and $\mathbf{t}^*$ cross.

We have proved that each $\mathbf{t} \in \mathcal{M}^*$ (the set of maximising sequences not asymptotic to the boundary) is rotationally ordered. We will now list immediate consequences, including existence of the *mean spacing* (rotation number) of the sequence $\mathbf{t}$:

Theorem 3.9 *There exists a continuous map $\alpha : \mathcal{M}^* \to \mathbb{R}$ with the properties:*

(i) For all $\mathbf{t} \in \mathcal{M}^*$ we have

$$|t_i - t_0 - iT\alpha(\mathbf{t})| < T, \tag{20}$$

in particular $\alpha(\mathbf{t}) = \lim_{|i| \to \infty} \frac{t_i}{i}$.

(ii) If $\mathbf{t} \in \mathcal{M}^*$ is periodic with period (p, q), then $\alpha(\mathbf{t}) = \frac{p}{q}$.

(iii) α is invariant under translation T; i.e. $\alpha(T_{(a,b)}(\mathbf{t})) = \alpha(\mathbf{t})$ for all integer pairs (a, b).

Proof: The existence of a mean spacing and relation (20) follow from the fact that $\mathbf{t}$ is rotationally ordered. Continuity of α follows from (20) (see Theorem 3.15 of [BAN], or [BAE]).

3.3 Quasiperiodic maximising sequences

We will now construct maximising sequences with irrational mean spacing α, for any $\rho^- < \alpha < \rho^+$. We shall always use the product topology on the space of sequences $\mathbb{R}^{\mathbb{Z}}$ (or semi-sequences $\mathbb{R}^{\mathbb{Z}}+$). The main tool will be a simple version of Tychonow's theorem: for every $a \in \mathbb{R}^{\mathbb{Z}}$ the set $\{x \in \mathbb{R}^{\mathbb{Z}} : |x_i| \leq a_i\}$ is compact. In Definition 1.3 we have defined the notion of a maximising sequence only for feasible sequences. However, since the limit of feasible sequences is not always a feasible sequence, the notion of a maximising but not necessarily feasible sequence or semi-sequence (a schedule) will be useful. We can again define the function $V : \mathbb{R}^{j-i+1} \to \mathbb{R} \cup \{-\infty\}$, by defining $V(u_i, ..., u_j) = -\infty$ if the segment $(u_i, ..., u_j)$ is not feasible.

Definition 3.4 *The schedule $\mathbf{t}$ is* **weakly-maximising** *if for all $m, p \in S$ with $m < p$, and all segments $\mathbf{u}$ with $u_m = t_m$, $u_p = t_p$, then $V(t_m, ..., t_p) \geq V(u_m, ..., u_p)$.*

Lemma 3.10 *Let $(\mathbf{u^n})_{n\in\mathbb{N}}$ be a sequence of weakly-maximising configurations which converge to a configuration $\mathbf{u}$. Then $\mathbf{u}$ is weakly-maximising.*

Proof: Since $\mathbf{u^n}$ is weakly-maximising, for each $(t_{i+1},...,t_{j-1}) \in \mathbb{R}^{j-i-1}$ we have

$$V(u_i^n, u_{i+1}^n, ...u_{j-1}^n, u_j^n) \geq V(u_i^n, t_{i+1}, ..., t_{j+1}, u_j^n). \tag{21}$$

Since $V : \mathbb{R}^{j-i+1} \to \mathbb{R} \cup \{-\infty\}$ is continuous, we can take the limit as $n \to \infty$ and obtain

$$V(u_i, u_{i+1}, ...u_{j-1}, u_j) \geq V(u_i, t_{i+1}, ..., t_{j+1}, u_j). \tag{22}$$

Lemma 3.11 *Assume that $\mathbf{t} \in \mathbb{R}^{\mathbb{Z}}$ or $\mathbf{t} \in \mathbb{R}^{\mathbb{Z}_+}$ is a weakly-maximising sequence or semi-sequence with a well defined mean spacing $\alpha \in (\rho^-, \rho^+)$. Assume that there exists a constant C such that $|t_n - t_0 - n\alpha T| \leq C$ for each $n \in \mathbb{Z}$, respectively $n \in \mathbb{Z}_+$. Then $\mathbf{t}$ is feasible (and therefore maximising).*

Remark: In Theorem 3.9 we have shown that the constant C can be taken to be T for all maximising sequences.

Proof: Assume that $\mathbf{t}$ is not feasible, and that for some $i \geq 0$, $(t_i, t_{i+1}) \notin A$ (the case $i < 0$ can be proved in the same way). We can find $n > i$ large enough such that

$$\frac{1}{n}(\frac{C}{T} + 1) < \min\{\rho^+ - \alpha, \alpha - \rho^-\}. \tag{23}$$

We can choose integers p_1 and p_2 such that $\alpha n - \frac{C}{T} \geq p_1 \geq \alpha n - \frac{C}{T} - 1$ and $\alpha n + \frac{C}{T} \leq p_2 \leq \alpha n + \frac{C}{T} + 1$. From (23) and the definition of p_1 we obtain

$$\frac{p_1}{n} \geq \alpha - \frac{1}{n}(\frac{C}{T} + 1) > \rho^-. \tag{24}$$

Applying results from [RT2] (see Corollary 2.8) we obtain $t_0 + p_1 T > (\beta^-)^n(t_0) = t_n^-$. Using our assumptions we see that:

$$t_n \geq t_0 + \alpha n T - C \geq t_0 + p_1 T > t_n^-. \tag{25}$$

In the same way we show that

$$t_n \leq t_0 + \alpha n T + C \leq t_0 + p_2 T < t_n^+. \tag{26}$$

Lemma 2.7 and relations (25), (26) imply there exists a feasible segment $t_0, u_1, u_2, ..., u_{n-1}, t_n$. Then we have

$$-\infty = V(t_0, t_1, ..., t_{n-1}, t_n) < V(t_0, u_1, ..., u_{n-1}, t_n)$$

which contradicts $\mathbf{t}$ being weakly-maximising.

Theorem 3.12 *Assume that $\alpha \in (\rho^-, \rho^+)$ is irrational. Then there exists a maximising sequence with mean spacing α .*

Proof: We choose a sequence of rationals $\alpha_k \in (\rho^-, \rho^+)$, $k \in \mathbb{N}$ such that $\alpha_k \to \alpha$, and we take a sequence of maximising configurations $(\mathbf{u^k})_{k\in\mathbb{N}}$ with mean spacings α_k for each k. Since

$$|u_i^k - u_0^k - iT\alpha_k| \leq T, \tag{27}$$

u_i^k is bounded for each i. We can apply Tychonow's theorem in an appropriate compact space and we find a convergent subsequence $\mathbf{u}^{m_k} \to \mathbf{u}$ (pointwise), which converges to some weakly-maximising (Lemma 3.10) configuration $\mathbf{u}$. Mean spacing is a continuous function on the space of maximising configurations (a consequence of (20)), so $\mathbf{u}$ must have mean spacing α. Taking the limit as $n \to \infty$ in (27) we obtain $|u_i - u_0 - i\alpha T| \leq T$. Lemma 3.11 implies that $\mathbf{u}$ is feasible sequence.

3.4 Structure of the set of maximising sequences

We will now analyse the structure of the set $\mathcal{M}_\alpha$ of maximising sequences with mean spacing $\alpha \in (\rho^-, \rho^+)$. Obviously $\mathbf{t} \in \mathcal{M}_\alpha$ is not asymptotic to the boundary, otherwise it would have the wrong mean spacing.

Definition 3.5 *We call a sequence* $\mathbf{t}$ **recurrent** *if there exists a sequence* $(a_i, b_i) \in \mathbb{Z}^2$*, with* $a_i \to \infty$*, such that* $\mathbf{t} = \lim_{i\to\infty} T_{(a_i,b_i)}\mathbf{t}$*.*

We define $\overline{B(\mathbf{t})}$ to be the closure of $B(\mathbf{t}) = \{T_{(a,b)}(\mathbf{t}),\ (a,b) \in \mathbb{Z}^2\}$, and p_i to be projection $\mathbf{t} \mapsto t_i$. For $t \in \mathcal{M}_\alpha$, we will show that $\overline{B(\mathbf{t})} \subset \mathcal{M}_\alpha$, and that for each irrational α it contains the set $\mathcal{M}_\alpha^{rec}$ of all recurrent maximising sequences of mean spacing α.

Lemma 3.13 *Suppose* $\mathbf{t} \in \mathcal{M}^*$*. Each element of* $\overline{B(\mathbf{t})}$ *is a maximising sequence with the same mean spacing as* $\mathbf{t}$*. The projection* p_i *is continuous, open and order preserving and maps* $\overline{B(\mathbf{t})}$ *homeomorphically onto a closed subset of* $\mathbb{R}$*.*

Proof: Assume that $\mathbf{t}^n \in B(\mathbf{t})$, $n \in \mathbb{N}$, is a sequence converging to $\mathbf{t}$. Then $\mathbf{t}$ is weakly-maximising because of Proposition (3.10), has the same mean spacing because of continuity of mean spacing, and Proposition (3.11) implies that it is feasible. Since $B(\mathbf{t})$ is totally ordered, $\overline{B(\mathbf{t})}$ is totally ordered. Hence $p_i|\overline{B(\mathbf{t})}$ is injective, and $p_i|\overline{B(\mathbf{t})} : \overline{B(\mathbf{t})} \to p_i(\overline{B(\mathbf{t})})$ is a homeomorphism. We now have to prove that $p_i(\overline{B(\mathbf{t})})$ is closed. Suppose $\mathbf{t}^n \in \overline{B(\mathbf{t})}$ is a sequence such that t_i^n converges. Then it is easy to show using the fact that $\overline{B(\mathbf{t})}$ is rotationally ordered that t_k^n is bounded for each k. Using Tychonow's theorem we find a convergent subsequence of $\mathbf{t}^n$ converging to $\mathbf{t}^*$ and $p_i(\mathbf{t}^*) = \lim_{n\to\infty} p_i(\mathbf{t}^n)$.

Lemma 3.14 *For every* $\mathbf{t} \in \mathcal{M}^*$*, there exists a strictly increasing map* $f : \mathbb{R} \to \mathbb{R}$*,* $f(x+T) = f(x) + T$ *such that* $t_{i+1} = f(t_i)$*.*

Proof: We define f on the closed set $S := p_0(\overline{B(\mathbf{t})})$ by $f := p_1 \circ (p_0|\overline{B(\mathbf{t})})^{-1}$ and we extend it from S to $\mathbb{R}$ affinely.

The structure of the set $\mathcal{M}_\alpha$ is a consequence of Lemmata 3.13 and 3.14, and is a standard result of Aubry-Mather theory (for the proofs see [BAN] or [KH]).

Theorem 3.15 *If $\alpha \in (\rho^-, \rho^+)$ is irrational, $\mathcal{M}_\alpha$ is totally ordered. The set $\mathcal{M}_\alpha^{rec}$ projected onto the circle S_1 (using periodicity) is homeomorphic either to a Cantor set or to the whole of S_1. Every $\mathbf{t} \in \mathcal{M}_\alpha$ is α- and ω-asymptotic to sequences in $\mathcal{M}_\alpha^{rec}$.*

Theorem 3.16 *If $\alpha \in (\rho^-, \rho^+)$ is rational, the set of periodic sequences $\mathcal{M}_\alpha^{per} = \mathcal{M}_\alpha^{rec}$. Every $\mathbf{t} \in \mathcal{M}_\alpha$ is α- and ω-asymptotic to sequences in $\mathcal{M}_\alpha^{per}$.*

4 Maximising semi-sequences

In this section we will extend some of the results from the previous one to maximising semi-sequences, i.e. sequences $t_0, t_1, ...$ maximal for all variations of any segment $t_m, \dots, t_n$ with $0 \leq m < n$ and fixed ends. Similar results in the context of Aubry-Mather theory have already been obtained by Katok ([KAT] and section 13.3d of [KH]) and Bangert [BAN2], and their continuous-time analogues form the subject of recent work of Fathi [F].

In particular, we shall show that for each t_0 and each $\alpha \in (\rho^-, \rho^+)$, we can find a maximising semi-sequence through t_0 with mean spacing α. This should be contrasted with maximising sequences: sometimes there do not exist maximising sequences through a fixed t_0 (for example, in the Frenkel-Kontorova model $V(t,t') = -(\frac{1}{2}(t'-t)^2 + \frac{k}{2\pi}\cos 2\pi t)$, for t_0 too low in the potential).

Proposition 4.1 *Assume that $\mathbf{t}$ is a maximising semi-sequence. Then it has a well defined mean spacing $\alpha = \lim_{n\to\infty} \frac{t_n - t_0}{nT}$, $\alpha \in [\rho^-, \rho^+]$.*

Proof: Let $O_n = \frac{t_n - t_0}{nT}$. Assume that the mean spacing of $\mathbf{t}$ is neither ρ^+ nor ρ^- (otherwise the Proposition is proved).

We will first prove the case when ρ^+, ρ^- are finite. By Lemma 2.10, $\liminf O_n < \rho^+$, $\limsup O_n > \rho^-$. It cannot happen that $\limsup O_n = \rho^+$ and $\liminf O_n = \rho^-$ at the same time (because $\mathbf{t}$ would then intersect every maximising sequence with rotation number in the rotation interval infinitely many times; a consequence of (20)). Hence there exists a $\delta > 0$, and a sequence of integers $n_i \to \infty$ such that for each i

$$\rho^- + \delta \leq O_{n_i} \leq \rho^+ - \delta \tag{28}$$

We will prove that O_n is a Cauchy sequence. Without loss of generality we can assume that in the sequence O_{n_i} there are infinitely many positive numbers. For fixed $0 < \epsilon < \delta$ there exists an i such that $\frac{2}{n_i} < \frac{\epsilon}{2}$ and $\frac{1}{n_i T} < \frac{\epsilon}{2}$. Then $\psi = O_{n_i} + \frac{\epsilon}{2}$ is in the rotation interval and there exists a maximising sequence $\mathbf{t}^*$ with the mean spacing ψ and $t_0 - T \leq t_0^* < t_0$. Because of relation (20), $t_{n_i}^* \geq t_0^* + \psi n_i T - T$, and then

$$\begin{aligned} t_{n_i}^* \geq t_0 - T + \psi n_i T - T &= t_0 - 2T + O_{n_i} n_i T + \frac{\epsilon}{2} n_i T \\ &> t_0 + O_{n_i} n_i T \\ &= t_{n_i}. \end{aligned}$$

Hence $\mathbf{t}$ and $\mathbf{t}^*$ cross between 0 and n_i, and Aubry's Fundamental Lemma 3.3 then implies that they do not cross for $n > n_i$. Hence for $n > n_i$ we have $t_n^* > t_n$, and

$$O_n = \frac{t_n - t_0}{nT} \quad < \quad \frac{t_n^* - t_0^*}{nT} < \psi + \frac{1}{nT} < O_{n_i} + \frac{\epsilon}{2} + \frac{1}{nT} < O_{n_i} + \epsilon.$$

Choosing a maximising sequence with $\psi = O_{n_i} - \frac{\epsilon}{2}$ we prove in the same way that for each $n > n_i$ we have $O_n > O_{n_i} - \epsilon$. Hence for each $n, m > n_i$ we have $|O_m - O_n| < 2\epsilon$.

The case when β^+ or β^- is infinite is even simpler (δ-bounds are not needed on that side of the domain).

We will show that the problem of finding a maximising semi-sequence $\mathbf{t}$ for fixed initial condition t_0 and mean spacing α in the rotation interval (ρ^-, ρ^+) has a solution.

Proposition 4.2 *For fixed t_0 and $\alpha \in (\rho^-, \rho^+)$ there exists a maximising semi-sequence $t_0, t_1, ...$ with mean spacing α.*

Proof: For α rational or irrational, we can find a sequence of rationals $\frac{p_n}{n} \in (\rho^-, \rho^+)$ which approximate α, choosing p_n to be the smallest integer such that $p_n \geq \alpha n$ for n large enough, i.e.

$$\alpha n \leq p_n < \alpha n + 1. \tag{29}$$

There exist maximising sequences $\overline{\mathbf{t}}$ and $\underline{\mathbf{t}}$ with mean spacing α, such that $t_0 - 3T \leq \underline{t}_0 < t_0 - 2T$ and $t_0 + 2T < \overline{t}_0 \leq t_0 + 3T$.

Define $t_n^n = t_0 + p_n T$, $t_0^n = t_0$. Since $\frac{p_n}{n} \in (\rho^-, \rho^+)$, applying Lemma (2.8), there exists a feasible segment $t_0^n, u_1, ..., u_{n-1}, t_n^n$. Applying Proposition (3.2), we can find a maximising segment $t_0^n, t_1^n, t_{n-1}^n, t_n^n$. Define $t_m^n = \overline{t}_m$ for $m > n$. We will show that a subsequence of $\mathbf{t}^n$ converges to the desired semi-sequence.

Using relations (29) and (20) we get

$$t_n^n = t_0 + p_n T \leq t_0 + (\alpha n + 1)T \leq t_0 + T + (\overline{t}_n - \overline{t}_0 + T) < \overline{t}_n.$$

By Proposition 3.3, maximising segments $\overline{t}_0, ..., \overline{t}_n$ and $t_0^n, ..., t_n^n$ do not cross (otherwise they would cross at least twice, which is impossible). So we obtain the estimates:

$$t_k^n < \overline{t}_k < \overline{t}_0 + \alpha kT + T < t_0 + \alpha kT + 4T.$$

Similarly, we obtain estimates from below:

$$t_k^n > \underline{t}_k > \underline{t}_0 + \alpha kT - T > t_0 + \alpha kT - 4T.$$

Hence we conclude that

$$|t_k^n - t_0^n - \alpha kT| < 4T \tag{30}$$

for each n. For each k, t_k^n is bounded, and we can apply Tychonow's theorem again to find a convergent subsequence of $\mathbf{t}^n$ which converges to $\mathbf{t}$. In the same way as in the proof of Lemma 3.11 we conclude that $\mathbf{t}$ is maximising (for each $k > n$, the segment $t_m^k, ...t_n^k$ is maximising). Taking the limit as $n \to \infty$ in (30), we get $|t_k - t_0 - \alpha kT| \leq 4T$. Hence $\mathbf{t}$ has rotation number α, and Lemma 3.11 implies that $\mathbf{t}$ is feasible.

Remark: Choosing bounds more carefully (which would complicate the proof), we could improve (30) and obtain $|t_k - t_0 - \alpha kT| \leq T$ ([BAN2], Corollary 3.10).

Next we prove that maximising semi-sequences are ω-asymptotic to recurrent maximising sequences.

Proposition 4.3 *Every maximising semi-sequence* $\mathbf{t}$ *with rotation number* $\alpha \in (\rho^-, \rho^+)$ *is* ω-*asymptotic to a recurrent maximising sequence with the same rotation number.*

Proof: (i) If $\alpha = p/q$ then either $\mathbf{t}$ is periodic of type (p,q) and so recurrent, or for all $n \geq 0, t_{n+q} > t_n + p$, or for all $n \geq 0, t_{n+q} < t_n + p$. Suppose the case $>$. Then t_n^k, defined to be $t_{n+kq} - kp$ for all $n > -kq$, is increasing with k and bounded above, so it goes to a limit u_n. The sequence $\mathbf{u}$ is maximising of type (p,q) and $\mathbf{t}$ is ω-asymptotic to it.

(ii) If α is irrational, choose a sequence of integer pairs (p_k, q_k) with $q_k > 0$ and $p_k - q_k\alpha$ decreasing to 0. Define $t_n^k = t_{n+q_k} - p_k$ for $n > -q_k$. It is increasing with k and bounded above by t_n, so goes to a limit u_n. The sequence $\mathbf{u}$ is maximising and of mean spacing α and $\mathbf{t}$ is ω-asymptotic to it.

We say that a semi-sequence $\mathbf{t}$ *starts* on a sequence $\mathbf{u}$ if $t_0 = u_0$ but $t_i \neq u_i$ for some $i > 0$. Note that for stationary sequences, this implies that $t_1 \neq u_1$.

Proposition 4.4 *A maximising semi-sequence* $\mathbf{t}$ *with irrational mean spacing can neither cross nor start on a maximising sequence* $\mathbf{u}$ *with the same mean spacing. A maximising semi-sequence* $\mathbf{t}$ *with rational mean spacing* p/q *can neither cross nor start on a maximising periodic sequence* $\mathbf{u}$ *of type* (p,q).

Proof: (i) Irrational case. Construct $M(\mathbf{t})$ and $M(\mathbf{u})$, the sets of limit points of forward translations of $\mathbf{t}, \mathbf{u}$, respectively. As is standard in Aubry-Mather theory, since they have the same mean spacing they coincide, else some pair of orbits (one from each) would cross more than once. $\mathbf{t}$ and $\mathbf{u}$ can not cross their limit sets, so the common limit set has a gap $\mathbf{v}^- < \mathbf{v}^+$ which contains them (if $\mathbf{t}$ or $\mathbf{u}$ is recurrent, they coincide with one or other boundary). But $\mathbf{v}^\pm$ are ω-asymptotic so the same must hold for $\mathbf{t}$ and $\mathbf{u}$. Then they can not cross. A refinement of the proof of the latter, as follows, shows that $\mathbf{t}$ can not start on $\mathbf{u}$. If $t_0 = u_0$ then $t_1 \neq u_1$, else stationary would imply that $t_i = u_i$ for all $i > 1$. Since (u_{-1}, u_0, u_1) is maximising, (u_{-1}, u_0, t_1) is not, and so its profit can be increased by say $\delta > 0$ by pushing u_0 to some u_0'. The semi-sequence $\mathbf{t}$ is maximising, so for all $N > 0$, $V(t_0, \ldots, t_N, t_{N+1}) \geq V(u_0, \ldots, u_N, t_{N+1})$. Thus

$$V(t_0, \ldots, t_N, u_{N+1}) - V(u_0, \ldots, u_N, u_{N+1}) \geq V(t_N, u_{N+1}) - V(t_N, t_{N+1}) + V(u_N, t_{N+1}) - V(u_N, u_{N+1}). \tag{31}$$

By taking N large enough we can make the right hand side smaller than δ. Then

$$V(u_{-1}, u_0', t_1, \ldots, t_N, u_{N+1}) > V(u_{-1}, u_0, \ldots, u_N, u_{N+1}),$$

contradicting $\mathbf{u}$ maximising.

(ii) Rational case. Suppose the mean spacing is p/q. Let $\mathbf{v} = \lim_{k\to\infty} T_{-kp,-kq}\mathbf{t}$. It is a maximising periodic sequence of type (p,q). Form a semisequence $\mathbf{w}$ starting from t_0 by following $\mathbf{t}$ to its first crossing with $\mathbf{u}$ (skip this if t_0 is already on $\mathbf{u}$), switching to $\mathbf{u}$ for one period and then switching to $T_{p,q}\mathbf{t}$, and then pushing w_j at any places where $u_j = t_j, j > 0$ or $u_j = t_{j-q} + p$ to increase the profit. Then there is a $\delta > 0$ such that $V(w_0, \ldots, w_N) \geq V(t_0, \ldots, t_{N-q}) + P + \delta$, where P is the profit per period on $\mathbf{u}$. All maximising sequences of type (p,q) have the same profit per period (otherwise one could exploit the difference by jumping to a better one, taking K periods and then switching back, and taking K large enough to compensate for the cost of switching). Since $\mathbf{t}$ is asymptotic to $\mathbf{v}$, $V(t_0, \ldots, t_N) - V(t_0, \ldots, t_{N-q}) - P \to 0$ as $N \to \infty$, and $V(w_0, \ldots, w_{N-1}, t_N) - V(w_0, \ldots, w_{N-1}, w_N) \to 0$. Hence for N large enough, $V(w_0, \ldots, w_{N-1}, t_N) > V(t_0, \ldots, t_N)$, contradicting $\mathbf{t}$ maximising.

Remark: As can be seen from the proofs, we do not really need that the whole of $\mathbf{u}$ is maximising, only that $(u_{-1}, u_0, \ldots)$ is maximising.

Proposition 4.4 has the following

Corollary 4.5 *If for fixed t_0 and mean spacing α there exists a recurrent maximising sequence $\mathbf{t}$, then $(t_i)_{i\geq 0}$ is the only maximising semi-sequence through t_0 with that mean spacing.*

If there is no recurrent maximising sequence through t_0, we distinguish two cases. For rational α and given t_0, there exist two maximising semi-sequences which are in the "gap" between two periodic sequences, one ω-asymptotic to each, typically only one of each. For irrational α and given t_0, typically there is a unique maximising semi-sequence $\mathbf{t}$ which is in the "gap" between two recurrent maximising sequences, ω-asymptotic to both of them. For proofs, see [BAN2].

5 Optimising semi-sequences

In this section we prove the main results of the paper. Our principal goal is to find which maximising semi-sequences are optimising. A large part of the answer is given by maximising the "average profit rate". We define the *average profit rate* of a (semi-) sequence $\mathbf{t}$ as

$$L(\mathbf{t}) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} V(t_{i-1}, t_i)}{|t_n - t_0|}, \tag{32}$$

if it converges. When $\mathbf{t}$ is periodic of period (p,q) with $p, q \neq 0$, it is obvious that $L(\mathbf{t})$ converges. We will prove that the average profit rate is well defined for all maximising sequences $\mathbf{t}$.

We recall that the sequence (semi-sequence) $\mathbf{t}$ is optimising if it is feasible and for all integers $m < n$ $(0 \leq m < n)$ and any segment $(u_0, u_1, ..., u_k)$ of arbitrary length $(k > 0)$ with the same boundary conditions as the segment $(t_m, ...t_n)$ $(u_0 = t_m,\ u_k = t_n)$ we have

$$V(t_m, t_{m+1}, ..., t_n) \geq V(u_0, u_1, ..., u_k).$$

In other words, a sequence is optimising if each segment $(t_m, t_{m+1}, ..., t_n)$, $m < n$ is maximising with respect to variations of its elements and inserting or removing elements in the segment.

We make the following additional standing assumption:

(H5) Non-triviality condition: If for some $t \in \mathbb{R}$, $(t,t) \in A$, then $V(t,t) < 0$.

This assumption is necessary for the average profit rate to be bounded, and is natural in applications. Otherwise, if for some t the value of $V(t,t)$ were positive, the average profit rate of the sequence $t_i = t$ would be ∞. Of course, in most applications we expect (t,t) not to be feasible, since this would allow an infinite number of actions to take place simultaneously, which is not realistic. Thus, for instance, in the example from the introduction, scheduling an arbitrarily large number of buses all at the same time is not a feasible course of action.

5.1 Average profit rate

We will prove some basic properties of the average profit rate function. The function $W(t,z) = V(t,t+z)$ will be useful; W is periodic in the first argument, $W(t+T,z) = W(t,z)$.

Lemma 5.1 *(i) The function V is bounded above: there exists C such that $V(t,t') \leq C$.*

(ii) There exists a $\delta > 0$ such that $|t - t'| < \delta$ implies that $V(t,t') < 0$.

Proof:(i) Let $D = V(t,t')$ for some $(t,t') \in A$. Let $z = t - t'$. If β^- is finite, then from (10) there exists a D_2 such that $(t,t') \in A$ implies $t - t' \leq D_2$. If β^- is infinite, uniform convergence (H2) implies that there exists a D_2 such that $t - t' > D_2$ implies $V(t,t') < D$. If β^+ is finite, then from (9) there exists a D_1 such that $(t,t') \in A$ implies $t - t' \geq D_1$. If β^+ is infinite, uniform convergence (H2) implies that there exists a D_1 such that $t - t' < D_1$ implies $V(t,t') < D$. Let $S = [0,T] \times [D_1, D_2]$, which is compact. The function W attains its maximum C on S, which is the maximum of W and V on the whole domain A.

(ii) Define a function $f : [0,T] \to \mathbb{R}^+$ by $f(t) = \inf\{|z|,\ W(t,z) \geq 0\}$. It is continuous, and defined on compact set. Condition (H5) implies that $f(t) > 0$ and hence f attains a minimum $\delta > 0$. From the definition and periodicity it follows that $|z| < \delta$ implies $W(t,z) < 0$ for each t.

Lemma 5.2 *The partial sums in the definition of average profit rate are bounded on the set of all sequences such that $t_0 < t_1 < ...$ or $t_0 > t_1 > ...$.*

Proof: Without loss of generality, assume that $t_0 < t_1 < ... < t_n$. Let $S = \{i > 0 : V(t_{i-1}, t_i) \geq 0\}$ and $k = |S|$, the cardinality of S, $k \leq n$. We see:

$$\sum_{i=1}^{n} V(t_{i-1}, t_i) \leq \sum_{i \in S} V(t_{i-1}, t_i) \leq kC.$$

Lemma 5.1 shows that $(t_{i-1}, t_i) \in S$ implies $t_i - t_{i-1} \geq \delta$, so we estimate:

$$|t_n - t_0| = \sum_{i=1}^{n} |t_i - t_{i-1}| \geq \sum_{i \in S} |t_i - t_{i-1}| \geq k\delta.$$

From these estimates we obtain

$$\sum_{i=1}^{n} \frac{V(t_{n-1}, t_n)}{|t_n - t_0|} \leq \frac{C}{\delta}.$$

Remark: Since we have shown that all maximising sequences are monotone, Lemma 5.2 implies that the average profit rate is bounded by $\frac{C}{\delta}$ for all maximising sequences.

The following lemma will make possible switching between sequences with the same mean spacing, and will be used heavily in the proofs of the basic properties of the average profit rate function. We will show that maximising segments are always at a safe distance from the boundary.

Lemma 5.3 *(i) Let $(t_0, t_1, ..., t_n)$ be a maximising segment such that $\alpha = \frac{t_n - t_0}{nT} \in (\rho^-, \rho^+)$. Then for each $0 < i < n$:*

$$|t_i - t_0 - \alpha i T| \leq 3T. \tag{33}$$

(ii) Let $[\alpha^-, \alpha^+] \subset (\rho^-, \rho^+)$, and N large enough such that there exist integers p_1, p_2 and

$$\alpha^+ + \frac{3}{N} \leq \frac{p_2}{N} < \rho^+, \;\; \alpha^- - \frac{3}{N} \geq \frac{p_1}{N} > \rho^-. \tag{34}$$

Then there exists $\epsilon > 0$ such that each maximising sequence $(t_0, ..., t_m)$ with $m \geq N$ and $\frac{t_m - t_0}{mT} \in [\alpha^-, \alpha^+]$ is at least 2ϵ from the boundary (for each $0 \leq i < m$, $d((t_i, t_{i+1}), B) \geq 2\epsilon$).

Proof: (i) There exist maximising sequences $\overline{\mathbf{t}}$ and $\underline{\mathbf{t}}$ with mean spacing α, such that $t_0 - 2T \leq \underline{t}_0 < t_0 - T$ and $t_0 + T < \overline{t}_0 \leq t_0 + 2T$. Applying relation (20) to $\overline{t}$ we obtain

$$\overline{t}_0 + (n\alpha - 1)T < \overline{t}_n < \overline{t}_0 + (n\alpha + 1)T,$$

and so

$$t_n < \overline{t}_n < t_n + 3T.$$

By Proposition 3.3, maximising segments $\overline{t}_0, ..., \overline{t}_n$ and $t_0, ..., t_n$ do not cross (otherwise they would cross at least twice, which is impossible). Thus, using (20) again on $\overline{t}$, we obtain the estimate:

$$t_k < \overline{t}_k < \overline{t}_0 + \alpha k T + T < t_0 + \alpha k T + 3T,$$

for $k = 0, \ldots, n$. Similarly, by comparing segments $(t_0, ..., t_n)$ and $(\underline{t}_0, ..., \underline{t}_n)$, we obtain: $t_k > \underline{t}_k > \underline{t}_0 + \alpha k T - T > t_0 + \alpha k T - 3T$.

(ii) Assume the contrary: that there exists a sequence of maximising segments $\mathbf{t^n}$ with properties as assumed, such that there exists a subsequence i_n with $d((t^n_{i_n}, t^n_{i_n+1}), B) < \frac{1}{n}$. We can assume that all these segments have length N and that $t^n_0 \in [0, T]$ (otherwise we cut them and translate them by an integer multiple of T). By (33) we conclude that t^n_i is uniformly bounded for fixed n. Therefore $\mathbf{t^n}$ is a sequence in a compact subset of $\mathbb{R}^{N+1}$, so it has a convergent subsequence which converges to a maximising segment (Lemma 3.10) $\mathbf{t} = (t_0, ..., t_N)$. We can find an index i such that (t^n_i, t^n_{i+1}) converges to the boundary as $n \to \infty$, and thus $(t_i, t_{i+1}) \in B$. By (34), however, we have

$$t_N \leq t_0 + \alpha^+ NT + 3T < t_0 + p_2 T < t^+_N$$

and

$$t_N \geq t_0 + \alpha^- NT - 3T > t_0 - p_1 T > t_N^-$$

(see Corollary 2.8). Lemma 2.7 then implies that there exists a feasible segment $(t_0, ..., t_N)$, which contradicts the fact that $\mathbf{t}$ is maximising.

Remark: We can easily see that for maximising semi-sequences $t_0, t_1, ...$ with well defined mean spacing α, the inequality (33) is still valid. We prove this by taking the limit $n \to \infty$.

For fixed a, b, n, let $(t_0, ..., t_n)$ be a maximising segment with $t_0 = a$, $t_n = b$. As in [KH], let

$$M(a, b, n) = \sum_{i=0}^{n-1} V(t_i, t_{i+1}).$$

Such a maximising segment always exists. If there is more than one such maximising segment from a to b, then any two such segments must have the same action, and hence M is still well defined. If there is no feasible segment $\{t_0, ..., t_n\}$ then $M(a, b, n) := -\infty$. We will use the function M to calculate the average profit rate. M has the following obvious property:

$$M(x, z, n+m) \geq M(x, y, n) + M(y, z, m). \tag{35}$$

The next Lemma will show that the average profit rates for maximising sequences with the same mean spacing are close.

Lemma 5.4 *Let $[\alpha^-, \alpha^+] \subset (\rho^-, \rho^+)$ and N large enough such that (34) is valid.*

(i) Then there exists a constant C (depending only on the interval $[\alpha^-, \alpha^+]$ such that for all $n > N$, $u, u' \in [0, T]$, $v, v' \in \mathbb{R}$ with $|v - v'| \leq 8T$, and $\frac{v-u}{nT}, \frac{v'-u'}{nT} \in [\alpha^-, \alpha^+]$ we have:

$$|M(u, v, n) - M(u', v', n)| < C. \tag{36}$$

(ii) If $\frac{v-u}{nT} \in (\alpha^-, \alpha^+)$, then $M(u, v, n)$ is continuous in (u, v) (keeping n fixed). Furthermore, $M(u, v, n)$ is continuous uniformly in n, for each $n > N$.

Proof: (i) Using Lemma 5.3 we can find an $\epsilon > 0$ such that all these sequences are at least 2ϵ away from the boundary. Without loss of generality we assume $u < u'$, $v < v'$. We will divide the segment $[u, u']$ into subsegments of equal length $[u^0, u^1], [u^1, u^2], ..., [u^{m-1}, u^m]$, $u^0 = u, u^m = u'$ such that $u^i - u^{i-1} < \epsilon$, and $[v, v']$ into subsegments of equal length $[v^0, v^1], [v^1, v^2], ..., [v^{m-1}, v^m]$, $v^0 = v, v^m = v'$ such that $v^i - v^{i-1} < \epsilon$. Here m depends only on ϵ (which depends only on the interval $[\alpha^-, \alpha^+]$). Let $a = \max\{|\alpha^-|, |\alpha^+|\}$ and S be the set $S = [0, T] \times [-a - 4T - \epsilon, a + 4T + \epsilon] \cap \{(t, t') \in A \, : \, d((t, t'), B) \geq \epsilon\}$. Since S is a compact subset of A, the function V is bounded on S and there is a constant C_0 such that for each $(t, t') \in S$,

$$|V(t, t')| < C_0.$$

We claim that

$$|M(u^i, v^i, n) - M(u^{i+1}, v^{i+1}, n)| < 4C_0. \tag{37}$$

Let $(t_0, t_1, ..., t_n)$, $t_0 = u^i$, $t_n = v^i$, and $(t'_0, t'_1, ..., t'_n)$, $t'_0 = u^{i+1}$, $t'_n = v^{i+1}$ be maximising sequences (they are feasible because of the assumptions and Lemma 2.7, and at least 2ϵ away from the boundary because of Lemma 5.3). We calculate:

$$\begin{aligned} M(u^i, v^i, n) &= M(t_0, t_n, n) = V(t_0, t_1, ..., t_{n-1}, t_n) \\ &= V(t'_0, t_1, t_2, ..., t_{n-1}, t'_n) + V(t_0, t_1) + V(t_{n-1}, t_n) - V(t'_0, t_1) - V(t_{n-1}, t'_n) \\ &\leq M(t'_0, t'_n, n) + 4C_0 \\ &= M(u^{i+1}, v^{i+1}, n) + 4C_0, \end{aligned}$$

since $(t_0, t_1), (t_{n-1}, t_n), (t'_0, t_1), (t_{n-1}, t'_n)$ are in S (after translating by an integer multiple of T, if necessary), and thus bounded by C_0. In the same way we get

$$M(u^{i+1}, v^{i+1}, n) \leq M(u^i, v^i, n) + 4C_0$$

and we have proved the claim. Applying the triangle inequality and using (37) we get:

$$|M(u, v, n) - M(u', v', n)| \leq m \cdot 4C_0 = C,$$

where m, C_0 (and thus C) depend only on the interval $[\alpha^-, \alpha^+]$.

(ii) Again, using Lemma 5.3 we can find an $\epsilon > 0$ such that all the segments of length n and mean spacing in $[\alpha^-, \alpha^+]$ are at least 2ϵ away from boundary. Define $S \subset D$ as above. Since S is compact, V is uniformly continuous on S. Then for each λ there exists a $\delta > 0$ such that:

$$(x, y), (x', y') \in S,\ |(x', y') - (x, y)| < \delta,\ \Rightarrow |V(x', y') - V(x, y)| < \frac{\lambda}{2},$$

$$|(x, y) - (u, v)| < \delta \Rightarrow \frac{y - x}{nT} \in [\alpha^-, \alpha^+].$$

Choose (x, y) such that $|(x, y) - (u, v)| < \delta$, and then $(x, y) \in S$. We calculate:

$$\begin{aligned} M(x, y, n) &= V(x, t_1, ..., t_{n-1}, y) \\ &= V(u, t_1, t_2, ..., t_{n-1}, v) + V(x, t_1) - V(u, t_1) + V(t_{n-1}, y) - V(t_{n-1}, v) \\ &\leq M(u, v, n) + 2\frac{\lambda}{2} \\ &= M(u, v, n) + \lambda \end{aligned}$$

and similarly $M(u, v, n) \leq M(x, y, n) + \lambda$. Therefore $|M(u, v, n) - M(x, y, n)| < \lambda$. Since λ was arbitrary, M is continuous in (u, v). Note that the choice of δ did not depend on n, and therefore M is continuous uniformly in $n > N$.

In the proof of the next proposition we use the fact that a maximising semi-sequence $\mathbf{t}$ has well defined mean spacing $\alpha = \lim_{n\to\infty} \frac{t_n - t_0}{nT}$ (Proposition 4.1).

Proposition 5.5 *Let* $\mathbf{t} = t_0, t_1, t_2, ...$ *be a maximising semi-sequence with mean spacing* $\alpha \neq 0$. *Then the average profit rate is well defined (convergent).*

Proof: Assume that $t_0 \in [0, T]$ (else translate $\mathbf{t}$ by a multiple of T). Write $a_n = M(t_0, t_n, n)$. Then the average profit rate is $\lim_{n\to\infty} \frac{a_n}{|t_n - t_0|}$. It is enough to prove that $\frac{a_n}{n}$ is convergent.

Choose $\delta > 0$ such that $[\alpha - \delta, \alpha + \delta] \in (\rho^-, \rho^+)$, define $\alpha^- = \alpha - \delta$, $\alpha^+ = \alpha + \delta$. Choose N large enough such that (34) is satisfied and for each $m \geq N$, $\frac{t_m - t_0}{mT} \in [\alpha^-, \alpha^+]$. For each $m \geq N$:

$$\begin{aligned} a_{m+n} &= M(t_0, t_{m+n}, m+n) \\ &= M(t_0, t_n, n) + M(t_n, t_{n+m}, m) \\ &\geq M(t_0, t_n, n) + M(t_n - kT, t_{n+m} - kT, m) \\ &\geq M(t_0, t_n, n) + M(t_0, t_m, m) - C \\ &= a_n + a_m - C. \end{aligned} \tag{38}$$

Here k is an integer such that $t_n - kT \in [0, T]$. We have applied Lemma 5.4 to $M(t_0, t_m, m)$ and $M(t_n - kT, t_{n+m} - kT, m)$.

We now recall that inequality (38) implies convergence of $\frac{a_n}{n}$ (super-additivity of the sequence $a_n - C$). Let $L_0 = \limsup \frac{a_n}{n}$, then $L_0 < \infty$ because of Lemma 5.1. Given $\delta' > 0$, let m be large enough such that $\frac{a_m}{m} > L_0 - \delta'$ and $\frac{C}{m} < \delta'$. Each $n > m$ can be written in the form $n = mk + r$ with $0 \leq r < m$. Now $\frac{a_n}{n} \geq \frac{ka_m + a_r - kC}{n} \geq L_0 - 2\delta'$ for n (i.e. k) large enough. Therefore $\frac{a_n}{n}$ converges to L_0. The average profit rate is now $L(\mathbf{t}) = \frac{L_0}{\alpha}$.

Proposition 5.6 *Let $\mathbf{u}$, $\mathbf{t}$ be two maximising semi-sequences with the same mean spacing $\alpha \neq 0$. Then $L(\mathbf{u}) = L(\mathbf{t})$.*

Proof: We can assume that $u_0, t_0 \in [0, T]$ (otherwise we translate them). Again we choose a $\delta > 0$ such that $[\alpha - \delta, \alpha + \delta] \in (\rho^-, \rho^+)$, and define $\alpha^- = \alpha - \delta$, $\alpha^+ = \alpha + \delta$. Choose N large enough that (34) is satisfied and for each $m \geq N$, $\frac{u_m - u_0}{mT} \in [\alpha^-, \alpha^+]$, $\frac{t_m - t_0}{mT} \in [\alpha^-, \alpha^+]$. By the remark after Lemma 5.3, $|u_m - u_0 - \alpha mT| \leq 3T$, $|v_m - v_0 - \alpha mT| \leq 3T$, and thus $|v_m - u_m| \leq 7T$. Hence we can apply Lemma 5.4 to $M(u_0, u_m, m)$ and $M(t_0, t_m, m)$ and obtain for each $m \geq N$,

$$\left|\frac{M(u_0, u_m, m)}{mT} - \frac{M(t_0, t_m, m)}{mT}\right| \leq \frac{C}{mT} \to 0.$$

Since $L(\mathbf{t}) = \frac{1}{\alpha} \lim_{n\to\infty} \frac{M(t_0, t_n, n)}{nT}$, the claim is proved.

5.2 Maximal average profit rate

We have proved that the average profit rate is well defined for all maximising semi-sequences and depends only on the mean spacing. Let $\mathbf{t}$ be a maximising sequence with mean spacing α. We will use the notation $L(\alpha) = L(\mathbf{t})$. A closely related average is:

$$A(\alpha) = \lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} V(t_i, t_{i+1})}{nT}.$$

Obviously, $A(\alpha)$ is well defined and $L(\alpha) = \frac{A(\alpha)}{\alpha}$. The function $A(\alpha)$ is known in the dynamical systems community as the mean action (e.g. [KH]) and in the solid state physics community as the mean energy per atom (e.g. [ALD],[GRI]).

The following two propositions are well-known properties of the mean action.

Proposition 5.7 *The mean action $A(\alpha)$, $A : (\rho^-, \rho^+) \to \mathbb{R}$ is a concave function, i.e. for each $\lambda \in (0,1)$, $\alpha, \beta \in (\rho^-, \rho^+)$, we have $A(\lambda\alpha + (1-\lambda)\beta) \geq \lambda A(\alpha) + (1-\lambda)A(\beta)$.*

Proof: Let $\mathbf{t}$, $\mathbf{u}$, $\mathbf{v}$ be maximising sequences with mean spacings $\lambda\alpha+(1-\lambda)\beta$, α, β respectively and $t_0, u_0, v_0 \in [0,T]$. Let N be large enough, such that for each $n > N$ there exists an m such that $\frac{\lambda\alpha n}{m}$ and $\frac{(1-\lambda)\beta n}{(n-m)}$ are in the rotation interval (ρ^-, ρ^+) ($\frac{m}{n}$ is an approximation to t). Then applying Lemma 5.4, there exist constants C_1, C_2, C_3 such that for each $n > N$:

$$|M(0, n(\lambda\alpha + (1-\lambda)\beta), n) - M(t_0, t_n, n)| < C_1,$$

$$|M(0, n\lambda\alpha, m) - M(u_0, u_m, m)| < C_2,$$

$$|M(n\lambda\alpha, n(\lambda\alpha + (1-\lambda)\beta), n-m) - M(v_0, v_{n-m}, n-m)| < C_3.$$

We can apply inequality (35):

$$M(0, n(\lambda\alpha + (1-\lambda)\beta), n) \geq M(0, n\lambda\alpha, m) + M(n\lambda\alpha, n(\lambda\alpha + (1-\lambda)\beta), n-m).$$

Using all the inequalities above we obtain:

$$\begin{aligned} A(\lambda\alpha + (1-\lambda)\beta) &= \lim_{n\to\infty} \frac{M(t_0, t_n, n)}{n} \\ &\geq \lim_{n\to\infty} \frac{M(u_0, u_m, m)}{n} + \frac{M(v_0, v_{n-m}, n-m)}{n} - \frac{C_1 + C_2 + C_3}{n} \\ &= \lim_{n\to\infty} \frac{m}{n} \cdot \frac{M(u_0, u_m, m)}{m} + \lim_{n\to\infty} \frac{n-m}{n} \cdot \frac{M(v_0, v_{n-m}, n-m)}{n-m} \\ &= \lambda A(\alpha) + (1-\lambda)A(\beta). \end{aligned}$$

Proposition 5.7 can be strengthened to:

Proposition 5.8 *The mean action is a strictly concave function, i.e. $A(\lambda\alpha+(1-\lambda)\beta) > \lambda A(\alpha) + (1-\lambda)A(\beta)$, for $\lambda \in (0,1)$.*

Proof: Suppose not, then there exists an interval $[a,b]$ where A is affine. Choose maximising periodic sequences $\mathbf{u}$, $\mathbf{v}$ with distinct periods (P_1, Q), (P_2, Q) in the interval (find any two and choose a common denominator), without loss of generality $P_2 > P_1$. Since $P_1 \neq P_2$, sequences $\mathbf{u}$ and $\mathbf{v}$ cross between some indices i and $i+1$, or at some index i. Construct two new periodic sequences with period $(P_1 + P_2, 2Q)$ by switching between $\mathbf{u}$, $\mathbf{v}$ and their translates $\mathbf{u}'$, $\mathbf{v}'$, by $\pm NT$, where $N = (P_2 - P_1)$, as drawn in Fig.1 for the case of a crossing between i and $i+1$. In this case, we apply (H3) to obtain a contradiction:

$$\begin{aligned} 2 \cdot A(\frac{P_1 + P_2}{2Q}) &\geq \frac{1}{2Q}(V(u_i, v_{i+1}, ..., v_{i+Q}, u_{i+1} + NT, ..., u_{i+Q} + NT) \\ &\quad + V(v_i, u_{i+1}, ..., u_{i+Q}, v_{i+1} - NT, ..., v_{i+Q} - NT)) \\ &> \frac{1}{2Q}(V(v_i, v_{i+1}, ..., v_{i+Q}) + V(u_{i+Q}, u_{i+1} + NT, ..., u_{i+Q} + NT) \\ &\quad + V(u_i, u_{i+1}, ..., u_{i+Q}) + V(v_{i+Q}, v_{i+1} - NT, ..., v_{i+Q} - NT)) \\ &= 2 \cdot (A(\frac{P_1}{Q}) + A(\frac{P_2}{Q})). \end{aligned}$$

The case of a crossing at i is treated similarly, but using (H4).

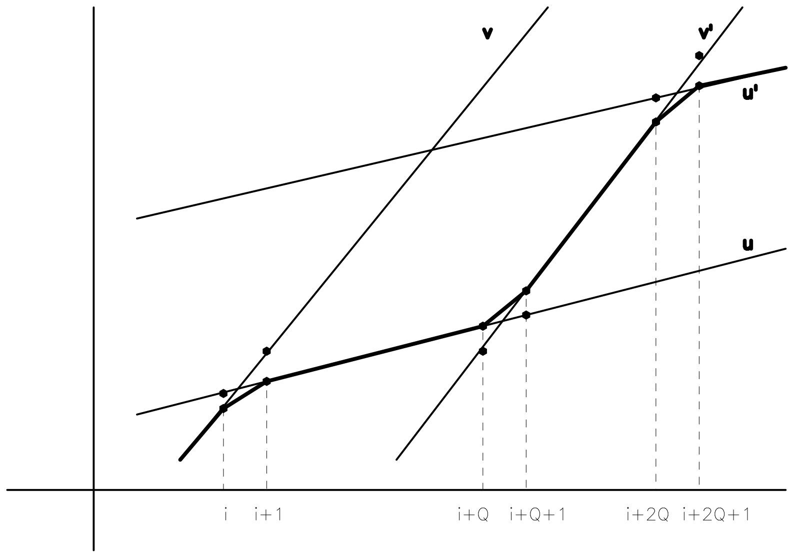

Fig 1: Proof of Proposition 5.8 (Schematic Aubry diagram). The straight lines represent the graphs of sequences, so the slopes represent their mean spacing.

Now we list the basic properties of the average profit rate function. For simplicity, and since this is the only case which arises in application, we restrict analysis to positive mean spacings, i.e. we assume that the rotation interval $(\rho^-, \rho^+) \subseteq (0, \infty)$.

Lemma 5.9 *The mean action function $A(\alpha) = \alpha L(\alpha)$ converges to $-\infty$ on the boundaries (ρ^-, ρ^+).*

Proof: We will prove that $\lim_{x\to\infty} A(x) = -\infty$ in the case $\rho^+ = \infty$. Since $V(x, x+t)$ converges uniformly in x to the boundary, for each $-N < 0$ we can find a $C > 0$ such that $y - x > C$ implies $V(x, y) < -N$. Take any mean spacing $\alpha > \frac{C}{T} + 1$, and maximising sequence $\mathbf{t}$ with rotation number α. Because of $t_{i+1} - t_i \geq \alpha T - T > C$ (20), we obtain $V(t_i, t_{i+1}) < -N$ and $A(\alpha) < \frac{-N}{T}$. The case with finite ρ^-, ρ^+ is similar.

Theorem 5.10 *(i) The average profit rate $L(\alpha)$, $L : (\rho^-, \rho^+) \to \mathbb{R}$ is a continuous, almost everywhere differentiable function. Left and right derivatives exist at each point.*

(ii) If $L(\alpha)$ has a local maximum α_M, it is unique and a strict global maximum.

(iii) $L(\alpha)$ has no local minima,

(iv) Function $L(\alpha)$ is either: a) strictly increasing on (ρ^-, ρ^+); or

b) strictly increasing on $(\rho^-, \alpha_M]$ and strictly decreasing on $[\alpha_M, \rho^+)$.

Proof: (i) Since the function A is strictly concave, it is continuous, almost everywhere differentiable and left and right derivatives exist at each point. Hence $L(\alpha) = \frac{A(\alpha)}{\alpha}$ has all these properties. The left and right derivatives of A, A^- and A^+, are strictly decreasing, and $A^-(\alpha) \geq A^+(\alpha)$.

(ii) If α_M is a local maximum,

$$0 \leq L^-(\alpha_M) = \frac{A^-(\alpha_M)}{\alpha_M} - \frac{A(\alpha_M)}{\alpha_M^2} \Rightarrow L(\alpha_M) \leq A^-(\alpha_M).$$

Similarly, we obtain $L(\alpha_M) \geq A^+(\alpha_M)$. Since A is strictly concave, the graph of the function A is strictly below any tangent to the graph at any point. We draw a tangent with the slope $L(\alpha_M) \in [A^+(\alpha_M), A^-(\alpha_M)]$ through α_M. For each $x \neq \alpha_M$ we have:

$$\begin{aligned} 0 &< (L(\alpha_M)(x - \alpha_M) + A(\alpha_M)) - A(x) \Rightarrow \\ 0 &< L(\alpha_M)\frac{x - \alpha_M}{x} + \frac{A(\alpha_M)}{x} - \frac{A(x)}{x} = L(\alpha_M) - \frac{A(\alpha_M)}{x} + \frac{A(\alpha_M)}{x} - L(x), \end{aligned}$$

and thus $L(x) < L(\alpha_M)$. Since this holds for each local maximum, the local maximum is unique.

(iii) Lemma 5.9 implies that $\lim_{x\to\rho^-} L(x) = -\infty$. Assume that α_m is a local minimum in (ρ^-, α_M) (or that there is no α_M). Then there must be a local maximum in (ρ^-, x_m), which is a contradiction with (ii) (or the assumption). Assume that $\alpha_m > \alpha_M$. Then $L(x)$ is decreasing on some interval left of α_m and increasing on $[\alpha_m, \rho^+)$, i.e. $L^-(\alpha_m) \leq 0 \leq L^-(x)$ for $x > \alpha_m$. Then for $x > \alpha_m$ is $A^-(x) - L(x) \geq A^-(\alpha_m) - L(\alpha_m)$. But the function $A^-(x)$ is strictly decreasing on the domain, and $L(x)$ is increasing, which is impossible.

(iv) If $L(x)$ is not strictly monotone on any of these intervals, there must be another local extremum in the intervals, which is a contradiction with either (ii) or (iii).

Corollary 5.11 *If there exists a* **profitable schedule***, i.e. a mean spacing α for which the average profit rate $L(\alpha)$ is positive, then the average profit rate has a strict global maximum α_M.*

Proof: Lemma 5.9 implies that there are $a < \alpha < b$ such that $L(a) < 0$, $L(b) < 0$. Since $L(\alpha) \geq 0$, there must be a local maximum α_M in (a, b), which is then a strict and global maximum.

It is easy to see that another sufficient condition for L to have a maximum is that there exist a constant $C > 0$ such that

$$V(t, t') \leq -C(t' - t)$$

for $t' - t$ large enough. We will show that the following condition is sufficient and necessary for existence of optimising semi-sequences:

(H6) The average profit rate has a local maximum (Theorem 5.10 then implies that it is a strict global maximum).

5.3 Existence

Following the same idea as in the construction of a maximising semi-sequence with fixed initial condition and mean spacing, we construct an optimising semi-sequence for each fixed initial condition t_0. We call a segment $(t_0, t_1, ..., t_n)$ optimising if for each variation $(u_0, ..., u_m)$ of arbitrary length $(m > 0)$, with $u_0 = t_0$, $u_m = t_n$:

$$V(t_0, t_1, ..., t_n) \geq V(u_0, ..., u_m).$$

From now on, we scale the period T to 1.

Lemma 5.12 *If there exists a feasible segment $(t_0, ..., t_0+p)$ for given $t_0 \in \mathbb{R}$ and integer p, then there exists an optimising segment $(t_0, t_1, ..., t_q)$, $t_q = t_0 + p$.*

Proof: For each $\frac{p}{q} \in (\rho^-, \rho^+)$ and $t_0 \in \mathbb{R}$ there exists a maximising sequence $\{t_0, t_1, ..., t_q\}$, $t_q = t_0 + p$ (Corollary 2.8 and its proof). We only have to show that the function $q \mapsto V(t_0, t_1, ..., t_q) = M(t_0, t_0+p, q)$ reaches a maximum on this set of sequences.

Let q_0 be an integer such that L is maximal, i.e. $L(\frac{p}{q_0}) = \max\{L(\frac{p}{q}),\ q \in \mathbb{N}\}$. There exists a maximising sequence $\mathbf{u}$ with period $\frac{p}{q_0}$; we can choose it such that $|t_0 - u_0| < T$. Let $C = M(u_0, u_0+p, q_0) - M(t_0, t_0+p, q_0) \geq 0$. Using properties of the average profit rate function, we can find an interval $[a,b] \subset (\rho^-, \rho^+)$, $a > 0$, such that for each $\frac{p}{q} \notin [a,b]$ the value of $L(\frac{p}{q})$ is too small, i.e. $L(\frac{p}{q_0}) > L(\frac{p}{q}) + \frac{C}{p}$. Then we calculate:

$$\begin{aligned} M(t_0, t_0+p, q_0) &= M(u_0, u_0+p, q_0) - C \\ &= p \cdot L(\frac{p}{q_0}) - C \\ &> p \cdot L(\frac{p}{q}) + C - C \\ &> M(t_0, t_0+p, q). \end{aligned}$$

Therefore for a given p the maximum of $M(t_0, t_0+p, q)$ is obtained for one of the finitely many q such that $\frac{p}{q} \in [a,b]$.

Proposition 5.13 *Assume that the average profit rate function has a maximum α_M. Then for any t_0 there exists an optimising semi-sequence with this initial condition.*

Proof: We will again use the same technique applying a limiting procedure and Tychonow's theorem. Choose the interval $[\alpha^-, \alpha^+] \subset (\rho^-, \rho^+)$ such that $\alpha_M \in [\alpha^-, \alpha^+]$ and a sequence of rationals $\frac{p_k}{q_k} \in [\alpha^-, \alpha^+]$, such that $\frac{p_k}{q_k} \to \alpha_M$. We choose $q_n > N$ large enough to apply Lemma 5.4 and let C be the resulting constant. Find an interval $[a,b]$ in the same way as in the proof of Lemma 5.12 such that for each k the maximum of $M(t_0, t_0+p_k, q)$ is obtained for some $\frac{p_k}{q_k^{max}} \in [a,b]$. It is easy to see that

$$\frac{p_k}{q_k^{max}} \to \alpha_M. \tag{39}$$

Now we are ready to construct a convergent sequence of schedules. We choose an optimising segment (using Lemma 5.12) $(t_0^k, t_1^k, ..., t_{q_k^{max}}^k)$, where $t_0^k = t_0$, $t_{q_k^{max}}^k = t_0 + p_k$, and define $t_i^k = i$ for $i > q_k^{max}$. Since $\frac{p_k}{q_k^{max}} \in [a,b]$, relation (33) (Lemma 5.3) implies that for fixed i, t_i^k is bounded, and hence the sequence $\mathbf{t}^k$ (applying Tychonow's theorem) has a convergent subsequence, which converges to a semi-sequence $\mathbf{t}$.

The optimising segments get longer for larger k ($q_k^{max} \to \infty$), and therefore we can ignore the tail $t_i^k = i$ when we analyse the properties of $\mathbf{t}$. By (39) and (33), the semi-sequence $\mathbf{t}$ has mean spacing α_M. Lemma 3.10 implies that $\mathbf{t}$ is weakly-maximising, and Lemma 3.11 shows that $\mathbf{t}$ is feasible and therefore maximising. Since a segment of arbitrary length for k large enough in $\mathbf{t}^k$ is optimising, as in the proof of Lemma 3.10 it follows that $\mathbf{t}$ is optimising.

In the proof we have constructed an optimising semi-sequence as a limit of periodic sequences with mean spacings which converge to α_M. However, each such semi-sequence approximates only a finite part of the limiting optimising semi-sequence with given initial condition. There is a better strategy (cf. [BAN2]) which is more suitable for numerical calculations. We use the property that each maximising semi-sequence is ω-asymptotic to a recurrent sequence. We find an approximate recurrent optimising sequence $\mathbf{u}$, $|u_0 - t_0| < T$ (using some of the procedures referred to in Section 7). Then we find a maximising segment between t_0 and u_n for n large enough and glue on the rest of the $\mathbf{u}$ sequence.

5.4 Asymptotically optimising sequences

We wish to characterise all optimising semi-sequences. For this it is convenient first to slightly extend the definition.

Definition 5.1 *The semi-sequence* $\mathbf{t}$ *is* **asymptotically optimising** *if for each* $\delta > 0$ *there exists an* N *such that for any integers* $n > m \geq N$ *and any segment* $(u_0, u_1, ..., u_k)$ *of arbitrary length with the same boundary conditions as the segment* $(t_m, ...t_n)$*,i.e.* $u_0 = t_m$, $u_k = t_n$, *we have*

$$V(u_0, u_1, ..., u_k) - V(t_m, t_{m+1}, ..., t_n) < \delta.$$

Clearly, optimising is a particular case of asymptotically optimising.

Proposition 5.14 *If* $\mathbf{t}$ *is a maximising semi-sequence with a mean spacing* α *which is not a maximum of* L*, then* $\mathbf{t}$ *is not asymptotically optimising.*

Proof: Suppose that the average profit rate function is strictly increasing on some neighbourhood $[\alpha - d, \alpha + d] \subset (\rho^-, \rho^+)$ of α (the strictly decreasing case is analogous). We apply Lemma 5.4 to the interval $[\alpha^-, \alpha^+] = [\alpha - d, \alpha + d]$ and find N, C such that for each $|u - u'| \leq T$, $|v - v'| \leq 8T$ and $n > N$,

$$|M(u, v, n) - M(u', v', n)| < C. \tag{40}$$

Choose $\delta = d/4$ and find a constant $D > 0$ such that $x, y \in [\alpha - d, \alpha + d]$, $x - y > \delta$ implies $L(x) - L(y) > D$ (this is possible since $L(x) - L(y)$ is a continuous function on a compact set bounded away from 0).

Approximate α by a rational p/q, with p, q large enough such that $0 < |\alpha - p/q| < 1/q$, $pTD > 3C$, $q > 2N$ and $p/q \in [\alpha - d/4, \alpha + d/4]$. Then using (33) we get for any i,

$$|t_{i+q} - t_i - pT| \quad \leq \quad |t_{i+q} - t_i - \alpha q T| + |(\alpha - \frac{p}{q})qT| \leq 2T. \tag{41}$$

Choose k such that $\frac{p}{q-k} - \frac{p}{q} > \delta$, $\frac{p}{q-k} \in [\alpha - d, \alpha + d]$, $q - k > N$.

If $\mathbf{u}, \mathbf{v}$ are periodic maximising sequences with periods $\frac{p}{q}$, $\frac{p}{q-k}$ respectively such that $|u_i - t_i| < T$, $|v_i - t_i| < T$, then relations (40) and (41) imply that

$$M(t_i, t_{i+q}, q) < C + M(u_i, u_{i+q}, q) = pTL(\frac{p}{q}) + C,$$

$$pTL(\frac{p}{q-k}) = M(u_i, u_{i+q}, q-k) < M(t_i, t_{i+q}, q-k) + C.$$

We now calculate:

$$\begin{aligned} M(t_i, t_{i+q}, q) &< pTL(\frac{p}{q}) + C \\ &\leq pT(L(\frac{p}{q-k}) + D) + C \\ &\leq -2C + pTL(\frac{p}{q-k}) \\ &\leq M(t_i, t_{i+q}, q-k) - C. \end{aligned}$$

Since i was arbitrarily large and C fixed (depending only on the interval $[\alpha - d, \alpha + d]$), $\mathbf{t}$ is not asymptotically optimising.

Lemma 5.15 *Assume that two maximising semi-sequences* $\mathbf{t}$, $\mathbf{u}$, *with* $t_0 = u_0$, *are* ω*-asymptotic to each other. If one of them is optimising, then so is the other one.*

Proof: Assume that $\mathbf{u}$ is optimising and $\mathbf{t}$ is not, i.e. there exists $i > 0$ and a segment $(v_0, \ldots, v_j)$ with $v_0 = t_0, v_j = t_i$, such that

$$V(v_0, \ldots, v_j) - V(t_0, \ldots, t_i) = C > 0. \tag{42}$$

Since $\mathbf{u}$ is maximising, $|u_{n+1} - u_n|$ is bounded by a constant K. Continuity of V and the condition (H1) imply that V is uniformly continuous on the set $\{(x,y) : |y - x| \leq K\}$, therefore we can find $n > i$ large enough that $|V(u_{n-1}, u_n) - V(u_{n-1}, t_n)| < C/2$. Relation (42) implies that there exists a segment $(v'_0, \ldots, v'_{n-i+j})$ with $v_0 = t_0, v_{n-i+j} = t_n$, such that $V(v'_0, \ldots, v'_{n-i+j}) - V(t_0, \ldots, t_n) \geq C > 0$, and therefore

$$\begin{aligned} V(u_0, v'_1, \ldots, v'_{n-i+j-1}, u_n) &> V(v'_0, \ldots, v'_{n-i+j}) - C/2 && (43) \\ &\geq V(t_0, \ldots, t_n) + C/2 && (44) \\ &> V(u_0, \ldots, u_n), && (45) \end{aligned}$$

which contradicts $\mathbf{u}$ being optimising.

Now we can obtain a close relationship between being optimising and maximising the average profit rate.

Theorem 5.16 *Let* $\mathbf{t}$ *be a maximising semi-sequence with the mean spacing* α_M *which maximises the average profit rate* L*. Then* $\mathbf{t}$ *is asymptotically optimising. If furthermore,* α_M *is irrational, or* α_M *is rational and* $\mathbf{t}$ *is recurrent, then* $\mathbf{t}$ *is optimising.*

Proof: (i) If α_M is rational, p/q in lowest terms, and maximising semi-sequence $\mathbf{t}$ is recurrent, then it is periodic with period (p, q). Suppose $\mathbf{t}$ is not optimising. Then there exist $m < n$ and $(v_m, \ldots, v_k)$ with $v_m = t_m$ and $v_k = t_n$ such that $V(v_m, \ldots, v_k) - V(t_m, \ldots, v_n) = C > 0$. Let $j > 0$ be such that $jq > n-m$. Then $V(t_m = v_m, \ldots, v_k = t_n, \ldots, t_{m+jq}) - V(t_m, \ldots, t_{m+jq}) = C$. Form a sequence $\mathbf{u}$ by repeating $t_m = v_m, \ldots, v_k = t_n, \ldots, t_{m+jq}$ periodically, with translation

jp after each period. Then $L(\mathbf{u}) - L(\mathbf{t}) = \frac{C}{jp} > 0$. So $\mathbf{t}$ does not maximise the average profit rate, contradicting Proposition 5.6.

(ii) Proposition 5.13 implies that there exists an optimising semi-sequence $\mathbf{v}$ with $v_0 = t_0$. Proposition 5.14 implies that $\mathbf{v}$ has mean spacing α_M. Since $\mathbf{v}$ is optimising, $\mathbf{v}$ is maximising. Thus both $\mathbf{t}$ and $\mathbf{v}$ fit in the same lamination of maximising sequences of mean spacing α_M. If α_M is irrational, this implies that they are asymptotic to each other. Thus by Lemma 5.15, $\mathbf{t}$ is optimising.

(iii) If α_M is rational p/q, and $\mathbf{t}$ is not recurrent, then $\mathbf{t}$ is either asymptotic to $\mathbf{v}$, and hence optimising, or $\mathbf{v}, \mathbf{t}$ are asymptotic to different maximising periodic sequences of mean spacing p/q. Let $\mathbf{w}$ be the periodic sequence to which $\mathbf{t}$ is asymptotic. Given $\delta > 0$ choose N large enough that $|V(t_j, t_{j+1}) - V(t_j, w_{j+1})|$, $|V(w_j, w_{j+1}) - V(w_j, t_{j+1})| < \delta/3$ for all $j \geq N$. Given $j > i > N$, the profit for segment $(t_i, \dots, t_j)$ can not be less than that for $(w_i, \dots, w_j)$, by more than $2\delta/3$, else one could switch from $\mathbf{t}$ to $\mathbf{w}$ and back, incurring two switching costs of at most $\delta/3$ but making a net gain, contradicting $\mathbf{t}$ maximising. If $(u_i, \dots, u_k)$ were a variation of a segment $(t_i, \dots, t_j)$ with $j > i \geq N$, $u_i = t_i, u_k = t_j$, which increased the profit by more than δ, then we could switch from $(w_{i-1}, \dots, w_{j+1})$ to $(w_{i-1}, u_i, \dots, u_k, w_{j+1})$. This would produce a segment of profit at least $\delta - 2\delta/3 = \delta/3$ greater, and so contradict the optimality of $\mathbf{w}$ which we proved in part (i).

By the same technique, we also obtain the following result for two-sided sequences.

Theorem 5.17 *Let* $\mathbf{t}$ *be a recurrent maximising sequence with the mean spacing* α_M *which maximises the average profit rate* L*. Then* $\mathbf{t}$ *is an optimising sequence.*

Remark: We have shown (Proposition 4.3) that each maximising semi-sequence is ω-asymptotic to a recurrent one. We conclude that in the case that L has no maximum (i.e. (H6) is not satisfied), there is no optimising semi-sequence. This can happen only when for each x, $L(x)$ is negative. The average profit is then always an average loss, and the optimal strategy is to schedule no buses at all (see Example 7.1)!

We have obtained the following characterisations of asymptotically optimising semi-sequences, and of optimising semi-sequences if α_M is irrational, which are consequences of Theorem 5.16 and Proposition 5.14.

Theorem 5.18 *Let* α_M *maximise the average profit rate. A semi-sequence* $\mathbf{t}$ *is asymptotically optimising if and only if it is maximising and has mean spacing* α_M*. If* α_M *is irrational,* $\mathbf{t}$ *is optimising if and only if it is maximising of mean spacing* α_M*.*

We conclude this subsection with:

Proposition 5.19 *If the maximising mean spacing* α_M *is irrational, the set* $\mathcal{T} \subset \mathbb{R}$ *of initial conditions* t_0 *for which there exists more than one optimising semi-sequence is countable.*

Proof: Choose $t_0 \in \mathcal{T}$, and two optimising semi-sequences $\underline{\mathbf{t}} \neq \overline{\mathbf{t}}$, with $t_0 = \underline{t}_0 = \overline{t}_0$. Since $\underline{t}_1 \neq \overline{t}_1$, we can choose a rational number $q(t_0)$ between $\underline{t}_1$ and $\overline{t}_1$, and define a function η :

$\mathcal{T} \to \mathbb{Q}$; $t_0 \mapsto q(t_0)$. The same argument as in Lemma 3.7 implies that two optimising (and hence maximising) semi-sequences asymptotic to the same sequence can not cross. This, and the description of the set of optimising semi-sequences in Theorem 5.18, imply that the function η is injective.

5.5 Relative profit

The problem remaining is how in the case of rational α_M to characterise those sequences which are optimising and not just asymptotically optimising. So suppose α_M is rational, p/q in lowest terms. For given t_0, if it is on a maximising periodic sequence of mean spacing p/q, then that is the only maximising semi-sequence from t_0 (Corollary 4.5) and thus the only optimising one.

If t_0 is not on a maximising periodic sequence of mean spacing p/q, then every maximising semi-sequence starting at t_0 is either *advancing* ($t_{n+q} > t_n + p$ for all n) and asymptotic to the nearest maximising periodic sequence of mean spacing p/q starting after t_0, or *retreating* ($t_{n+q} < t_n + p$ for all n) and asymptotic to the nearest maximising periodic sequence of mean spacing p/q starting before t_0. There exist maximising semi-sequences from t_0 of each type [BAN2]. There exists an optimising semi-sequence among them, and all maximising semi-sequences with the same asymptotic behaviour as it are also optimising. The question is how to decide which of the two possibilities, or both, are optimising.

The answer lies in comparing "relative profits".

Definition 5.2 *For a maximising semi-sequence* $\mathbf{t}$ *of mean spacing* p/q*,* ω*-asymptotic to a maximising periodic sequence* $\mathbf{u}$*, the relative profit is*

$$R(\mathbf{t}) = \sum_{i \geq 0} V(t_i, t_{i+1}) - V(u_i, u_{i+1}). \tag{46}$$

The sum is convergent because given $\delta > 0$ there exists N such that the switching cost $|V(t_n, u_{n+1}) + V(u_n, t_{n+1} - V(t_n, t_{n+1}) - V(u_n, u_{n+1})| < \delta$ for all $n \geq N$. Thus

$$|\sum_{i=m}^{n-1} V(t_i, t_{i+1}) - V(u_i, u_{i+1})| < 2\delta$$

for all $n > m \geq N$, else one of $\mathbf{t}$ and $\mathbf{u}$ would not be maximising. So the sequence of partial sums for $R(\mathbf{t})$ is Cauchy and hence converges.

Similarly, for a backwards semi-sequence $\mathbf{t} = (t_i)_{i \leq 0}$, α-asymptotic to $\mathbf{v}$, we define

$$R^-(\mathbf{t}) = \sum_{i<0} V(t_i, t_{i+1}) - V(v_i, v_{i+1}). \tag{47}$$

Now suppose that $\mathbf{a}$ and $\mathbf{r}$ are respectively an advancing and a retreating maximising semi-sequence of the maximising mean spacing p/q, from the same point $a_0 = r_0$. Denote the maximising periodic sequences to which they are asymptotic by $\mathbf{A}$ and $\mathbf{R}$, respectively.

Theorem 5.20 *The advancing maximising semi-sequence* $\mathbf{a}$ *is optimising iff*

$$R(\mathbf{a}) \geq R(\mathbf{r}) + V(A_{-1}, A_0) + \sum_{j \in J} (R(\rho^j) + R^-(\rho^j)), \tag{48}$$

where J is the set of gaps in the set of periodic maximising sequences of mean spacing p/q from $\mathbf{R}$ *to* $T_{1,0}\mathbf{A}$ *(empty in the case* $\mathbf{R} = T_{1,0}\mathbf{A}$*), and ρ^j is a retreating heteroclinic maximising sequence in gap j.*

Similarly, the retreating maximising semi-sequence $\mathbf{r}$ *is optimising iff*

$$R(\mathbf{r}) \geq R(\mathbf{a}) - V(R_0, R_1) + \sum_{j \in J}(R(\alpha^j) + R^-(\alpha^j)), \tag{49}$$

where α^j is an advancing heteroclinic maximising sequence in gap j.

Remark: Generically, J is finite (and empty if $p = 1$), but if J is infinite, the sums (48) and (49) still make sense, at least if V is Lipschitz, because the contribution from a given gap is then at most of order its width, and the sum of the widths of the gaps is bounded.

Proof: If (48) is not true, then for N large enough, instead of following $\mathbf{a}$ from a_0 to a_N it is better to follow $\mathbf{r}$, switch successively to $T_{-k_j p, -k_j q}\rho^j$ for some k_j (or a finite subsequence of them in case there are infinitely many) chosen to make the total switching gain small, and finally switch to $T_{1,0}\mathbf{a}$, ending at a_N after $N+1$ actions with more profit.

Conversely, if (48) holds but $\mathbf{a}$ is not optimising, then one can do better from a_0 to a_N for some N by a different route $\mathbf{b}$ than $\mathbf{a}$. It must involve a different number of actions because $\mathbf{a}$ is maximising. It can not be achieved by fewer actions, else $\mathbf{b}$ crosses $\mathbf{A}$ and then we could switch at the crossing to make a schedule from a_0 to A_N in N steps, and a schedule from A_0 to a_N in $N-1$ steps with greater total profit. Extend each to A_{kq} and a_{kq} respectively for k large enough that switching them over would not lose more than has been gained. The first schedule can not be better than $\mathbf{a}$ since it joins the same points in the same number of steps. Thus the second one is better than $\mathbf{A}$. But it connects A_0 to $A_0 + kp$ in $kq - 1$ steps, so has mean spacing different from p/q, contradicting p/q maximising the average profit rate.

Thus the better route from a_0 to a_N must involve one or more extra actions. A similar argument to the above shows that without loss of generality it involves just one extra action. But a better route from a_0 to a_N with one more action implies the same for all larger N, with at least as much gain, by simply extending by $\mathbf{a}$. Comparison with the route given by following $\mathbf{r}$, switching to $T_{-k_j p, -k_j q}\rho^j$ successively, and ending at a_N after $N+1$ actions, yields a contradiction.

The proof for (49) is analogous.

Theorem 5.20 has a nice geometric interpretation in the case that V is C^2 and all maximising periodic sequences of mean spacing p/q are hyperbolic (which is C^2-generic). Following [MMP] (the proof there can be streamlined), the relative profit of an advancing stationary semi-sequence $\mathbf{a}$ ω-asymptotic to a periodic sequence $\mathbf{A}$ is the area under the stable manifold of A_0 from a_0, for the associated area-preserving twist map $(t, p) \mapsto (t', p')$. The relative profit of a retreating stationary semi-sequence $\mathbf{r}$ ω-asymptotic to a periodic sequence $\mathbf{R}$ is minus the area under the stable manifold of R_0 from r_0. The relative profit of a backwards retreating semi-sequence ρ^- α-asymptotic to a periodic sequence $\mathbf{u}$ is minus the area under the unstable manifold of u_0 from ρ_0^-, and that for a backwards advancing semi-sequence α^- α-asymptotic to a periodic sequence $\mathbf{u}$ is the area under the unstable manifold of u_0 from α_0.

Thus **a** is optimising iff the area under the chain formed by the stable manifold of A_0 from a_0, the stable manifold of R_0 from r_0, the unstable manifold of the appropriate periodic point up to ρ_0^j and the stable manifold from ρ_0^j, ending at A_{-1} (see Fig.2), is at least $V(A_{-1}, A_0)$.

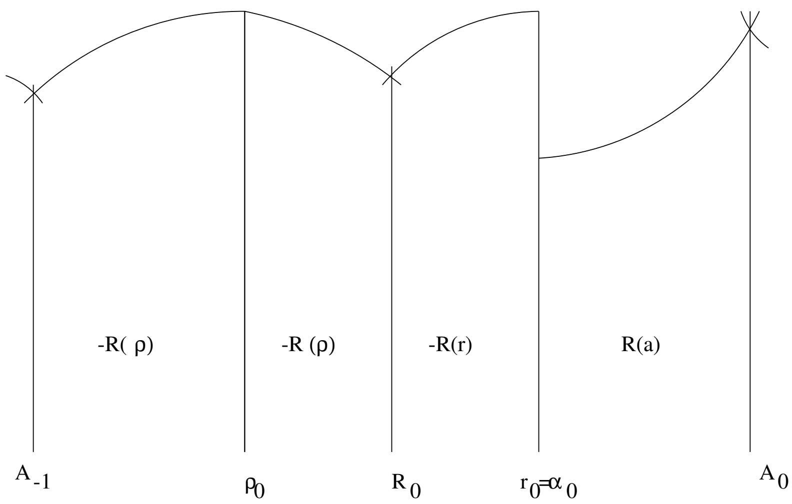

Figure 2: The area entering the criterion for an advancing maximising semi-sequence to be optimising.

Similarly, **r** is optimising iff the area under the chain formed by the stable manifold of R_0 from r_0, the stable manifold of A_0 from a_0, the unstable manifold up to α_0^j and the stable manifold from α_0^j, ending at R_1, is at most $V(R_0, R_1)$.

Remarks:

1. It can happen that both **a** and **r** are optimising, even for a whole interval of initial condition.
2. In the special case $A_0 = R_1$, optimising semi-sequences can not cross (this is easily proved by contradiction).
3. The set of initial conditions for which there exist more than one advancing or more than one retreating optimising semi-sequence is countable (proof analogous to that of Prop 5.19).
4. One might ask which two-sided maximising sequences are optimising, beyond the periodic ones. Using the same notation as in Theorem 5.20, an advancing maximising sequence **a** is optimising iff

$$R(\mathbf{a}) + R^-(\mathbf{a}) \geq V(A_{-1}, A_0) + \sum_j (R(\rho^j) + R^-(\rho^j)), \tag{50}$$

   and a retreating maximising sequence is optimising iff

$$R(\mathbf{r}) + R^-(\mathbf{r}) \geq -V(R_0, R_1) + \sum_j (R(\alpha^j) + R^-(\alpha^j)). \tag{51}$$

5.6 Summary

We can summarise this section by giving the following characterisation of optimising semi-sequences, which is the main result of the paper:

Theorem 5.21 *A semi-sequence* $\mathbf{t}$ *is optimising iff it is maximising with a mean spacing* α_M *which maximises the average profit rate, and either (i)* α_M *is irrational, or (ii)* α_M *is rational and* $\mathbf{t}$ *is periodic or it satisfies (48) if advancing or (49) if retreating.*

6 Robustness

The set M_α^{rec} of recurrent optimising sequences (where α is the mean spacing) is totally ordered and (modulo translation by an integer multiple of T) is topologically either a circle or a closed non-empty subset of a circle which in the case of irrational mean spacing is a Cantor set and for rational mean spacing $\frac{p}{q}$ is composed of periodic sequences of type (p, q) and is typically finite. In the case of a circle, there is unique optimising semi-sequence for every $t_0 \in \mathbb{R}$, namely the one on the circle with this initial condition, and it can be written $t_n = g(n\alpha + \theta)$ for some function g and some initial phase θ. As every optimising semi-sequence is asymptotic to an optimising recurrent sequence it is important to study the difference between the two cases.

The case of a circle with rational mean spacing is highly exceptional. On the other hand by KAM theory (e.g. [MOS]), for smooth enough "transversal" one-parameter families of profit functions $V_\mu(t, t') = W_\mu(t, t' - t)$ with weak dependence of W on t, circles with irrational mean spacing occur for a set of nearly full measure of parameter values μ (corresponding to mean spacing satisfying a Diophantine condition).

Suppose that there is an external effect that disrupts the schedule, for example, a traffic jam holds up the shuttle buses. Then if the set M_α^{rec} is a circle, the new optimising schedule is phase-shifted, by resetting θ.

If M_α^{rec} is not a circle, however, then the new optimising schedule may well be asymptotic to the old one. This is the case if α is rational, M_α^{rec} consists of finitely many periodic schedules and the perturbation is small enough.

7 Illustrations

We will construct several examples and calculate, either analytically or numerically, the average profit rate function and its maximum.

Example 7.1 Integrable limit

The profit function is of the form $V(x, y) = f(y - x)$. The function f can be defined on some interval $(a, b) \subset (0, +\infty)$. Then the domain is $D = \{(x, y) \in \mathbb{R}^2,\ a < y - x < b\}$, and the twist interval is (a, b). Conditions (H1)-(H5) are satisfied if and only if f is continuous, concave and converges to $-\infty$ on the boundaries a, b.

The maximising sequences are all sequences $\mathbf{t}$ such that $t_{i+1} - t_i = \alpha$ for each i, where $\alpha \in (a, b)$ is the mean spacing. Indeed, if $(t_m, u_{m+1}, ..., u_{n-1}, t_n)$ is some variation of a segment $(t_m, t_{m+1}, ..., t_{n-1}, t_n)$, applying the well-known Jensen inequality for concave functions we get:

$$\begin{aligned} V(t_m, t_{m+1}, ..., t_{n-1}, t_n) &= (n-m)f(\alpha) \\ &= (n-m)f(\frac{\sum_{k=m}^{n-1}(t_{k+1}-t_k)}{n-m}) \\ &= (n-m)f(\frac{\sum_{k=m}^{n-1}(u_{k+1}-u_k)}{n-m}) \\ &\geq \sum_{k=m}^{n-1} f(u_{k+1}-u_k) \\ &= V(t_m, u_{m+1}, ..., u_{n-1}, t_n). \end{aligned}$$

Equality holds if and only if segments $\mathbf{t}$ and $\mathbf{u}$ are equal (a maximising sequence is strictly maximal for each finite variation). The average profit rate is $L(\alpha) = \frac{f(\alpha)}{\alpha}$. If f is differentiable, we can calculate the mean spacing which maximises the average profit rate by solving the equation $L'(x) = 0$, i.e.

$$f'(x) = L(x) = \frac{f(x)}{x}.$$

If there is more than one solution, we take the smallest (since $f'(x)$ is decreasing).

We will now construct an example where there is no optimising sequence. Define $f(x) = -1/x - x$, so $V(x, y) = -1/(y-x) - (y-x)$. The average profit rate is now $L(x) = -1/x^2 - 1$, and it is a strictly increasing function. Proposition 5.14 implies that there is no optimising sequence. There is no optimising segment in any sense. It is easy to see that removing an arbitrary t_i from an arbitrary segment $(t_0, t_1, ..., t_n)$ or postponing the last action t_n, one increases the value of the function $V(t_0, ..., t_n)$. The average profit rate $L(x)$ is always less than -1, and the best strategy is to never take any action.

Example 7.2 *:* **"Standard" profit functions**

$$V(x, y) = f(y-x) + g(x), \tag{52}$$

where $f : \mathbb{R}^+ \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous, f is concave, $\lim_{x\to 0} f(x) = \lim_{x\to\infty} f(x) = -\infty$, and g is periodic with period T. Then the conditions (H1) - (H5) are satisfied. The domain is $D = \{(x, y) \in \mathbb{R}^2,\ y > x\}$, and the rotation interval is $(0, \infty)$. A three-parameter ($\mu > 0$, b, K) family of such profit functions is:

$$V(x, y) = \log(y-x) - \mu \cdot (y-x)^2 + \frac{b}{2\pi}\sin(2\pi x) + K(y-x). \tag{53}$$

For a fixed rational p/q we can find numerically a maximising sequence with this mean spacing. We used a simple gradient method, but the minimisation eigenvalue method [GRI] would be more subtle. On Fig.3 we plot maximising sequences with mean spacing $\frac{7}{19}$, for parameters $\mu = 1$ and $b = 0.5$ or 5. We plot $(x_i, b\sin 2\pi x_i)$, which has physical meaning (configuration in the potential). Neither the maximising sequences nor the stationary ones depend on K, since the term $K(x-y)$ is a "coboundary" (its sum along a sequence telescopes to a difference between end terms). The mean action depends on K as $A_{\mu,b,K}(\alpha) = A_{\mu,b,0}(\alpha) + K\alpha$ and the average profit rate as $L_{\mu,b,K}(\alpha) = L_{\mu,b,0}(\alpha) + K$, so all the numerics can be done at one value of K.

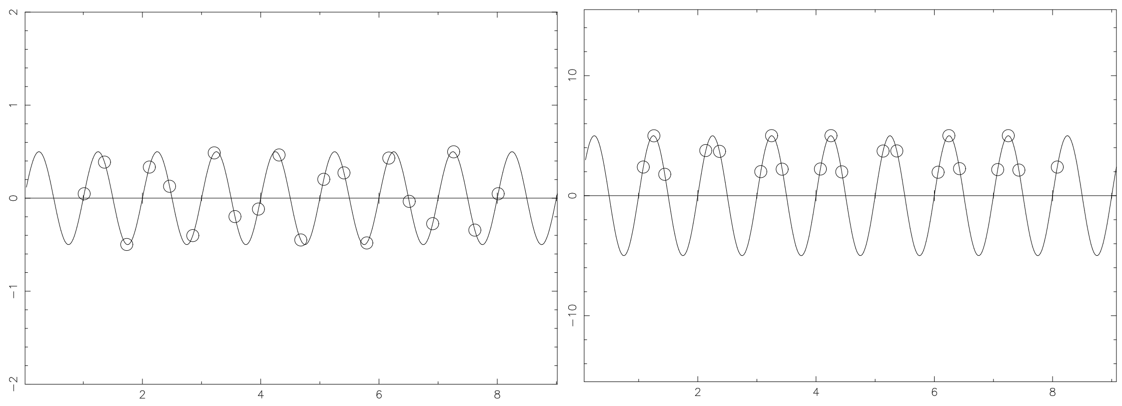

Figure 3: Maximising configuration with period $\frac{7}{19}$*, for* $\mu = 1$ *and* $b = 0.5$ *(left),* 5.0 *(right).*

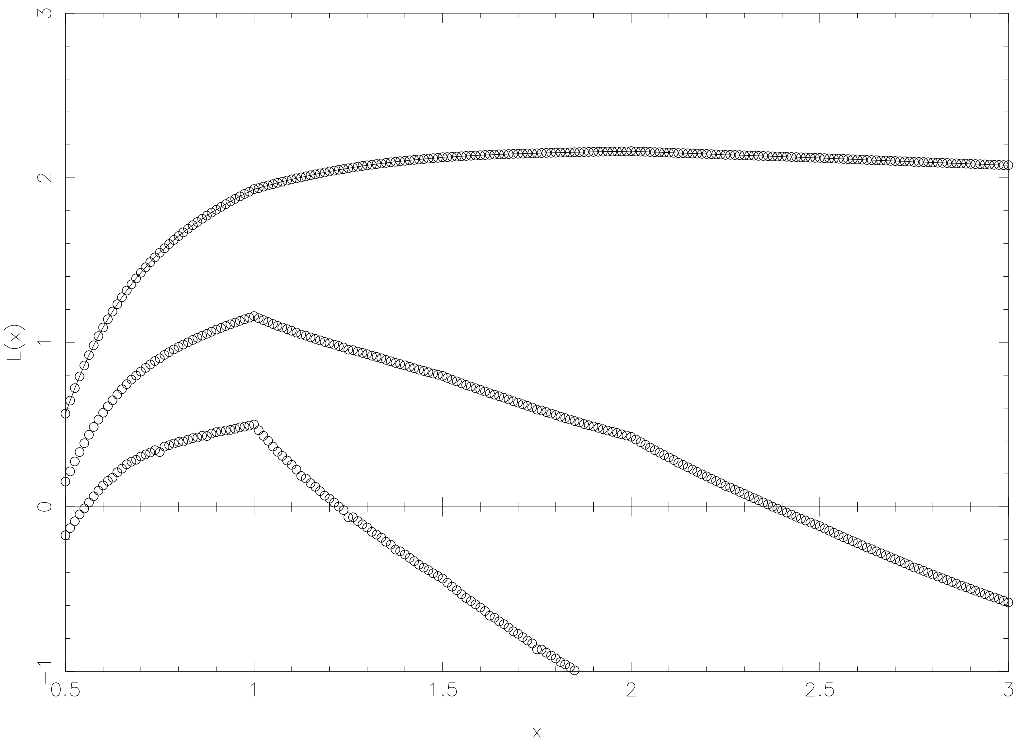

Figure 4: Average profit rate $L(x)$ *against mean spacing* x*, for (53), parameters* $K = 2$ *and* $\mu = 0.1$*,* $b = 0.2$*, (upper curve);* $\mu = 1$*,* $b = 1$ *(middle curve);* $\mu = 1.85$*,* $b = 2.2$ *(lower curve).*

We can calculate the average profit rate function by choosing many rational numbers, finding a corresponding maximising sequence and computing its average profit rate. We see in Fig.4 that it has a unique maximum, is almost everywhere differentiable, and is not necessarily a concave function (as opposed to the mean action A).

The final step in the numerical procedure is to find the maximal mean spacing for fixed parameters. We have used an adapted bisection procedure always keeping three rational points $\frac{p_l}{q_l} < \frac{p_m}{q_m} < \frac{p_r}{q_r}$, such that the value of the function L is always largest at the middle point. We choose a new rational point $\frac{p_n}{q_n} = \frac{p_b+p_m}{q_b+q_m}$, where b is either l or r. This procedure ensures that the denominator does not grow too fast, and we can safely find a maximising sequence with mean spacing $\frac{p_n}{q_n}$. We repeat the procedure until $|\frac{p_r}{q_r} - \frac{p_l}{q_l}| < \epsilon$, where ϵ is the desired accuracy.

In Fig.5 we plot the mean spacing α which maximises the average profit rate $L(\alpha)$ for different values of parameters μ, b of function (53) (the result does not depend on K).

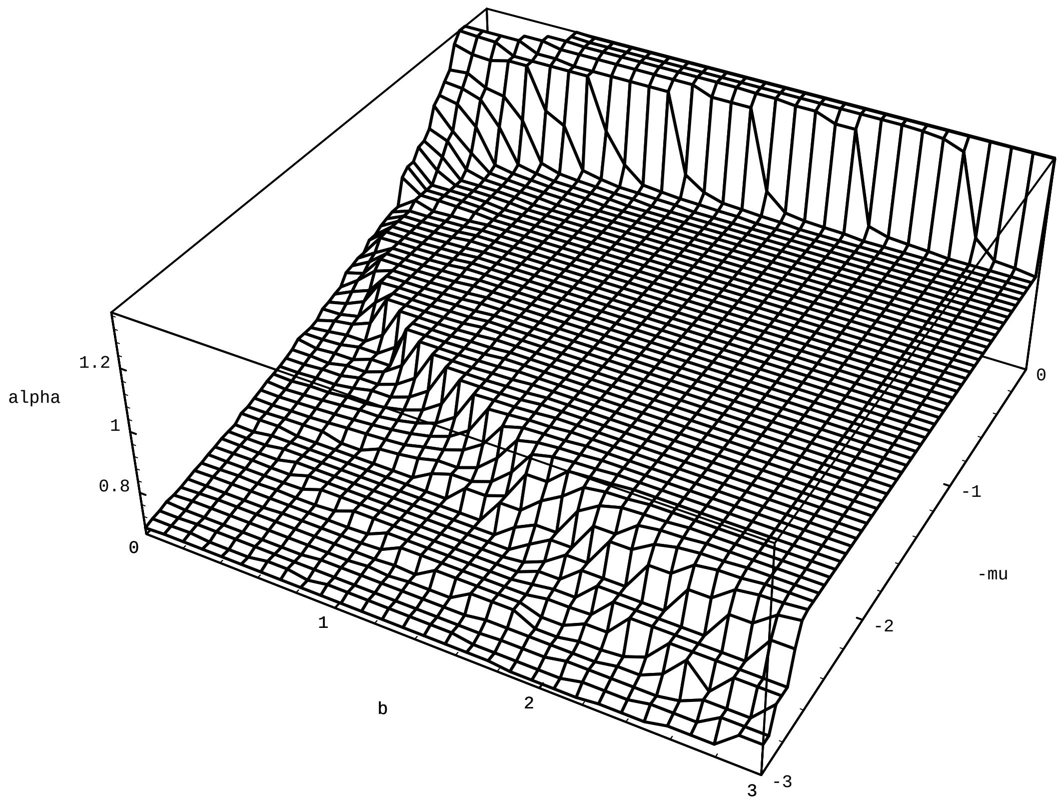

Figure 5: Maximising mean spacing α for (53), for parameters $0 < b < 3$, $-3 < -\mu < 0$.

The following theorem justifies Fig.5. In the case that we have a continuous family of profit functions, the maximal average profit rate is continuous as well. We have added a weak condition to avoid technical complications in the proof.

Theorem 7.3 *Assume that V_μ is a continuous family of profit functions, defined on the same domain D, with corresponding rotation interval (ρ^-, ρ^+). Assume that there is an interval $[a,b] \subset (\rho^-, \rho^+)$ and that for each μ the average profit function has a strict maximum $\alpha_{M,\mu} \in [a,b]$. Then the function $\mathcal{A} : \mu \mapsto \alpha_{M,\mu}$ is continuous.*

Proof: We write $\mathcal{A}$ as a composition $\mathcal{A} = \mathcal{X} \circ \mathcal{L}$. To do this we first define $\mathcal{S}$ to be the subspace of the space $(C[a,b], ||.||_\infty)$ of continuous functions on $[a,b]$ consisting of those functions with a unique strict maximum in $[a,b]$. The function $\mathcal{L} : \mathbb{R} \to \mathcal{S}$ assigns to each parameter μ the corresponding average profit function $L \in \mathcal{S}$. The function $\mathcal{X} : \mathcal{S} \to \mathbb{R}$ assigns to each function in $\mathcal{S}$ its maximum α_M. It is easy to check that $\mathcal{X}$ is a continuous function.

We prove that $\mathcal{L}$ is continuous. Assume that μ_n is a sequence of parameters, $\mu_n \to \mu_0$, and fix a rational mean spacing $\frac{p}{q} \in [a,b]$. Since $L_\mu(\frac{p}{q}) = \max\{V_\mu(t_0, t_1, ..., t_{q-1}, t_0 + pT)\}$, then $\mu \mapsto L_\mu(\frac{p}{q})$ is continuous and $L_{\mu_n}(\frac{p}{q}) \to L_{\mu_0}(\frac{p}{q})$. Since L_μ is continuous for each μ, and since $[a,b]$ is compact, $L_{\mu_n} \to L_{\mu_0}$ in sup norm. Hence $\mathcal{L}$ is continuous.

The maximising mean spacing typically locks at rationals (where the average profit rate is not differentiable). This is expected whenever a periodic maximising sequence does not exist for every t_0 because then the derivative of the mean action is discontinuous at rationals ([MAT2]). In Fig.4 it is obvious that for some high values of parameters the maximum $\alpha_M = 1$ is very sharp (the left derivative is much bigger than the right derivative), and this explains the large plateau of parameter values for which the maximising mean spacing is 1. On the other hand, the maximising mean spacing never locks at irrationals because the mean action is differentiable at irrationals ([MAT2]).

8 Discussion

We have analysed the problem of optimising

$$\sum_{n \geq 1} V(t_{n-1}, t_n)$$

for a T-periodic function $V(t+T, t'+T) = V(t,t')$, given t_0. We have argued that the appropriate definition of an optimising schedule $\mathbf{t}$ is that $\forall 0 \leq m < n$, $k > 0$ and $u_0, ..., u_k$ with $u_0 = t_m$, $u_k = t_n$,

$$\sum_{j=m+1}^{n} V(t_{j-1}, t_j) \geq \sum_{j=1}^{k} V(u_{j-1}, u_j).$$

We have proved that for each t_0, there exists an optimising schedule. All such schedules have the same mean spacing α_M, which maximises the "average profit rate" $L(\alpha)$, and we have characterised them. The maximising mean spacing is typically a devil's staircase function of parameters, continuous but locally constant at each rational value.

It would be very interesting to apply the theory to realistic profit functions V for some real-world problems. Of course, the model has its limitations. Firstly, V might not be exactly periodic, e.g. weekends disrupt 24-hour periodicity, and public holidays disrupt weekly periodicity.

Secondly, the profit for the n^{th} action might depend on more than just t_n and t_{n-1}, e.g. new investment would be required to send buses at closer time intervals than the round trip time divided by the current number of buses owned. Nonetheless, we believe it is a class of models worth applying as a first step for dynamic optimisation problems in a periodic environment.

Acknowledgements

The paper was born in July 1992, during a workshop on "Nonlinear Dynamics in Economics", organised by Mark Salmon in Florence, and the principal results were presented at the British Mathematics Colloquium in April 1996. RSM and JS are grateful to L. Montrucchio for stimulating their interest in dynamic optimisation and to the UK Centre for Economic Policy Research for financial support to attend the workshop. JS would like to thank the Nuffield Foundation for the provision of computing facilities, to the Newton Institute, Cambridge, for its hospitality and financial support during the summer of 1995 during which most of the work in Section 2 was carried out, and for NATO collaborative research grant CRG 921181. SS would like to thank *Agrokor d.d.*, Zagreb, for a scholarship which partially supported his stay in Cambridge from 1996, where the paper was completed.

References:

(A) S.Aubry, Complete devil's staircase in the one-dimensional lattice gas, J Physique Lettres **44** (1983) L247–L250.

(ALD) S.Aubry, P.LeDaeron, The discrete Frenkel-Kontorova model and its extensions. I. Exact results for the ground states, *Physica D*, **8**, 381-422 (1983)

(BAE) C.Baesens, Area preserving maps and Frenkel-Kontorova models, Lecture notes, unpublished;

(BAN) V.Bangert, Mather sets for twist maps and geodesics on tori, *Dyn. Reported*, Vol. 1, eds. U.Kirchgraber, H.O.Walter, Wiley&Sons (1988), 1–56

(BAN2) V.Bangert, Geodesic rays, Busemann functions and monotone twist maps, *Calc. of Variations and Part.Dif.Eq.*, **2**, 49-63 (1994)

(BM) M.Boldrin and L.Montrucchio, On the indeterminancy of capital accumulation paths, *J.Econ.Theory*, **40**, 26-39 (1986)

(CHL) D.A.Carlson, A.B.Haurie, A.Leizarowitz, Infinite horizon optimal control (2^{nd} ed.), Springer-Verlag (1991)

(F) A.Fathi, Théorème KAM faible et théorie de Mather sur les systèmes Lagrangiens, *C.R.Acad. Sci.Paris*, **324** (série I), 1043-1046 (1997)

(GRI) R.B.Griffiths, Frenkel-Kontorova models of commensurate-incommensurate phase transitions, in *Fundamental problems in statistical mechanics VII* , 69-109, Elsevier Sci.Pub., (1990)

(KH) A.Katok, B.Hasselblatt, Introduction to the modern theory of dynamical systems, Cambridge University Press (1995)

(KAT) A.Katok, More about Birkhoff periodic orbits and Mather sets for twist maps, IHES preprint $M/82/35$, (June 1982)

(KS) M.Kamien, N.L.Schwartz, Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management, North-Holland Elsevier Science, New York (1991)

(MMP) R.S.MacKay, J.D.Meiss, I.C.Percival, Resonances in area-preserving maps, Physica D 27 (1987) 1–20.

(MS) R.S.MacKay, J.Stark: Lectures on Orbits of Minimal Action for Area-Preserving Maps, Preprint, University of Warwick (1985)

(MAT) J.Mather, Existence of quasi-periodic orbits for twist homeomorphisms of the annulus, *Topology*, **21**, 457-467 (1982)

(MAT2) J.Mather, Differentiability of the Minimal Average Action as a Function of the Rotation Number, *Bol.Soc.Brasil.Math.*, **21** (n.s. 1), 59-70, (1990)

(MOS) J.K.Moser: Stable and random motions in dynamical systems (with special emphasis on celestial mechanics), Princeton University Press (1973)

(RT1) F.Rhodes, C.L.Thompson, Rotation numbers for monotone functions of the circle, *J.Lond. Math. Soc.*, **34**, 360-368, (1986)

(RT2) F.Rhodes, C.L.Thompson, Topologies and rotation numbers for families of monotone functions on the circle, *J.Lond. Math. Soc.*, **43**, 156-170, (1991)

(SL) S.L.Stokey and R.E.Lucas, Recursive methods in economic dynamics, Harvard University Press (1989)

(W) P.Whittle, Optimization Over Time: Dynamic Programming and Stochastic Control, 2 vols, Wiley (1982-3)

(YK) Yakovenko S.Yu., Kontorer L.A., Nonlinear Semigroup and infinite horizon optimization, in: Idempotent analysis, eds Maslov VP, Samborskii SN, Adv Sov Math **13** (1992) 167–210.