Dynamics of relative phases: generalised multibreathers

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Abstract

For small Hamiltonian perturbation of a Hamiltonian system of arbitrary number of degrees of freedom with a normally non-degenerate submanifold of periodic orbits we construct a nearby submanifold and an "effective Hamiltonian" on it such that the difference between the two Hamiltonian vector

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fields is small. The effective Hamiltonian is independent of one coordinate, the "overall phase", and hence the corresponding action is preserved. Unlike standard averaging approaches, critical points of our effective Hamiltonian subject to given action correspond to exact periodic solutions. We prove there has to be at least a certain number of these critical points given by global topological principles. The linearisation of the effective Hamiltonian about critical points is proved to give the linearised dynamics for the full system to leading order in the perturbation. Hence in the case of distinct eigenvalues which move at non-zero speed with ε, the linear stability type of the periodic orbit can be read off from the effective Hamiltonian. Our principal application is to networks of oscillators or rotors where many such submanifolds of periodic orbits occur at the uncoupled limit – simply excite a number $N \geq 2$ of the units in rational frequency ratio and put the others on equilibria, subject to a non-resonance condition. The resulting exact periodic solutions for weak coupling are known as multibreathers. We call the approximate solutions given by the effective Hamiltonian dynamics, "generalised multibreathers". They correspond to solutions which look periodic on a short time scale but the relative phases of the excited units may evolve slowly. Extensions are sketched to travelling breathers and energy exchange between degrees of freedom.

Keywords: breathers, phases, Hamiltonian, dynamics

1 Introduction

Since the seminal papers of Takeno et al. [1, 2], several papers have dealt with the mathematical analysis of "discrete breathers": spatially localised time-periodic solutions of networks of weakly coupled oscillators [3, 4, 5, 6, 7, 8, 9, 10, 11]. Discrete breathers can be constructed by continuation of solutions from the decoupled limit, for which some units are oscillating and the remainder are at rest. A discrete breather resulting from one oscillating unit at the uncoupled limit we call a "single breather"; those resulting from more than one oscillating unit we call "multibreathers".

The present paper firstly provides a variation on the general Hamiltonian proof of existence of "multibreathers" that was promised in [4], given independently in [5], and later worked out in [11]. Secondly, it derives their linear stability types more generally than [5]. Thirdly, and most significantly, it constructs approximate solutions which look like multibreathers but whose relative phases evolve slowly and are found to be governed by an effective Hamiltonian. Thus we enlarge the concept of "multibreather" to "generalised multibreather" solutions which are not necessarily exactly time-periodic, but which we expect to be of physical importance.

The simplest example of system to which our results apply is a chain of weakly coupled pendula in which at the uncoupled limit N pendula, not necessarily neighbours, are excited with approximately equal frequencies, non-resonant with the frequencies of linearised motion of the remaining pendula about stable equilibrium. Our method constructs a $\mathbb{T}^N \times \mathbb{D}^N$ in the phase space of the full system, which is approximately invariant and on which the vector field is close to that of an effective Hamiltonian H^{eff} on $\mathbb{T}^N \times \mathbb{D}^N$. The coordinates on the $\mathbb{T}^N \times \mathbb{D}^N$ can be thought of as a global phase, $N-1$ relative phases, a global action and $N-1$ relative momenta. The effective Hamiltonian does not depend on the global phase, thus the global action A is preserved and for each A, $H^{eff}_A = H^{eff}(A, \cdot)$ describes slow evolution of the relative phases and their momenta on a $\mathbb{T}^{N-1} \times \mathbb{D}^{N-1}$.

The idea of constructing an effective Hamiltonian for approximate dynamics of slow variables is old. For example, Chirikov [12] constructed an effective Hamiltonian for the dynamics of resonant phases $m \cdot \theta$ of a Hamiltonian system $H(I, \theta)$ on $T^*\mathbb{T}^N$ near the resonant surfaces where $m \cdot \frac{\partial H}{\partial I}$ is small, for one or more $m \in \mathbb{Z}^N \backslash \{0\}$. Earlier, Poincaré [13] introduced an effective Hamiltonian for the dynamics of the Laplace vectors - eccentricities and inclinations with their associated angles, for the planetary ellipses in the solar system by averaging over the relatively rapid orbital motion (for a history, see [14]).

Similarly, Whitham [15] introduced an avergaed Lagrangian to describe the evolution of the envelope of oscillatory solutions of some Lagrangian PDEs, and under the name of "collective coordinates" this idea was applied to slow evolution of solitons and breathers for perturbations of integrable PDEs by various people, e.g. [16, 17].

Our approach is weaker than that of Poincaré and Chirikov, in that we require periodic reference dynamics whereas they allow quasiperiodic. But our approach is stronger than theirs in two respects:

(i) critical points of our H^{eff}, including degenerate ones, are in one-to-one correspondence with exactly time-periodic solutions,

(ii) we allow the zone where the reference dynamics is good to be a neighbourhood of a torus of lower dimension than the total number of degrees of freedom, and obtain an effective Hamiltonian for just the reduced number of degrees of freedom, subject to a non-resonance condition on the normal dynamics.

Moser [18] obtained an effective Hamiltonian for the motion of relative phases on perturbing a vector field on a manifold consisting entirely of periodic solutions of constant period, but only to first order in the perturbation. Thus to deduce existence of periodic orbits for the perturbed system he requires non-degenerate critical points of his effective Hamiltonian. Also he did not consider the questions of perturbation of a vector field with only a submanifold of periodic solutions, nor of uniformity of the estimates with respect to system size.

Persistence of some number of periodic solutions from a submanifold of periodic solutions was obtained in [19], but no properties of the non-equiibrium solutions of the effective Hamiltonian are considered there, nor is uniformity with respect to system size considered.

In section 2, we specify the kind of Hamiltonian under study. In section 3 we give a method to find multibreather solutions. Section 4 presents the main result which enables one to describe approximate dynamics of generalised multibreathers. In section 5, we investigate the relation between the linear stability of equilibria of the approximate dynamics and of the multibreathers. In section 6 we apply the method to a range of situations in oscillator networks. Conclusions and discussions end the paper in section 7.

2 Relative momenta and relative phases for multibreathers

We consider the following type of C^2 Hamiltonian describing networks of coupled units (which may be either oscillators or rotors)

$$H_\varepsilon(\underline{p},\underline{q}) = \sum_{s\in S} H_s(p_s,q_s) + \varepsilon H^1(\underline{p},\underline{q}), \qquad (\underline{p},\underline{q}) = (p_s,q_s)_{s\in S} \in (\mathbb{R}^2)^S, \tag{1}$$

where S is a countable index set and ε is a coupling constant. For each $s \in S$ we assume that $H_s(p_s,q_s)$ is a 1 DOF Hamiltonian describing an oscillator or rotor and has an elliptic equilibrium (p_s^e,q_s^e). Without loss of generality, we set $(p_s^e,q_s^e) = (0,0)$ and $H_s(p_s^e,q_s^e) = 0$. We endow the phase space with the l_2 norm

$$||(\underline{p},\underline{q})||_2 := \sqrt{\sum_{s\in S} |(p_s,q_s)|^2},$$

so that the Hamiltonian in (1) is convergent.

At $\varepsilon = 0$ we choose a finite subset $S' \subset S$ of size $N+1 \geq 2$ and consider periodic solutions for which the units belonging to S' exhibit periodic motions with frequencies in the irreducible ratio $\underline{m} \in \mathbb{Z}^{N+1}$ and the remaining units are on elliptic equilibria. These solutions give multibreathers. The state of the units belonging to S' can be represented by action-angle variables $(I_s, \psi_s \bmod 1)$. We divide the variables as follows:

$$(\underline{p},\underline{q}) = (\underline{I},\underline{\psi},p_r,q_r),$$

where $\underline{I} = (I_s)_{s\in S'}$, $\underline{\psi} = (\psi_s)_{s\in S'}$, and r runs through $S'' = S \setminus S'$. The multibreathers of period T and ratio $\underline{m}$ at $\varepsilon = 0$ are given by

$$\underline{I}(t) = I_0, \qquad \underline{\psi}(t) = ft\,\underline{m} + \psi_0, \qquad (p_r(t),q_r(t)) = (\underline{0},\underline{0}),$$

where

$$f = \frac{1}{T}, \qquad (\underline{0},\underline{0}) := (0,0)_{s\in S''},$$

and I_0 is defined so that

$$\frac{dH_s}{dI_s}(I_{0s}) = f m_s, \qquad \forall s \in S'.$$

Thus the multibreathers with given period form an invariant $(N+1)$-torus $\mathbb{T}^{N+1}$. If the units belonging to S' are locally anharmonic, i.e. they satisfy $d^2H_s/dI_s^2 \neq 0$ (actually, it would suffice to have all but one of the units in S' anharmonic), then this $\mathbb{T}^{N+1}$ is part of an invariant set $E \times \mathbb{T}^{N+1}$ for some interval E in $\mathbb{R}$, filled by periodic orbits of the same frequency ratio with possibly different period. The projection of $E \times \mathbb{T}^{N+1}$ to the action space is a resonant curve $\underline{I} = \underline{I}^{res}(A)$ parametrised by the *area* $A = \underline{m} \cdot \underline{I}$, satisfying

$$\frac{\partial H_0}{\partial \underline{I}}(\underline{I}^{res}(A), \underline{0}, \underline{0}) = f(A)\, \underline{m},$$

for some function $f(A)$. It is easy to see that on $E \times \mathbb{T}^{N+1}$

$$\dot{\underline{I}} = 0, \qquad \dot{\underline{\psi}} = f(A)\, \underline{m}, \qquad \dot{A} = 0, \qquad \dot{p}_r = 0, \qquad \dot{q}_r = 0$$

and

$$f(A) = \frac{d}{dA} H_0(\underline{I}^{res}(A), \underline{0}, \underline{0}).$$

Since $m \in \mathbb{Z}^{N+1}$ is irreducible, there exists $M \in GL(N+1, \mathbb{Z})$ such that $Me_1 = m$ where e_1 is the first standard basis vector (see Appendix A of [20], for example). This induces a canonical change of coordinates for $(\underline{I}, \underline{\psi})$ given by

$$\underline{\psi} = M \begin{bmatrix} \varphi \\ \underline{\theta} \end{bmatrix}, \qquad \begin{bmatrix} A \\ \underline{J} \end{bmatrix} = M^t \underline{I}.$$

Then the new variables $(A, \varphi) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ are called the *area* and *overall phase* and $(\underline{J}, \underline{\theta}) \in \mathbb{R}^N \times \mathbb{R}^N/\mathbb{Z}^N$ are called the *relative momenta* and *relative phases*. The resonance curve $\underline{I} = \underline{I}^{res}(A)$ is transformed into $\{(A, \underline{J}^{res}(A)) : A \in E\}$. In the new variables we write the Hamiltonian as

$$H_\varepsilon = H^0(A, \underline{J}, p_r, q_r) + \varepsilon H^1(A, \varphi, \underline{J}, \underline{\theta}, p_r, q_r),$$

which satisfies

$$\begin{aligned} D_A H^0(A, \underline{J}^{res}(A), \underline{0}, \underline{0}) &= f(A) \\ D_{\underline{J}} H^0(A, \underline{J}^{res}(A), \underline{0}, \underline{0}) &= 0. \end{aligned}$$

The multibreathers at $\varepsilon = 0$ are represented by

$$\begin{array}{rclcl} A(t) = A, & & \varphi(t) = f(A)t + \varphi_0, & & \\ \underline{J}(t) = \underline{J}^{res}(A), & \underline{\theta}(t) & = & \underline{\theta}, & (p_r(t), q_r(t)) = (\underline{0}, \underline{0}), \end{array} \qquad (2)$$

for arbitrary $(A, \varphi_0, \underline{\theta})$, which parametrise the invariant $E \times \mathbb{T}^{N+1}$. Note that $(\varphi_0, \underline{\theta})$ are periodic variables of period 1. Since φ_0 is just the initial phase along the periodic orbits we have an $E \times \mathbb{T}^N$ of multibreathers, labelled by $(A, \underline{\theta})$. The phase space, in a neighbourhood of these multibreathers is as follows:

$$\Gamma = \{(A, \varphi_0, \underline{J}, \underline{\theta}, p_r, q_r) \in E \times \mathbb{T}^1 \times \Omega \times \mathbb{T}^N \times \mathbb{R}^{2S''}\},$$

where $\Omega \subset \mathbb{R}^N$ is a neighbourhood of $\underline{J}^{res}(A) \in \mathbb{R}^N$ and $||(p_r, q_r)||_2$ is finite. The tangent space of Γ denoted by Λ is a Banach space $\mathbb{R}^2 \times \mathbb{R}^{2N} \times \mathbb{R}^{2S''}$ with l_2 norm.

When the coupling constant ε is different from zero, there is generically no nearby invariant $E \times \mathbb{T}^{N+1}$ consisting entirely of periodic orbits of bounded period, because for generic Hamiltonian systems periodic orbits of given energy and period less than a given bound are isolated. Nonetheless, one can prove that at least N periodic solutions survive small perturbation (and at least 2^{N-1} if all non-degenerate). See [3, 4, 5, 11, 21] for various cases. We will present a variation on the methods of [4, 5, 11], which is essential for the extension to our generalised multibreathers.

3 Periodic orbits of the system under constraints

The starting point is a variational formulation of the problem of finding a periodic orbit. Let Z be a space of functions

$$z(s) = (A(s), \varphi(s), \underline{J}(s), \underline{\theta}(s), p_r(s), q_r(s)),$$

with the periodicity condition

$$z(s+1) = z(s) + (0, 1, \underline{0}, \underline{0}, \underline{0}, \underline{0}),$$

known as a "loop space". More precisely, we choose the loop space to be $Z = C^1(S^1, \Gamma)$ where $S^1 = \mathbb{R}/\mathbb{Z}$ is a circle of length 1. The tangent space $T_z Z$, $z \in Z$, is isomorphic to the Banach space $C^1(S^1, \Lambda)$ with the norm

$$||\xi||_2^2 = ||A||_{C^1}^2 + ||\varphi||_{C^1}^2 + ||\underline{J}||_{C^1}^2 + ||\underline{\theta}||_{C^1}^2 + \sum_{s \in S'} ||(p_s, q_s)||_{C^1}^2.$$

We define the *symplectic area* of a loop z by

$$\mathcal{A}(z) = \oint A d\varphi + \underline{J} \cdot d\underline{\theta} + p_r \cdot dq_r.$$

By the Poincaré-Cartan variational principle, $z(ft)$ is a periodic orbit of H of symplectic area $a \in \mathbb{R}$ if and only if $(z, f) \in Z \times \mathbb{R}$ is a critical point of the function from $Z \times \mathbb{R}$ to $\mathbb{R}$ defined by:

$$S_a(z, f) = \oint H \circ z \, ds - f\left(\mathcal{A}(z) - a\right). \tag{3}$$

This amounts to expressing the periodic orbits of H as critical points of the *averaged energy*

$$\mathcal{H}(z) = \oint H \circ z \, ds$$

in the space Z subject to fixed area $\mathcal{A}(z) = a$. It can be easily checked that the Lagrange multiplier f associated with the latter constraint is just the frequency of the periodic orbit (time frequency, not angular frequency, as we did not insert a factor $1/2\pi$ in the definition of $\mathcal{A}$). Let us note that since the system is autonomous, these periodic solutions are determined up to an arbitrary phase that we can fix by choosing $\varphi(0) = \varphi_0$.

Now, for $\varepsilon = 0$, $a \in E$, and fixed φ_0, we have critical points $(z_{a,\underline{\theta}}, f(a))$ of S_a corresponding to rescaled solutions in time given in (2) for each $\underline{\theta} \in \mathbb{T}^N$. These critical points are degenerate, so not guaranteed to persist to $\varepsilon \neq 0$. The non-trivial homology of the N-torus, however, obliges there to remain at least some number of nearby critical points for ε small, corresponding to particular choices of relative phase $\underline{\theta} \in \mathbb{T}^N$, thus giving some multibreathers for $\varepsilon > 0$. To prove this, our approach is to stationarize S subject to additional constraints

$$\int_0^1 \underline{\theta}(s) \, ds = \underline{\bar{\theta}}, \qquad \int_0^1 \underline{J}(s) \, ds = \underline{\bar{J}},$$

which is a non-degenerate variational problem, and then examine for which of the resulting critical points the constraints can be removed. It would suffice to constrain only the relative phases, not the relative momenta, but the above extension is essential to derive generalised multibreather solutions later. Let us introduce Lagrange multipliers $\underline{u}, \underline{v} \in \mathbb{R}^N$ and $k \in \mathbb{R}$ which represent artificial constraint forces and consider the *constrained energy* $W : Z \times \mathbb{R} \times \mathbb{R}^{2N} \times \mathbb{R} \to \mathbb{R}$

$$\begin{aligned} W_{a,\varphi_0,\underline{\bar{J}},\underline{\bar{\theta}},\varepsilon}(z, f, \underline{u}, \underline{v}, k) &= \mathcal{H}(z) - f(\mathcal{A}(z) - a) - \underline{v} \cdot \left(\int_0^1 \underline{J} \, ds - \underline{\bar{J}} \right) \\ &\quad - \underline{u} \cdot \left(\int_0^1 \underline{\theta} \, ds - \underline{\bar{\theta}} \right) - k \left(\int_0^1 (\varphi(s) - s - \varphi_0) \, ds \right). \end{aligned}$$

For the last term of W we lift φ to $\mathbb{R}$; it is introduced simply to eliminate the global phase shift degeneracy due to the autonomous nature of the system.

The constrained energy W is differentiable. The derivative DW is a mapping from $Z \times \mathbb{R}^{2N+2}$ to the dual $(TZ \times \mathbb{R}^{2N+2})^*$, the bounded linear maps from $C^1(S^1, \Lambda) \times \mathbb{R}^{2N+2}$ to $\mathbb{R}$. However, DW can equally well be regarded as a mapping from $Z \times \mathbb{R}^{2N+2}$ to $C^0(S^1, \Lambda) \times \mathbb{R}^{2N+2}$, and this is the point of view we shall take. Let us write

$$w = (z, f, \underline{u}, \underline{v}, k) \qquad \text{and} \qquad \alpha = (a, \varphi_0, \underline{\bar{J}}, \underline{\bar{\theta}}).$$

Taking a general variation $\delta w = (\delta z, \delta f, \delta \underline{u}, \delta \underline{v}, \delta k) \in C^1(S^1, \Lambda) \times \mathbb{R}^{2N+2}$ and using integration by parts, DW at w is given by

$$DW_{w,\alpha,\varepsilon}(\delta w) \;=\; \int_0^1 \Big[(D_A H(z) - f\dot{\varphi})\delta A + (D_\varphi H(z) + f\dot{A} - k)\, \delta\varphi$$

$$
\begin{aligned}
&+(D_{\underline{J}} H(z) - f\dot{\underline{\theta}} - \underline{v})\, \delta\underline{J} + (D_{\underline{\theta}} H(z) + f\dot{\underline{J}} - \underline{u})\, \delta\underline{\theta} \\
&+ (D_{p_r} H(z) - f\dot{q_r})\, \delta p_r + (D_{q_r} H(z) + f\dot{p_r})\, \delta q_r]\; ds \\
&-\delta f\, (\mathcal{A}(z) - a) - \delta k \int_0^1 (\varphi(s) - s - \varphi_0)\, ds \\
&-\delta\underline{u} \cdot \int_0^1 (\underline{\theta} - \bar{\underline{\theta}})\, ds - \delta\underline{v} \cdot \int_0^1 (\underline{J} - \bar{\underline{J}})\, ds.
\end{aligned} \tag{4}
$$

Note that $DW_{w,\alpha,\varepsilon}(\delta w), \delta w \in C^1(S^1,\Lambda)\times\mathbb{R}^{2N+2}$, is well-defined because $C^0(S^1,\Lambda)\times \mathbb{R}^{2N+2} \subset (TZ \times \mathbb{R}^{2N+2})^*$. The Euler-Lagrange equations for critical points are:

$$
\begin{array}{ll}
f\dot{A} + D_\varphi H(z) - k = 0 & \\
f\dot{\varphi} - D_A H(z) = 0 & a = \mathcal{A}(z) \\
f\dot{\underline{J}} + D_{\underline{\theta}} H(z) - \underline{u} = 0 & \bar{\underline{J}} = \int_0^1 \underline{J}(s)\, ds \\
f\dot{\underline{\theta}} - D_{\underline{J}} H(z) + \underline{v} = 0 & \bar{\underline{\theta}} = \int_0^1 \bar{\underline{\theta}}(s)\, ds \\
f\dot{p}_r + D_{q_r} H(z) = 0 & \varphi_0 = \int_0^1 \varphi(s)\, ds - \frac{1}{2}. \\
f\dot{q}_r - D_{p_r} H(z) = 0 &
\end{array} \tag{5}
$$

If $w_{\alpha,\varepsilon} = (z_{\alpha,\varepsilon}(s), f_{\alpha,\varepsilon}, \underline{u}_{\alpha,\varepsilon}, \underline{v}_{\alpha,\varepsilon}, k_{\alpha,\varepsilon})$ is a critical point of $W_{\alpha,\varepsilon}$ then, by taking constant functions for the variations with respect to $\underline{\theta}, \underline{J}, \varphi, A$ respectively, the Lagrange multipliers are given by

$$
\begin{aligned}
\underline{u}_{\alpha,\varepsilon} &= \int_0^1 D_{\underline{\theta}} H(z_{\alpha,\varepsilon}(s))\, ds, & \underline{v}_{\alpha,\varepsilon} &= \int_0^1 D_{\underline{J}} H(z_{\alpha,\varepsilon}(s))\, ds \\
k_{\alpha,\varepsilon} &= \int_0^1 D_\varphi H(z_{\alpha,\varepsilon}(s))\, ds, & f_{\alpha,\varepsilon} &= \int_0^1 D_A H(z_{\alpha,\varepsilon}(s))\, ds.
\end{aligned} \tag{6}
$$

Sometimes, we will denote $w_{\alpha,\varepsilon}$ by $w_\varepsilon(\alpha)$.

We see that (z, f) is a critical point of S_a if and only if $(z, f, 0, 0, 0)$ is a critical point of $W_{\alpha,\varepsilon}, \alpha = (a, \varphi_0, \bar{\underline{J}}, \bar{\underline{\theta}})$ for some $\varphi_0, \bar{\underline{J}}, \bar{\underline{\theta}}$. So we obtain multibreathers by finding α and critical points of $W_{\alpha,\varepsilon}$ for which $\underline{u}, \underline{v}, k = 0$. For $\varepsilon = 0$ and any $\alpha = (a, \varphi_0, \bar{\underline{J}}, \bar{\underline{\theta}})$, $W_{\alpha,0}$ has a critical point $w_{\alpha,0}$ given by

$$
\begin{array}{llllll}
A(s) = a, & \varphi(s) = s + \varphi_0, & \underline{J}(s) = \bar{\underline{J}}, & \underline{\theta}(s) = \bar{\underline{\theta}}, & p_r(s) = p_r^e, & q_r(s) = q_r^e, \\
f = D_A H^0(a, \bar{\underline{J}}, p_r^e, q_r^e), & & \underline{u} = 0, & \underline{v} = D_{\underline{J}} H^0(a, \bar{\underline{J}}, p_r^e, q_r^e), & & k = 0.
\end{array} \tag{7}
$$

It has $\underline{u}, \underline{v}, k = 0$ if and only if $\bar{\underline{J}} = J^{res}(a)$. The following proposition states that $w_{\alpha,0}$ is a non-degenerate critical point of the constrained energy $W_{\alpha,0}$ provided a non-resonance condition holds:

$$
2\pi n D_A H^0(a, \bar{\underline{J}}, p_r^e, q_r^e) \notin \overline{\bigcup_{s\in S''} \operatorname{spec} \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} D^2_{p_s q_s} H_s(p_s^e, q_s^e) \right\}} \qquad \text{for all } n \in \mathbb{Z}. \tag{8}
$$

So, it persists to some $w_{\alpha,\varepsilon}$ for ε small and $\bar{\underline{J}} \in \Omega$.

Proposition 1 *Suppose that the non-resonance condition (8) holds on $(a, \underline{J}) \in E \times \Omega$. Then $w_{\alpha,0}$, $\alpha \in E \times \mathbb{T}^1 \times \Omega \times \mathbb{T}^N$ is a non-degenerate critical point of $W_{\alpha,0}$ and there is $\varepsilon_0 > 0$ such that there is a 1-parameter family of critical points $w_{\alpha,\varepsilon}$ of $W_{\alpha,\varepsilon}$ for $0 \le \varepsilon \le \varepsilon_0$ for every $\alpha \in E \times \mathbb{T}^1 \times \Omega \times \mathbb{T}^N$.*

Proof. The proof is based on the implicit function theorem, which applies if DW is C^1 and its derivative $D^2W_{\alpha,0}$ with respect to $Z \times \mathbb{R}^{2N+2}$ has bounded inverse.

It is easy to check that DW is C^1, as a function from $Z \times \mathbb{R}^{2N+2}$ to $(TZ \times \mathbb{R}^{2N+2})^*$, or $C^0(S^1, \Lambda) \times \mathbb{R}^{2N+2}$ since $H \in C^2$. Its derivative $D^2W_{\alpha,0}$ at $w_{\alpha,0}$ is given by

$$\begin{aligned} D^2W_{\alpha,0}(w_{\alpha,0})(\delta w, \Delta w) &= \int_0^1 [(D_A D_z H^0(z)\,\Delta z - \Delta f - f\Delta\dot{\varphi})\,\delta A \\ &+ (f\dot{\Delta A} - \Delta k)\,\delta\varphi + (D_{\underline{J}} D_z H^0(z)\,\Delta z - f\dot{\Delta\underline{\theta}} - \Delta\underline{v})\cdot\delta\underline{J} \\ &+ (f\dot{\Delta\underline{J}} - \Delta\underline{u})\cdot\delta\underline{\theta} + (D_{p_r} D_z H^0(z)\,\Delta z - f\dot{\delta q_r})\cdot\delta p_r \\ &+ (D_{q_r} D_z H^0(z)\,\Delta z + f\dot{\delta p_r})\cdot\delta q_r]\,ds - \delta f\, D\mathcal{A}(z)\,\Delta z \\ &- \delta k\int_0^1 \Delta\varphi\,ds - \delta\underline{u}\cdot\int_0^1 \Delta\underline{\theta}\,ds - \delta\underline{v}\cdot\int_0^1 \Delta\underline{J}\,ds, \end{aligned}$$

where $\delta w, \Delta w \in C^1(S^1, \Lambda) \times \mathbb{R}^{2N+2}$. $D^2W_{\alpha,0}(w_{\alpha,0})$ has a bounded inverse if and only if given $\pi \in C^0(S^1, \Lambda) \times \mathbb{R}^{2N+2}$, there is a unique $\Delta w \in C^1(S^1, \Lambda) \times \mathbb{R}^{2N+2}$ such that

$$D^2W_{\alpha,0}(w_{\alpha,0})(\delta w, \Delta w) = \pi(\delta w) \tag{9}$$

for all $\delta w \in C^1(S^1, \Lambda) \times \mathbb{R}^{2N+2}$, and the ratio $||\Delta w||/||\pi||$ is bounded. Equation (9) is equivalent to

$$\begin{aligned} f\dot{\Delta A} - \Delta k &= \pi_\varphi & (10)\\ f\dot{\Delta\varphi} + \Delta f - D_A H^0(z)\Delta A - D_A D_{\underline{J}} H^0(z)\,\Delta\underline{J} &= -\pi_A & (11)\\ f\dot{\Delta\underline{J}} - \Delta\underline{u} &= \pi_{\underline{\theta}} & (12)\\ f\dot{\Delta\underline{\theta}} - D_{\underline{J}} D_A H^0(z)\,\Delta A - D^2_{\underline{J}} H^0(z)\,\Delta\underline{J} - \Delta\underline{v} &= -\pi_{\underline{J}} & (13)\\ f\dot{\Delta p_s} + D_{q_s} D_{p_s} H_s(0,0)\Delta p_s + D^2_{q_s} H_s(0,0)\Delta q_s &= \pi_{q_s} & (14)\\ f\dot{\Delta q_s} - D^2_{p_s} H_s(0,0)\Delta p_s - D_{p_s} D_{q_s} H_s(0,0)\Delta q_s &= -\pi_{p_s} \qquad \text{for } s \in S'' & (15)\\ \textstyle\int_0^1 (\Delta A + a\dot{\Delta\varphi} + \bar{\underline{J}}\cdot\dot{\Delta\underline{\theta}})\,ds &= -\pi_f & (16)\\ \textstyle\int_0^1 \Delta\varphi\,ds = -\pi_k, \quad \int_0^1 \Delta\underline{J}\,ds = -\pi_{\underline{v}}, \quad \int_0^1 \Delta\underline{\theta}\,ds &= -\pi_{\underline{u}}, & (17) \end{aligned}$$

where the subscripts on π represents the components of π.

For every $s \in S''$, (14) and (15) are solved for Δp_s and Δq_s as follows. Let us write

$$x = \begin{pmatrix} \Delta p_s \\ \Delta q_s \end{pmatrix}, \quad B = \begin{pmatrix} D_{q_s} D_{p_s} H_s(0,0) & D^2_{q_s} H_s(0,0) \\ -D^2_{p_s} H_s(0,0) & -D_{p_s} D_{q_s} H_s(0,0) \end{pmatrix}, \quad y = \begin{pmatrix} \pi_{q_s} \\ -\pi_{p_s} \end{pmatrix}$$

and rewrite (14) and (15) as

$$f\dot{x} + B\,x = y.$$

The solutions are given by

$$x(s) = \exp(-\frac{1}{f}Bs)\, x(0) + \frac{1}{f}\int_0^s \exp(-\frac{1}{f}B(s-s'))\, y(s')\, ds', \tag{18}$$

which satisfies $x(1) = x(0)$ if and only if

$$x(0) = \left(I - \exp\left(-\frac{1}{f}B\right)\right)^{-1} \frac{1}{f}\int_0^1 \exp\left(-\frac{1}{f}B(1-s')\right) y(s')\, ds'. \tag{19}$$

The resonance condition provides that $(I - \exp(-\frac{1}{f}B))$ has a bounded inverse. Thus $x(s)$ is uniquely determined by (18) and (19), and the ratio $||x||/||y||$ is bounded. Integrating (10), ΔA and Δk are solved by

$$\Delta k = -\int_0^1 \pi_\varphi(s)\, ds, \qquad \Delta A(s) = \frac{1}{f}\int_0^s \pi_\varphi(s')\, ds' + \Delta A(0) + \frac{s}{f}\Delta k,$$

where $\Delta A(0) = -\pi_f$ from (16). Similarly, $\Delta \underline{J}$, $\Delta \underline{u}$ are solved from (12) and (17), and $\Delta \underline{\theta}$, $\Delta \underline{v}$ are solved from (13) and (17). Finally, Δf and $\Delta \varphi$ are solved from (11) and (17). One can easily check that the ratio $||\Delta w||/||\pi||$ is bounded. □

As already remarked, from the Euler-Lagrange Equation (5) it can be seen that if $\underline{u}_{\alpha,\varepsilon} = 0$, $\underline{v}_{\alpha,\varepsilon} = 0$ and $k_{\alpha,\varepsilon} = 0$, then $z_{\alpha,\varepsilon}(ft)$ is a periodic solution of the free system. We prove existence of periodic solutions for H_ε by considering the function

$$H_\varepsilon^{eff}(\alpha) = \oint H_\varepsilon \circ z_{\alpha,\varepsilon}\, ds, \qquad \alpha = (a, \varphi_0, \underline{\bar{J}}, \underline{\bar{\theta}}).$$

The following lemma states that $\underline{u}_{\alpha,\varepsilon}$, $\underline{v}_{\alpha,\varepsilon}$ are given by derivatives of H^{eff}, H^{eff} is independent of φ_0, and $k_{\alpha,\varepsilon}$ is automatically zero.

Lemma 1 *If $(z_{\alpha,\varepsilon}, f_{\alpha,\varepsilon}, \underline{u}_{\alpha,\varepsilon}, \underline{v}_{\alpha,\varepsilon}, k_{\alpha,\varepsilon})$ is a critical point of $W_{\alpha,\varepsilon}$, $\alpha = (a, \varphi_0, \underline{\bar{J}}, \underline{\bar{\theta}})$, then*

$$\underline{u}_{\alpha,\varepsilon} = D_{\underline{\bar{\theta}}} H_\varepsilon^{eff}(\alpha), \qquad \underline{v}_{\alpha,\varepsilon} = D_{\underline{\bar{J}}} H_\varepsilon^{eff}(\alpha),$$

$D_{\varphi_0} H_\varepsilon^{eff}(\alpha) = 0$ and $k_{\alpha,\varepsilon} = 0$.

Proof. Let $w_{\alpha,\varepsilon} = (z_{\alpha,\varepsilon}, f_{\alpha,\varepsilon}, \underline{u}_{\alpha,\varepsilon}, \underline{v}_{\alpha,\varepsilon}, k_{\alpha,\varepsilon})$ be a critical point of $W_{\alpha,\varepsilon}$ and consider a variation of α, $\alpha' = (a, \varphi_0, \underline{\bar{J}}, \underline{\bar{\theta}} + \Delta\underline{\bar{\theta}})$ for $|\Delta\underline{\bar{\theta}}|$ small. Then we have

$$\lim_{|\Delta\underline{\bar{\theta}}|\to 0} \frac{W_{\alpha,\varepsilon}(z_{\alpha',\varepsilon}, f_{\alpha,\varepsilon}, \underline{u}_{\alpha,\varepsilon}, \underline{v}_{\alpha,\varepsilon}, k_{\alpha,\varepsilon}) - W_{\alpha,\varepsilon}(w_{\alpha,\varepsilon})}{|\Delta\underline{\bar{\theta}}|} = 0. \tag{20}$$

On the other hand, $z_{\alpha',\varepsilon}$ is a component of the critical point $w_{\alpha',\varepsilon}$ of $W_{\alpha',\varepsilon}$ which satisfies

$$\mathcal{A}(z_{\alpha',\varepsilon}) = a, \quad \int_0^1 \underline{J}_{\alpha',\varepsilon}\, ds = \underline{\bar{J}}, \quad \int_0^1 \underline{\theta}_{\alpha',\varepsilon}\, ds = \underline{\bar{\theta}} + \Delta\underline{\bar{\theta}}, \quad \int_0^1 (\varphi_{\alpha',\varepsilon} - s - \varphi_0)\, ds = 0.$$

Thus (20) implies

$$\lim_{|\Delta\underline{\bar\theta}|\to 0} \frac{\mathcal{H}(z_{\alpha',\varepsilon}) - \mathcal{H}(z_{\alpha,\varepsilon}) - \underline{u}_{\alpha,\varepsilon}\cdot\Delta\underline{\bar\theta}}{|\Delta\underline{\bar\theta}|} = 0,$$

or

$$D_{\underline{\bar\theta}} H^{eff}_{\varepsilon}(\alpha) = \underline{u}_{\alpha,\varepsilon}.$$

Similarly, we have

$$\begin{aligned} D_{\underline{\bar J}} H^{eff}_{\varepsilon}(\alpha) &= \underline{v}_{\alpha,\varepsilon} \\ D_{\varphi_0} H^{eff}_{\varepsilon}(\alpha) &= k_{\alpha,\varepsilon}. \end{aligned}$$

It is easily checked that $(z_{\alpha,\varepsilon}(s+\Delta\varphi), f_{\alpha,\varepsilon}, \underline{u}_{\alpha,\varepsilon}, \underline{v}_{\alpha,\varepsilon}, k_{\alpha,\varepsilon})$ is a critical point of $W_{\alpha',\varepsilon}$, where $\alpha' = (a, \varphi_0+\Delta\varphi, \underline{\bar J}, \underline{\bar\theta})$. Consequently, we have

$$\frac{\partial}{\partial\varphi_0} z_{\alpha,\varepsilon}(s) = \frac{\partial}{\partial s} z_{\alpha,\varepsilon}(s). \tag{21}$$

Then

$$\begin{aligned} \frac{\partial}{\partial\varphi_0} H^{eff}_{\varepsilon}(\alpha) &= \frac{\partial}{\partial\varphi_0}\int_0^1 H(z_{\alpha,\varepsilon}(s))\,ds = \int_0^1 DH(z_{\alpha,\varepsilon}(s))\,\frac{\partial}{\partial\varphi_0} z_{\alpha,\varepsilon}(s)\,ds \\ &= \int_0^1 DH(z_{\alpha,\varepsilon}(s))\,\frac{\partial}{\partial s} z_{\alpha,\varepsilon}(s)\,ds = \int_0^1 \frac{d}{ds} H(z_{\alpha,\varepsilon}(s))\,ds = 0. \end{aligned}$$

Thus H^{eff}_{ε} is independent of φ_0 and $k_{\alpha,\varepsilon} = 0$. $\square$

So periodic orbits of area a close to one of those of the unperturbed $E\times\mathbb{T}^N$ for ε small are in one-to-one correspondence with the critical points of $H^{eff}_{a,\varepsilon}(\underline{\bar J}, \underline{\bar\theta}) := H^{eff}_{\varepsilon}(a, \varphi_0, \underline{\bar J}, \underline{\bar\theta})$. Now under the anharmonic assumption that all but one excited unit is locally anharmonic we have that $D^2_{\underline{\bar J}} H^{eff}_{a,0}$ is invertible at $\underline{\bar J} = \underline{J}^{res}(a)$, and so the same is true for ε small. But $D_{\underline{\bar J}} H^{eff}_{a,0} = \underline{0}$ at $\underline{\bar J} = \underline{J}^{res}(a)$ for all $\underline{\bar\theta}$, so by the implicit function theorem for ε small there exists a locally unique $\underline{\bar J}_{a,\varepsilon}(\underline{\bar\theta})$ such that

$$D_{\underline{\bar J}} H^{eff}_{a,\varepsilon}(\underline{\bar J}_{a,\varepsilon}(\underline{\bar\theta}), \underline{\bar\theta}) = \underline{0}.$$

Furthermore $\underline{\bar J}_{a,\varepsilon}$ is C^1 in $\underline{\bar\theta}$. Let

$$V^{eff}_{a,\varepsilon}(\underline{\bar\theta}) = H^{eff}_{a,\varepsilon}(\underline{\bar J}_{a,\varepsilon}(\underline{\bar\theta}), \underline{\bar\theta}),$$

which we call an "effective potential". This is nearly the same as the "Melnikov-type" potential referred to in [4] and used in [11], and the "effective action" of [5], except that here we use constant area rather than constant period. V^{eff} could have been derived directly without introducing the constraint $\underline{\bar J}$, but we require H^{eff} in the following section, not just V^{eff}.

Theorem 1 *For all $a \in E$, and ε small enough, H_ε has at least N periodic orbits of area a near $E \times \mathbb{T}^{N+1}$ and at least 2^{N-1} if all are non-degenerate.*

Proof. $V_{a,\varepsilon}^{eff} : \mathbb{T}^N \to \mathbb{R}$ is C^2, so we obtain at least N critical points by the theory of Ljusternik-Schnirelman category [22]. If all critical points are non-degenerate (as is usually the case) then we obtain at least 2^{N-1} critical points by Morse theory [23]. We will show later that non-degenerate critical points of $H_{a,\varepsilon}^{eff}$ correspond to non-degenerate periodic orbits (only double Floquet multipliers $+1$), thus completing the proof. □

Note that the critical points do not necessarily depend continuously on ε: for example, it could be that some annihilate in pairs while other pairs are created, or that a continuous branch of critical points for $\varepsilon > 0$ does not tend to a unique limit as $\varepsilon \to 0$. The only constraint is that for each ε small there are at least N critical points. If, however,

$$V_{a,\varepsilon}^{eff}(\underline{\bar{\theta}}) = C + g(\varepsilon)\, V_a^0(\underline{\bar{\theta}}) + o(g(\varepsilon)) \qquad \text{as } \varepsilon \to 0$$

for some C^2 function V_a^0, and function $g(\varepsilon) \to 0$ as $\varepsilon \to 0$, with $g(\varepsilon) \neq 0$ for $\varepsilon \neq 0$, then each non-degenerate critical point of V_a^0 persists to a critical point of $V_{a,\varepsilon}^{eff}$ depending smoothly on ε (cf. Theorem 4 of [11]). Furthermore, if V_a^0 is a Morse function (all critical points non-degenerate) with K critical points ($K \geq 2^{N-1}$ by Morse theory) then the set of critical points of $V_{a,\varepsilon}^{eff}$ for ε small non-zero consists of precisely K points continuing the critical points of V_a^0 and they depend smoothly on ε. This is directly analogous to the variational approach to Melnikov's method for finding "subharmonic" periodic orbits in a periodically forced integrable system (cf. [24] for the case of homoclinic orbits). In the common case where the coupling between the phases is first order in ε, then $g(\varepsilon)$ can be taken to be ε and V_a^0 is given simply by evaluating $\int H_\varepsilon$ on the unperturbed orbits.

Note that weaker conditions on critical points of V_a^0 than non-degeneracy may suffice to control the location of some critical points of $V_{a,\varepsilon}^{eff}$, for example a strict local minimum or maximum, cf. [25].

If we start from an initial condition $z_{\alpha,\varepsilon}(0)$ where $\underline{u}$ and $\underline{v}$ do not vanish, the solution can not be periodic of nearby period. Indeed, the averaged values $\underline{\bar{J}}$ and $\underline{\bar{\theta}}$ over one revolution of the overall phase φ will slowly vary in time. In the next section we will prove that approximate solutions of the free system can be constructed by considering H_ε^{eff} as an $N+1$ DOF Hamiltonian system for $\alpha = (a, \varphi_0, \underline{\bar{J}}, \underline{\bar{\theta}})$.

4 Approximate dynamics of the relative variables

In the free system the rate of change of the relative variables is approximately given by the opposite of the constraint forces. Let us consider the following dynamical system:

$$\begin{aligned}
\dot{a} &= 0 \\
\dot{\varphi}_0 &= f_\varepsilon(\alpha) \\
\dot{\underline{\bar{J}}} &= -\underline{u}_\varepsilon(\alpha) \\
\dot{\underline{\bar{\theta}}} &= \underline{v}_\varepsilon(\alpha),
\end{aligned} \tag{22}$$

where $\alpha = (a, \varphi_0, \underline{\bar{J}}, \underline{\bar{\theta}})$. Actually, the right hand side does not depend on φ_0 as was proved in Lemma 1. We will show that the dynamics of (22) gives approximate solutions of the free system. First we prove that (22) is Hamiltonian with the Hamiltonian function

$$H_\varepsilon^{eff}(\alpha) = \oint H_\varepsilon(z_{\alpha,\varepsilon}(s))\, ds. \tag{23}$$

Proposition 2 *The dynamical system (22) is a Hamiltonian system with the Hamiltonian H_ε^{eff}.*

Proof. It is proved in Lemma 1 that

$$\underline{u}_\varepsilon(\alpha) = D_{\underline{\bar{\theta}}} H_\varepsilon^{eff}(\alpha), \qquad \underline{v}_\varepsilon(\alpha) = D_{\underline{\bar{J}}} H_\varepsilon^{eff}(\alpha)$$

and

$$D_{\varphi_0} H_\varepsilon^{eff}(\alpha) = 0.$$

By the same argument as in Lemma 1,

$$f_\varepsilon(\alpha) = D_a H_\varepsilon^{eff}(\alpha).$$

Therefore, (22) is Hamiltonian with the Hamiltonian function $H_\varepsilon^{eff}(\alpha)$. □

In the system (22), a is constant in time. For fixed $a \in E$, the relevant dynamics is for the relative variables $(\underline{\bar{J}}, \underline{\bar{\theta}})$

$$\begin{aligned}
\dot{\underline{\bar{J}}} &= -D_{\underline{\bar{\theta}}} H_{a,\varepsilon}^{eff}(\underline{\bar{J}}, \underline{\bar{\theta}}) \\
\dot{\underline{\bar{\theta}}} &= D_{\underline{\bar{J}}} H_{a,\varepsilon}^{eff}(\underline{\bar{J}}, \underline{\bar{\theta}})
\end{aligned} \tag{24}$$

which is also a Hamiltonian system with the Hamiltonian $H_{a,\varepsilon}^{eff}(\underline{\bar{J}}, \underline{\bar{\theta}}) = H_\varepsilon^{eff}(\alpha)$. Now we consider a surface

$$\mathcal{P}_\varepsilon = \{z_{\alpha,\varepsilon}(0) : \alpha \in E \times \mathbb{T}^1 \times \Omega \times \mathbb{T}^N\},$$

for $0 \leq \varepsilon \leq \varepsilon_0$. This surface is almost invariant under H_ε, and the dynamics on it is given approximately by (22). The following theorem states that the errors depend only on $\varepsilon|\underline{u}|$ and $\varepsilon|\underline{v}|$, thus they are especially small near a critical point of $H^{eff}_{a,\varepsilon}$, and the critical point corresponds to an exact periodic orbit of H_ε as was pointed out in the previous section.

Theorem 2 *Let $\alpha(t) = (a(t), \varphi_0(t), \underline{\bar{J}}(t), \underline{\bar{\theta}}(t))$ be a solution of (22). Then for $0 \leq \varepsilon \leq \varepsilon_0$, $z_{\alpha(t),\varepsilon}(0)$ is an approximate solution of H_ε in the sense that the deviation of its rate of change from the equation of motion is $\mathcal{O}(\varepsilon|\underline{u}|, \varepsilon|\underline{v}|)$. Furthermore, for fixed $a \in E$, equilibrium points $(\underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon)$ correspond to exact periodic orbits of H_ε with period $1/f_\varepsilon(a, \underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon)$.*

Proof. Evaluate $\frac{d}{dt} z_{\alpha(t),\varepsilon}(0)$ with (21) to obtain

$$\frac{d}{dt} z_{\alpha(t),\varepsilon}(0) = f\frac{\partial z}{\partial s} - \frac{\partial z}{\partial \underline{\bar{J}}}\underline{u} + \frac{\partial z}{\partial \underline{\bar{\theta}}}\underline{v}. \tag{25}$$

From the Euler-Lagrange equation and the fact $k_{\alpha,\varepsilon} = 0$, $\partial z/\partial s$ satisfies

$$\begin{aligned}
f\frac{\partial A}{\partial s} &= -D_\varphi H(z) \\
f\frac{\partial \varphi}{\partial s} &= D_A H(z) \\
f\frac{\partial \underline{J}}{\partial s} &= -D_{\underline{\theta}} H(z) + \underline{u} \\
f\frac{\partial \underline{\theta}}{\partial s} &= D_{\underline{J}} H(z) - \underline{v} \\
f\frac{\partial p_r}{\partial s} &= -D_{q_r} H(z) \\
f\frac{\partial q_r}{\partial s} &= D_{p_r} H(z).
\end{aligned} \tag{26}$$

From (25) and (26)

$$\frac{d}{dt}\begin{bmatrix} A \\ \varphi \\ \underline{J} \\ \underline{\theta} \\ p_r \\ q_r \end{bmatrix} = X_{H_\varepsilon} + \begin{bmatrix} 0 \\ 0 \\ \underline{u} \\ -\underline{v} \\ \underline{0} \\ \underline{0} \end{bmatrix} - \frac{\partial z}{\partial \underline{\bar{J}}}\underline{u} + \frac{\partial z}{\partial \underline{\bar{\theta}}}\underline{v}, \tag{27}$$

where X_{H_ε} is the Hamiltonian vector field of H_ε. Moreover from (7) we have

$$\frac{\partial}{\partial \underline{J}} z_{\alpha,0}(0) = \begin{bmatrix} 0 \\ 0 \\ Id \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \frac{\partial}{\partial \underline{\theta}} z_{\alpha,0}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ Id \\ 0 \\ 0 \end{bmatrix},$$

where Id is the $N \times N$ identity matrix. Therefore, for ε small enough,

$$\frac{\partial z}{\partial \underline{J}} \underline{u} = \begin{bmatrix} 0 \\ 0 \\ \underline{u} \\ 0 \\ \underline{0} \\ \underline{0} \end{bmatrix} + \mathcal{O}(\varepsilon|\underline{u}|), \quad \frac{\partial z}{\partial \underline{\theta}} \underline{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \underline{v} \\ \underline{0} \\ \underline{0} \end{bmatrix} + \mathcal{O}(\varepsilon|\underline{v}|). \tag{28}$$

In view of (25), (26) and (28), it is proved that $z_{\alpha(t),\varepsilon}(0)$ is an approximate solution of H_ε to order $\mathcal{O}(\varepsilon|\underline{u}|, \varepsilon|\underline{v}|)$, meaning that

$$\frac{d}{dt} z_{\alpha_\varepsilon(t),\varepsilon}(0) - X_{H_\varepsilon}(z_{\alpha_\varepsilon(t),\varepsilon}(0)) = \mathcal{O}(\varepsilon|\underline{u}|, \varepsilon|\underline{v}|). \tag{29}$$

If $(\underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon)$ is an equilibrium point of (24),

$$\alpha_\varepsilon(t) = (a, f_\varepsilon(a, \underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon)\, t + \varphi_{0\varepsilon}, \underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon)$$

is a periodic orbit of (22) with any $\varphi_{0\varepsilon}$, and

$$\underline{u}_\varepsilon(\alpha_\varepsilon(t)) = \underline{v}_\varepsilon(\alpha_\varepsilon(t)) = 0.$$

This means that $z_{\alpha_\varepsilon(t),\varepsilon}(0)$ is an exact periodic orbit of H_ε. □

5 Linear Stability of multibreathers

In this section we consider the relation between the linear stability of an equilibrium $(\underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon)$ of $H^{eff}_{a,\varepsilon}$ and of the corresponding periodic orbit γ_ε defined by $\gamma_\varepsilon(t) := z_{\alpha_\varepsilon(t),\varepsilon}(0)$. We obtain the linear stability type in the case that the equilibrium has distinct eigenvalues which move with non-zero speed with ε.

The linear stability of $(\underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon)$ under $H^{eff}_{a,\varepsilon}$ is determined by the eigenvalues $\mu_{\varepsilon,1}, \cdots, \mu_{\varepsilon,2N}$ of

$$B_\varepsilon = \begin{pmatrix} -D_{\underline{\bar{\theta}}} D_{\underline{\bar{J}}} H^{eff}_{a,\varepsilon}(\underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon) & -D^2_{\underline{\bar{\theta}}} H^{eff}_{a,\varepsilon}(\underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon) \\ D^2_{\underline{\bar{J}}} H^{eff}_{a,\varepsilon}(\underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon) & D_{\underline{\bar{J}}} D_{\underline{\bar{\theta}}} H^{eff}_{a,\varepsilon}(\underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon) \end{pmatrix}. \tag{30}$$

Since $(\underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon)$ is an equilibrium point of a Hamiltonian system, if μ is an eigenvalue of B_ε, then $-\mu$, $\bar{\mu}$, and $-\bar{\mu}$ are also eigenvalues. In the limit $\varepsilon \to 0$, one can see that (24) becomes degenerate and $\lim_{\varepsilon\to 0} \mu_{\varepsilon,i} = 0$ for all i.

On the other hand, the linear stability of $\gamma_\varepsilon(t)$ is determined by its Floquet multipliers. At $\varepsilon = 0$, they consist of a $(2N+2)$-fold degenerate eigenvalue $+1$ and other eigenvalues on the unit circle which are due to elliptic equilibria of the units in S''. For ε small, under certain "signature" conditions on the elliptic equilibria in S'' as in [10], all eigenvalues except $+1$ remain on the unit circle, but $+1$ turns into $2N+2$ eigenvalues near $+1$ of which $+1$ remains at least double. In this case the linear stability is determined by the eigenvalues around $+1$. In fact, the $2N$ eigenvalues near $+1$ not constrained to be $+1$ are in one-to-one correspondence with $\mu_{\varepsilon,1}, \cdots, \mu_{\varepsilon,2N}$ in the nondegenerate case. To prove this fact we need a general result on perturbations of eigenvalue problems in Banach spaces.

Let L_ε be a continuous one-parameter family of bounded linear operators on a Banach space X with a norm $||\cdot||$ for $0 \le \varepsilon \le \varepsilon_0$. Suppose L_ε has a two dimensional invariant subspace $\mathbb{E}_\varepsilon$ with eigenvalue $+1$. Let the spectrum of L_ε be separated into two parts Σ'_ε, Σ''_ε in such a way that a simple closed curve σ encloses an open set containing Σ'_ε in its interior and Σ''_ε in its exterior, and Σ'_ε consists of a finite number of eigenvalues including $+1$. Then one defines the spectral projection associated with Σ'_ε by

$$P_\varepsilon = -\frac{1}{2\pi i}\int_\sigma R_\varepsilon(\zeta)\, d\zeta,$$

where $R_\varepsilon(\zeta)$ is the resolvent of L_ε

$$R_\varepsilon(\zeta) = (L_\varepsilon - \zeta)^{-1}.$$

P_ε gives a decomposition $X = U_\varepsilon \oplus V_\varepsilon$, where $U_\varepsilon = \mathrm{Range} P_\varepsilon$ and $V_\varepsilon = \mathrm{Range}(1 - P_\varepsilon)$, such that the spectra of $L_\varepsilon|_{U_\varepsilon}$, $L_\varepsilon|_{V_\varepsilon}$ coincide with Σ'_ε, Σ_ε'' respectively [26]. Let $n+2$ be the dimension of U_ε. We will use the following lemma.

Lemma 2 *Suppose that there are nonzero $\mu_{\varepsilon,i} \in \mathbb{C}$, $\xi_{\varepsilon,i} \in X$, $i = 1, \cdots, n$, such that for $0 < \varepsilon \le \varepsilon_0$,*

$$L_\varepsilon\, \xi_{\varepsilon,i} - e^{\mu_{\varepsilon,i}}\, \xi_{\varepsilon,i} = \varepsilon\, \mu_{\varepsilon,i}\, \eta_{\varepsilon,i}, \tag{31}$$

$$\lim_{\varepsilon\to 0} \mu_{\varepsilon,i} = 0, \qquad \lim_{\varepsilon\to 0} ||\eta_{\varepsilon,i}|| < \infty \qquad \textit{for } 1 \le i \le n,$$

$\xi_{\varepsilon,i}$ satisfies

$$\lim_{\varepsilon\to 0} ||P_\varepsilon\, \xi_{\varepsilon,i}|| \ne 0 \qquad \textit{for } 1 \le i \le n, \tag{32}$$

and

$$U_0 = \lim_{\varepsilon\to 0} \mathbb{E}_\varepsilon \oplus \mathit{span}\{\lim_{\varepsilon\to 0} P_\varepsilon\, \xi_{\varepsilon,i}\}_{1\le i\le n}.$$

Then there are constants $b > 0$ and $\varepsilon_1 > 0$ such that for $\varepsilon \in [0, \varepsilon_1]$, the set

$$\mathcal{D}_{\varepsilon,i}(b) = \{\lambda \in \mathbb{C} \, : \, |\lambda - e^{\mu_{\varepsilon,i}}| \le \varepsilon |\mu_{\varepsilon,i}| b\}$$

has an eigenvalue of L_ε belonging to $\Sigma_\varepsilon{}'$ for all $1 \le i \le n$.

Proof. Let $\{\lambda_{\varepsilon,0} = 1, \lambda_{\varepsilon,1}, \cdots, \lambda_{\varepsilon,s}\}$ be the whole set of eigenvalues in $\Sigma_\varepsilon{}'$ and $P_{\varepsilon,i}$, $D_{\varepsilon,i}$ be the eigenprojection and the eigennilpotent associated to $\lambda_{\varepsilon,i}$ respectively. Then the resolvent $R_\varepsilon(\mu)$ has the following Laurent expansion [26]:

$$R_\varepsilon(\mu) = -\sum_{j=0}^{s} \left[\frac{P_{\varepsilon,j}}{\mu - \lambda_{\varepsilon,j}} + \sum_{k=1}^{m_j - 1} \frac{D_{\varepsilon,j}{}^k}{(\mu - \lambda_{\varepsilon,j})^{k+1}} \right] + R_\varepsilon(\mu)\,(1 - P_\varepsilon), \tag{33}$$

where m_j is the algebraic multiplicity of $\lambda_{\varepsilon,j}$. We note that $1 - P_\varepsilon$ is the projection onto V_ε and $R_\varepsilon(\mu)\,(1 - P_\varepsilon)$ is uniformly bounded for μ inside σ and $0 \le \varepsilon \le \varepsilon_0$.

First we show that under the conditions of the lemma, $\lambda_{\varepsilon,1}, \cdots, \lambda_{\varepsilon,s}$ are simple. Eq. (31) can be written as

$$\xi_{\varepsilon,i} = R_\varepsilon(e^{\mu_{\varepsilon,i}})\,(\varepsilon\,\mu_{\varepsilon,i}\,\eta_{\varepsilon,i}), \qquad \text{for } \ 1 \le i \le n. \tag{34}$$

If $\lambda_{\varepsilon,i}$ is not simple for some $1 \le i \le n$, then for some j, k,

$$\lim_{\varepsilon \to 0} P_{\varepsilon,i}\,\xi_{\varepsilon,j} \ne 0, \qquad \lim_{\varepsilon \to 0} P_{\varepsilon,i}\,\xi_{\varepsilon,k} \ne 0.$$

Taking the limit $\varepsilon \to 0$ in (33) and (34),

$$\lim_{\varepsilon \to 0} P_{\varepsilon,i}\,\xi_{\varepsilon,j} = \lim_{\varepsilon \to 0} \frac{D_{\varepsilon,i}{}^{m_i - 1}(\varepsilon \mu_{\varepsilon,j} \eta_{\varepsilon,j})}{(e^{\mu_{\varepsilon,j}} - \lambda_{\varepsilon,i})^{m_i}}.$$

A similar equation holds for k. However, the range of $D_{\varepsilon,i}{}^{m_i - 1}$ is the one-dimensional subspace spanned by the eigenvector of $\lambda_{\varepsilon,j}$. This contradicts the linear independence of $\lim_{\varepsilon \to 0} P_{\varepsilon,i}\,\xi_{\varepsilon,j}$ and $\lim_{\varepsilon \to 0} P_{\varepsilon,i}\,\xi_{\varepsilon,k}$.

Thus, $R_\varepsilon(\mu)$ can be written as:

$$R_\varepsilon(\mu) = -\sum_{j=1}^{n} \frac{P_{\varepsilon,j}}{\mu - \lambda_{\varepsilon,j}} - \frac{P_{\varepsilon,0}}{\mu - 1} - \frac{D_{\varepsilon,0}}{(\mu - 1)^2} + R_\varepsilon(\mu)\,(1 - P_\varepsilon).$$

Now we fix j and consider the equation

$$\lim_{\varepsilon \to 0} P_{\varepsilon,i}\,\xi_{\varepsilon,j} = \lim_{\varepsilon \to 0} P_{\varepsilon,i}\,R_\varepsilon(e^{\mu_{\varepsilon,j}})\,(\varepsilon \mu_{\varepsilon,j} \eta_{\varepsilon,j}) \qquad \text{for } 1 \le i \le n.$$

By the hypothesis, there is some i, $1 \le i \le n$ for which $\lim_{\varepsilon \to 0} P_{\varepsilon,i}\,\xi_{\varepsilon,j} \ne 0$. Then

$$\lim_{\varepsilon \to 0} P_{\varepsilon,i}\,\xi_{\varepsilon,j} = -\lim_{\varepsilon \to 0} \frac{P_{\varepsilon,i}(\varepsilon \mu_{\varepsilon,j} \eta_{\varepsilon,j})}{e^{\mu_{\varepsilon,j}} - \lambda_{\varepsilon,i}} = -\lim_{\varepsilon \to 0} \frac{\varepsilon \mu_{\varepsilon,j}}{e^{\mu_{\varepsilon,j}} - \lambda_{\varepsilon,i}} P_{\varepsilon,i}\,\eta_{\varepsilon,j}$$

implies that there is a constant $b^{(j)}$, $\varepsilon^{(j)}$ such that $\mathcal{D}_{\varepsilon^{(j)}}(b^{(j)})$ contain $\lambda_{\varepsilon,i}$ for $\varepsilon \in [0, \varepsilon^{(j)}]$. Choosing $b = \min b^{(j)}$ and $\varepsilon_1 = \min \varepsilon^{(j)}$, the proof is completed.

$\square$

The following theorem gives a condition for one-to-one correspondence between the eigenvalues of B_ε in (30) and the Floquet multipliers of γ_ε near 1.

Theorem 3 *Let $\mu_{\varepsilon,1}, \cdots, \mu_{\varepsilon,2N}$ be eigenvalues of B_ε and $\xi_{\varepsilon,1}, \cdots, \xi_{\varepsilon,2N}$ be corresponding unit eigenvectors. Let T_ε be the period of γ_ε. Suppose that $\mu_{\varepsilon,1}, \cdots, \mu_{\varepsilon,2N}$ are simple eigenvalues and $\lim_{\varepsilon\to 0}\{\xi_{\varepsilon,1}, \cdots, \xi_{\varepsilon,2N}\}$ form a basis of $\mathbb{R}^{2N}$. Then there exist constants $b > 0$ and ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$, there are disjoint balls of radius $\varepsilon\,|\mu_{\varepsilon,i}|\, b$ centred on $\exp(\mu_{\varepsilon,i} T_\varepsilon)$ containing a Floquet multiplier of γ_ε for each i.*

Proof. Let us write (22) as

$$\dot{\alpha} = g_\varepsilon(\alpha), \qquad \alpha \in E \times \mathbb{T}^1 \times \mathbb{R}^{2N} \tag{35}$$

and the Hamiltonian equation of H_ε as

$$\dot{z} = X_{H_\varepsilon}(z), \qquad z \in \Gamma. \tag{36}$$

The first variational equation around the periodic orbit α_ε defined by $\alpha_\varepsilon(t) := (a, f_\varepsilon\, t + \varphi_0, \underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon)$ is given by

$$\dot{\delta\alpha} = Dg_\varepsilon(\alpha_\varepsilon(t))\,\delta\alpha = \begin{pmatrix} 0 & 0 & 0 \\ C_\varepsilon & 0 & D_\varepsilon \\ E_\varepsilon & 0 & B_\varepsilon \end{pmatrix} \delta\alpha, \tag{37}$$

where

$$C_\varepsilon = \frac{\partial f}{\partial a}(a, \underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon), \quad D_\varepsilon = \begin{pmatrix} \frac{\partial f}{\partial \underline{J}}(a, \underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon) & \frac{\partial f}{\partial \underline{\theta}}(a, \underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon) \end{pmatrix}, \quad E_\varepsilon = \begin{pmatrix} -\frac{\partial \underline{u}}{\partial a}(a, \underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon) \\ \frac{\partial \underline{v}}{\partial a}(a, \underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon) \end{pmatrix}.$$

By standard Floquet theory [27] it is readily checked that the Floquet multipliers of α_ε are given by $\exp(\mu_1 T_\varepsilon), \cdots, \exp(\mu_{2N} T_\varepsilon)$ with corresponding eigenvectors

$$\tilde{\xi}_{\varepsilon,1} = \begin{pmatrix} 0 \\ 0 \\ \xi_{\varepsilon,1} \end{pmatrix}, \quad \cdots, \quad \tilde{\xi}_{\varepsilon,2N} = \begin{pmatrix} 0 \\ 0 \\ \xi_{\varepsilon,2N} \end{pmatrix}$$

and double eigenvalue $+1$.

Now let us consider the relation between the Floquet multipliers of α_ε and γ_ε. By Theorem 1 the map $\Phi_\varepsilon : \alpha \to z_{\alpha,\varepsilon}(0)$ gives approximate solutions to (36) and transforms periodic orbits of (35) into periodic orbits of (36). In other words, Φ_ε satisfies

$$\frac{d}{dt}\Phi_\varepsilon(\alpha(t)) = X_{H_\varepsilon}(\Phi_\varepsilon(\alpha(t))) + K_\varepsilon(\alpha(t)),$$

where K_ε is the error term given in the proof of Theorem 1, which vanishes for $\alpha_\varepsilon(t)$. Differentiating the left hand side with respect to t, we obtain

$$D\Phi_\varepsilon(\alpha)\, g_\varepsilon(\alpha) = X_{H_\varepsilon}(\Phi_\varepsilon(\alpha)) + K_\varepsilon(\alpha).$$

Differentiating both sides with respect to α,

$$D^2\Phi_\varepsilon(\alpha)(g_\varepsilon(\alpha),\,\cdot\,) + D\Phi_\varepsilon(\alpha)\, Dg_\varepsilon(\alpha) = DX_{H_\varepsilon}(\Phi_\varepsilon(\alpha))\, D\Phi_\varepsilon(\alpha) + DK_\varepsilon(\alpha). \tag{38}$$

Using (37) and (38)

$$\begin{aligned} \frac{d}{dt}(D\Phi_\varepsilon(\alpha_\varepsilon(t))\,\tilde{\xi}_{\varepsilon,i}) &= D^2\Phi_\varepsilon(\alpha_\varepsilon(t))(g_\varepsilon(\alpha_\varepsilon(t)),\tilde{\xi}_{\varepsilon,i}) \\ &= DX_{H_\varepsilon}(\gamma_\varepsilon(t))\, D\Phi_\varepsilon(\alpha_\varepsilon(t))\,\tilde{\xi}_{\varepsilon,i} - \mu_{\varepsilon,i} D\Phi_\varepsilon(\alpha_\varepsilon(t))\tilde{\xi}_{\varepsilon,i} \\ &\quad + DK_\varepsilon(\alpha_\varepsilon(t))\,\tilde{\xi}_{\varepsilon,i}. \end{aligned} \tag{39}$$

On the other hand, from (27) and (28)

$$\begin{aligned} K_\varepsilon(\alpha) &= \begin{bmatrix} 0 \\ 0 \\ \underline{u}(\alpha) \\ -\underline{v}(\alpha) \\ \underline{0} \\ \underline{0} \end{bmatrix} - \frac{\partial z_{\alpha,\varepsilon}}{\partial \underline{J}}\,\underline{u} + \frac{\partial z_{\alpha,\varepsilon}}{\partial \underline{\theta}}\,\underline{v} \\ &= \left(D\Phi_\varepsilon(\alpha) - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Id & 0 \\ 0 & 0 & 0 & Id \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{pmatrix} 0 \\ 0 \\ -\underline{u} \\ \underline{v} \end{pmatrix} := \varepsilon\, N_\varepsilon(\alpha) \begin{pmatrix} 0 \\ 0 \\ -\underline{u} \\ \underline{v} \end{pmatrix}, \end{aligned}$$

and $N_\varepsilon(\alpha)$ is a uniformly bounded linear operator for $\varepsilon \in [0,\varepsilon_0]$. Using the fact $\underline{u}(\alpha_\varepsilon(t)) = \underline{v}(\alpha_\varepsilon(t)) = 0$, it follows that

$$DK_\varepsilon(\alpha_\varepsilon(t)) = \varepsilon N_\varepsilon(\alpha) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ E_\varepsilon & 0 & B_\varepsilon \end{pmatrix}. \tag{40}$$

In view of (39) and (40), observe that

$$\begin{aligned} \frac{d}{dt}(e^{\mu_{\varepsilon,i}t} D\Phi_\varepsilon(\alpha_\varepsilon(t))\tilde{\xi}_{\varepsilon,i}) &= DX_{H_\varepsilon}(\gamma_\varepsilon(t))\, e^{\mu_{\varepsilon,i}t}\, D\Phi_\varepsilon(\alpha_\varepsilon(t))\tilde{\xi}_{\varepsilon,i} \\ &\quad + \varepsilon\, \mu_{\varepsilon,i}\, e^{\mu_{\varepsilon,i}t} N_\varepsilon(\alpha_\varepsilon(t))\,\tilde{\xi}_{\varepsilon,i}. \end{aligned} \tag{41}$$

Using the variation of constants formula and integrating (41) from 0 to T_ε we obtain

$$e^{\mu_{\varepsilon,i}T_\varepsilon} D\Phi_\varepsilon(\alpha_\varepsilon(0))\,\tilde{\xi}_{\varepsilon,i} = M_\varepsilon\, D\Phi_\varepsilon(\alpha_\varepsilon(0))\tilde{\xi}_{\varepsilon,i}$$

$$+ \varepsilon\mu_{\varepsilon,i} \int_0^{T_\varepsilon} M_\varepsilon(T_\varepsilon, s)\, N_\varepsilon(\alpha_\varepsilon(s))\, e^{\mu_{\varepsilon,i} s}\, ds\, \tilde{\xi}_{\varepsilon,i},$$

where $M_\varepsilon(t,s)$ is the fundamental map solution of the first variational equation around $\gamma_\varepsilon(t)$ and M_ε is its monodromy map. We note that the last integral is uniformly bounded in the operator norm from $\mathbb{R}^{2N+2}$ to Γ for $\varepsilon \in [0, \varepsilon_0]$.

The non-resonance condition (8) tells us that the spectrum of M_0 is composed of the isolated eigenvalue 1 and other eigenvalues on the unit circle. Since the spectrum is continuous with respect to ε, for ε small enough, we can choose some closed curve σ enclosing 1 containing all eigenvalues but 1 in its exterior. It is easily checked that if we set $L_\varepsilon := M_\varepsilon$, then $\mu_{\varepsilon,i} T_\varepsilon$, $D\Phi_\varepsilon(\alpha_\varepsilon(0))\tilde{\xi}_{\varepsilon,i}$, and

$$\eta_{\varepsilon,i} := -\frac{1}{T_\varepsilon} \int_0^{T_\varepsilon} M_\varepsilon(T_\varepsilon, s)\, N_\varepsilon(\alpha_\varepsilon(s))\, e^{\mu_{\varepsilon,i} s}\, ds\, \tilde{\xi}_{\varepsilon,i}$$

for $1 \leq i \leq 2N$, satisfy the conditions in Lemma 2. Applying Lemma 2 we obtain constants $b > 0$, $\varepsilon_1 > 0$, and $2N$ balls centered on $\exp(\mu_{\varepsilon,i} T_\varepsilon)$ with radius $\varepsilon|\mu_{\varepsilon,i}|b$ for $\varepsilon \in (0, \varepsilon_1]$. Each ball contains a Floquet multiplier of γ_ε and they are disjoint for ε_1 small enough. □

By the above theorem one can see that if the Floquet multipliers of γ_ε due to the elliptic equilibria of the units in S'' remain on the unit circle for ε small, $(\underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon)$ and $\gamma_\varepsilon(t)$ have same stability type. According to symplectic signature theory (see e.g. [10]) the Floquet multipliers can leave the unit circle only by a collision of eigenvalues having different symplectic signature or a collision with ± 1. The following theorem provides a condition on symplectic signatures of those Floquet multipliers to establish the coincidence of linear stability.

Theorem 4 *Suppose that the conditions in Theorem 3 are fulfilled. Let ω_s be the frequency of small oscillations about (p_s^e, q_s^e) for $s \in S''$, signed according to the sign of $D^2 H_s$. Let*

$$\Sigma = \{e^{i\omega_s T} : s \in S''\}$$

and $\bar{\Sigma}$ its complex conjugate and denote the breather period by T. Suppose there is a constant $C_1 > 0$ such that

$$d(\Sigma, \bar{\Sigma}) > C_1.$$

Then there is a constant $\varepsilon_2 > 0$ such that the linear stability types of $(\underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon)$ and of γ_ε coincide with each other for $\varepsilon \in [0, \varepsilon_2)$.

Proof. As proved in [10], the hypotheses imply that there exists $\varepsilon_2 > 0$ such that the symplectic signatures of eigenvalues continuing Σ and $\bar{\Sigma}$ are unchanged for $\varepsilon \in [0, \varepsilon_2]$, so that the eigenvalues continuing Σ and $\bar{\Sigma}$ remain on the unit circle and their Jordan blocks are diagonal. Therefore, the linear stability type of γ_ε is completely determined by the $2N$ eigenvalues near $+1$. By Theorem 3 they are contained in $2N$ disjoint balls of radius $\varepsilon|\mu_{\varepsilon,i}|b$ centered on $\exp(\mu_{\varepsilon,i}T_\varepsilon)$. If one of $\mu_{\varepsilon,i}$ has nonzero real part the ball centered on $\exp(\mu_{\varepsilon,i}T_\varepsilon)$ has no intersection with the unit ball for ε small enough. The corresponding Floquet multiplier can not be on the unit circle. If all $\mu_{\varepsilon,i}$ have zero real part, that is $(\underline{\bar{J}}_\varepsilon, \underline{\bar{\theta}}_\varepsilon)$ is linearly stable, all $2N$ balls have intersections with the unit circle and contain only one Floquet multiplier each. If μ is an eigenvalue of a symplectic map, $1/\bar{\mu}$ is also an eigenvalue. Thus, those Floquet multipliers inside the balls are on the unit circle. □

6 Examples and Extensions

6.1 Two-site generalised multibreathers

The simplest example to treat is a 1D chain of anharmonic oscillators $H_s = p_s^2/2m_s + V_s(q_s), s \in \mathbb{Z}$, with a nearest neighbour coupling $\epsilon W(q_{s+1}, q_s)$, and just two neighbouring oscillators excited at $\epsilon = 0$, say $s = 0, 1$, with equal period T. We will derive "pendulum dynamics" for the one relative phase $\bar{\theta}$.

Taking action-angle variables (I_s, ψ_s) for $s = 0, 1$, we choose to define the global and relative phase variables ϕ and θ respectively, and associated momenta A, J, by

$$\psi_1 = \phi + \theta \tag{42}$$

$$\psi_0 = \phi \tag{43}$$

$$A = I_0 + I_1 \tag{44}$$

$$J = I_1 \tag{45}$$

Thus we obtain unperturbed Hamiltonian of the form $H = H_0(I_0) + H_1(I_1) = H_0(A - J) + H_1(J)$ for the two oscillators. Let $J^{res}(A)$ be the value of J at which $\partial H/\partial J = 0$ (assumed to exist and depend smoothly on A). Applying our procedure (or [11]), we obtain effective Hamiltonian to first order in ϵ and in $j = J - J^{res}(A)$,

$$H_{\epsilon,a}^{eff}(j, \theta) = C(a) + \frac{1}{2}\beta j^2 + \epsilon < W(q_1(J^{res}(a), \theta + \phi), q_0(A - J, \phi)) > \tag{46}$$

where the average is taken over ϕ and

$$C(a) = H_0(a - J^{res}(a)) + H_1(J^{res}(a)) \tag{47}$$

$$\beta = H_1''(a - J^{res}(a)) + H_0''(J^{res}(a)) \tag{48}$$

In the case of two hard oscillators (i.e. H_i'' positive), then $\beta > 0$, so the effective dynamics oscillates about minima of the effective potential $\epsilon < W > (\theta)$ and librates for energy above its the maximum. For soft oscillators (H_i'' negative), the signs are reversed. In the mixed case, the sign of β depends on which oscillator has the stronger anharmonicity.

6.2 Stability criterion for multibreathers

The above stability criterion for 2-site multibreathers generalises to the case of more than two excited oscillators, giving a variant (because our effective potential is at constant action rather than constant period) and extension of a result of [5] (which was already anticipated in [3, 4]).

To first order in ϵ, if all oscillators are hard then the local minima of the effective potential are linearly stable and critical points of positive index are linearly unstable; if all oscillators are soft then the maxima of the effective potential are linearly stable and all critical points of lower index are linearly unstable.

In the mixed case with some hard and some soft oscillators, there is no easy statement. One has to analyse the linear stability of the critical points of the effective Hamiltonian. Many different sorts of dynamics are possible. For example, if there is one heavy near-isochronous oscillator, one light soft oscillator and one light hard oscillator, one can obtain an effective Hamiltonian like

$$H^{eff}(\underline{J}, \underline{\theta}) = \frac{1}{2}(J_1^2 - J_2^2) + \sin(\theta_1 + \theta_2).$$

Then

$$\ddot{\theta}_1 + \ddot{\theta}_2 = 0,$$

$$\ddot{\theta}_1 - \ddot{\theta}_2 = -2\cos(\theta_1 + \theta_2),$$

so there are solutions with $\theta_1 + \theta_2 = c$, constant, and for them $\theta_1 - \theta_2$ accelerates at rate $-2\cos c$. Alternatively, $\theta - 1 + \theta_2 = \alpha t$, some $\alpha \neq 0$, and then $\theta_1 - \theta_2 = \frac{2}{\alpha^2}\cos\alpha t + At + B$ for arbitrary constants A, B.

6.3 Travelling breathers

We believe that our approach may provide an explanation of numerical observations of travelling breathers, e.g. [28, 29, 30, 32]. In contrast to the "loop dynamics" proposed in [30], which we regard as of little use because it is unrealistic to expect the period to remain constant, our approach is based on deriving approximate effective dynamics for loops by elimination of dependence on a global phase, as done throughout our paper.

Consider a translation invariant lattice (e.g. 1D chain) and suppose there exists a 1-site breather solution and a 2-site breather with approximately the same energy and area as the 1-site breather. This is the regime studied by [32], for example, because using $A = \int dE/\omega$, one can transform Figure 4.3 of [32] to Figure 1 here.

Note that he also has a second 2-site breather which is in antiphase but it has significantly higher energy than the indicated in-phase 2-site breather and plays no role in what we will describe. In addition, he has a pair of asymmetric 2-site breathers for A in an interval (A_0, A_1), which are not essential to our argument but

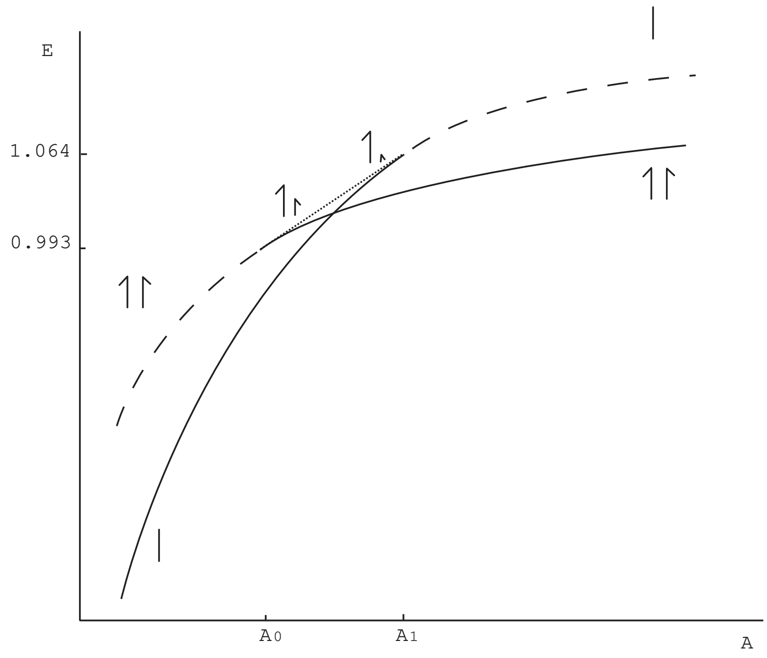

Figure 1: Transformation of Figure 4.3 of [32] to E versus A, using $A = \int dE/\omega$. The symbols $\uparrow$ etc. indicate the amplitudes and phases of oscillation of the relevant breathers. Solid lines are linearly stable, dashed and dotted are unstable, the dotted line corresponding to a pair of asymmetric breathers.

modify the resulting phase portraits. His oscillators are soft, so the kinetic energy term in H^{eff} is negative.

We can translate the two chosen breathers on the lattice to obtain other equivalent ones. It will be convenient, however, to identify configurations which differ by a lattice translation and hence study the interaction of just these two breathers. Suppose that they are part of an effective Hamiltonian dynamics that looks like one of the cases of Figure 2.

This would require theoretical justification, but the results of [32] strongly suggest case (c) for $A \in (A_0, A_1)$, (b) for $A < A_0$ and (a) for $A > A_1$. Then choosing orbits of H^{eff} just a little outside the separatrix yields travelling breathers.

This approximation is good as long as the dynamics it yields is slow compared with the time scale of the underlying oscillations. This restriction requires the dynamics to remain in a thin zone of $\bar{J}$ where $D_{\bar{J}} H^{eff}$ is small, and this in turn requires that the island widths be not too large. We believe this is the reason that case (c) and nearby cases of (a) and (b) are good for travelling breathers. When the energy difference between the 1-site and 2-site breathers is too large then the corresponding island is wide, so the dynamics of H^{eff} is a poor approximation, but when they are of roughly equal energy (for the same area) then the approximate dynamics is good.

Note that our interpretation of travelling breathers gives substance to the idea

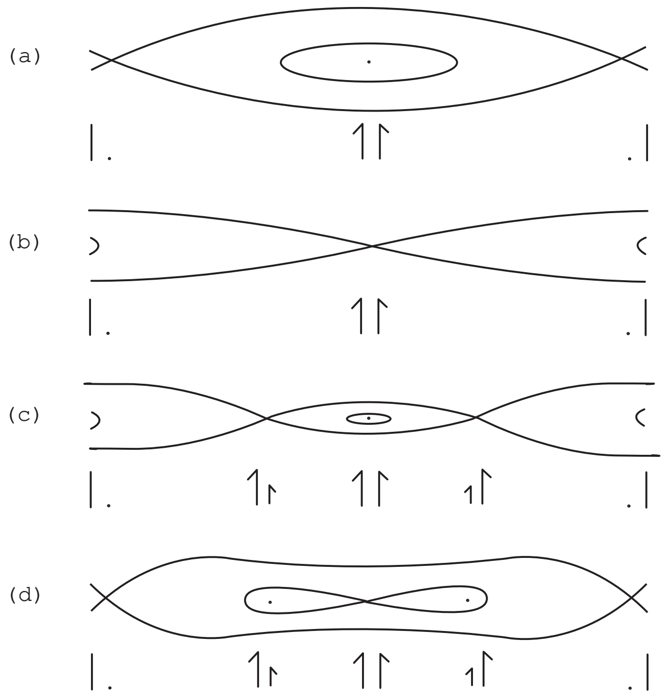

Figure 2: Four possible phase portraits for the relative phase and momentum for the interaction between a 1-site and a 2-site breather. The symbols indicate the amplitudes and phases of oscillation of the two most important sites at the equilibrium points of this effective dynamics.

of a "movability separatrix" proposed in [33]. It also clarifies the applicability of the concept of Peierls-Nabarro (PN) barrier to moving breathers, as follows. Let us define the relative kinetic energy $K_a^{eff} = H^{eff}(a) - V_a^{eff}$, and suppose it has definite sign. As one of us has argued since 1993, the appropriate definition of PN barrier for breathers is the energy difference between certain breathers with the same area. More precisely, the PN barrier for breathers of area a is the minimum change in V_a^{eff} to move a breather by one lattice site, because this is the minimum relative kinetic energy that one has to supply to put the most stable breather onto a travelling trajectory of H_a^{eff}.

Note that in [32], the relative kinetic energy is negative (soft oscillators), so the most stable breather is the one with the highest V^{eff}. Thus, denoting the 1-site breather by 1, the 2-site in-phase symmetric breather by 2 and the asymmetric 2-site breathers by $2a$, the PN barriers for the three cases of [32] are

(a) $V^{eff}(2) - V^{eff}(1)$

(b) $V^{eff}(1) - V^{eff}(2)$

(c) $\max(V^{eff}(2), V^{eff}(1)) - V^{eff}(2a)$.

If the relative kinetic energy is indefinite (this requires at least 3 excited states, so it is not immediately relevant to the above scenario) then it would be hard to define a useful concept of PN barrier.

6.4 Complete energy exchange

Our ideas can be extended to situations in which the amplitude of oscillation on one excited site can go to zero. This could be relevant to ideas of Aubry [31] on energy transfer in biomolecules like myosin.

One might think that we already permitted the amplitude of oscillation on one excited oscillator to go to zero in the previous subsection, but the picture there was that the coupling was strong enough to move us into a regime where one 2-site breather had the same energy as a 1-site breather and so the amplitude of oscillation on the neighbouring site to the 1-site breather is not that small.

The problem with the amplitude of oscillation of an excited oscillator going to zero is that this is a coordinate singularity in the action-angle variables on which our analysis was based. It is not difficult, however, to get round this singularity. If one has $N+1$ oscillators of roughly equal frequency ω (and non-negative energies) for which any of the amplitudes is allowed to go through zero then one simply replaces the space $\mathbb{T}^N \times \mathbb{R}^N$ of relative phases and momenta by $\mathbb{C}P^N$, complex projective N-space. It is the space of 1-complex dimensional subspaces of $\mathbb{C}^{N+1}$. To see this, think of each oscillator phase space as $\mathbb{C}$ rather than $\mathbb{R}^2$. Constant action gives a sphere $\mathbb{S}^{2N+1}$ in the phase space $\mathbb{C}^{N+1}$ of $N+1$ oscillators, and then we eliminate a global phase using the reference dynamics $\dot{z} = i\omega z$, to obtain $\{z \in \mathbb{C}^{N+1} : |z| = 1\}/(z \sim e^{i\phi}z) = \mathbb{C}P^N$. It has a natural symplectic structure.

To derive an effective Hamiltonian on $\mathbb{C}P^N$, instead of the constraints $\underline{\bar{J}}, \underline{\bar{\theta}}$, we introduce constraint $\underline{w} = \int e^{-2\pi i s}\gamma(s)ds \in \mathbb{C}^{N+1}$ and see that $H^{eff}(\underline{w})$ is independent of global phase rotation. Thus it preserves area A. For each $a \in \mathbb{R}$, the set $\{A = a\}$ is homeomorphic to a $(2N+1)$-sphere in $\mathbb{C}^{N+1}$ thus H^{eff}_a is a Hamiltonian on $\mathbb{C}P^N$.

For example, the standard physics experiment of two weakly coupled near-identical pendula has relative phase space $\mathbb{C}P^1$, homeomorphic to $\mathbb{S}^2$. Let us draw it so that the “West” pole corresponds to the right hand pendulum at rest, the “East” pole to the lefthand pendulum at rest, the North pole to equal amplitude in antiphase and the South pole to equal amplitude in phase. Then coupling of the form $\epsilon(q_1 - q_0)^2$ yields relative phase portraits like Figure 3, familiar to chemists in the transition from normal to local modes in molecules with two relevant degrees of freedom, and to mechanical engineers in the interaction of nearly resonant undamped nonlinear oscillations.

We see solutions of H^{eff} which completely exchange energy between the two oscillators provided a is not too large, viz. near the equator. For a large, however, there are two 1-site breathers, born at some value a_0, and the separatrix around

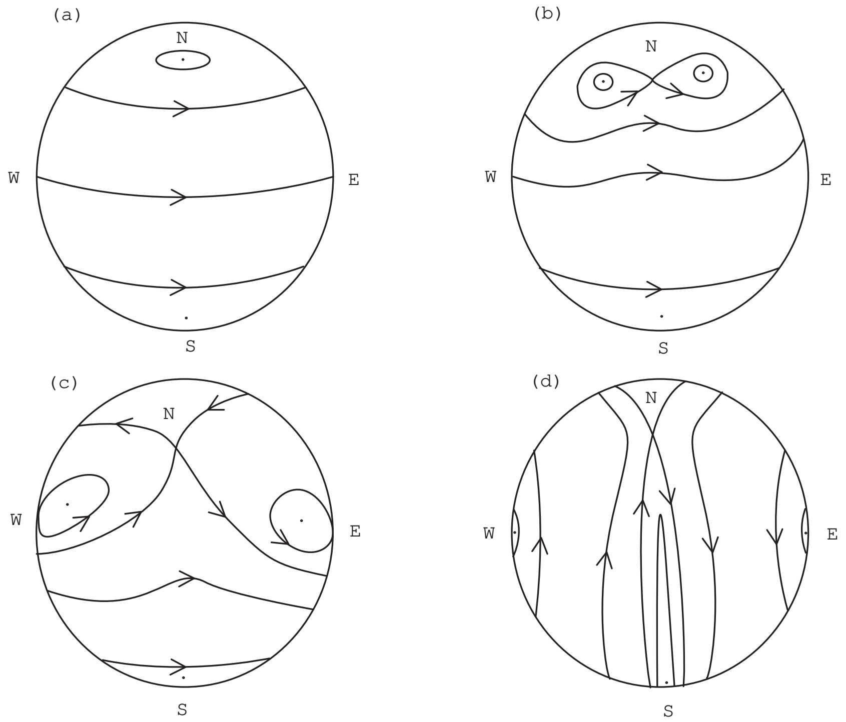

Figure 3: Four phase portraits for weakly coupled pendula, for the following ranges of values of the action a: (a) small, (b) between a_0 and a_1, (c) a little above a_1, (d) large.

them crosses the East and West poles as a increases through a higher value a_1, leading to a fraction of the energy remaining for ever on the initially excited pendulum. We could say there is an "exchange separatrix".

The connection with the rest of this paper can be seen in the picture for large a. If one restricts attention to a band around the Greenwich meridian and the date line, one obtains a pendulum portrait for 2-site generalised multibreathers.

The approximation by H^{eff} is best if the frequencies of the 2-site oscillators remain nearly resonant for all points on the $\mathbb{C}P^1$ for some a: then H_a^{eff} is nearly constant on it and so the relative phase dynamics is very slow. This can be achieved by coupling a soft and hard oscillator with nearly equal but opposite anharmonicities $d\omega/dI$. This "isochronous" regime is very interesting, as already pointed out by Chirikov [12]. Aubry proposes [31] that it may play a role in energy transfer in biomolecules. This may be true but we wish to point out that many phase portraits are possible in this case, as it is a small perturbation of the zero flow, the only constraint being that it is Hamiltonian.

Note that Liusternik-Schnirelman category theory gives at least $N+1$ critical points of a smooth function on $\mathbb{C}P^N$ (and assuming non-degeneracy does not give any more), so we have at least $N+1$ exactly periodic orbits. The lower bound is achieved in the familiar case of linear systems with distinct normal mode frequencies, but what is interesting here is that one can not reduce the number further.

7 Conclusions

For networks of oscillators, in parts of phase space where some number $N+1$ of oscillators are moving in nearly rational frequency ratio and the remainder are approximately at rest, we have derived an effective Hamiltonian $H_a^{eff}(\underline{J},\underline{\theta})$ which describes approximately the evolution of the relative phases $\underline{\theta}$ and momenta $\underline{J}$ of the excited oscillators, a being an adiabatic invariant corresponding to the area of the instantaneous approximate periodic orbit.

Our derivation has the special feature that critical points $(\underline{\bar{J}},\underline{\bar{\theta}})$ of H_a^{eff} correspond to exact periodic orbits, with mean values of $(\underline{J},\underline{\theta})$ given by $(\underline{\bar{J}},\underline{\bar{\theta}})$ and period near that of the unperturbed case. Under an anharmonicity assumption, topological principles allow us to deduce existence of at least $N+1$ such periodic orbits and generically at least 2^N.

The linear stability of these periodic orbits is closely related to that of the critical point of H_a^{eff} under the associated Hamiltonian dynamics. We prove that the linear stability type of the critical point determines that of the periodic orbit in the case that the eigenvalues of the former move at distinct non-zero speeds with the coupling ϵ.

We believe that the construction of H^{eff} can be slightly modified to make an exact correspondence between the linearised dynamics of the two problems, but have not yet completely worked out the details. The idea is that the remainder term in Eq. (29) can be reduced, by near-identity coordinate changes and/or modification of the constraints, to $\mathcal{O}(\epsilon\tau^2)$ where $\tau = \max(|u|,|v|)$ plus an $\mathcal{O}(\epsilon\tau)$ multiple of the Hamiltonian vector field, because the remainder has zero mean with respect to φ_0 in the $A,\underline{J}$ and $\underline{\theta}$ directions. In fact, we believe that one could iterate the procedure to reduce the remainder to $\mathcal{O}(\epsilon\tau^n)$) plus an $\mathcal{O}(\epsilon\tau)$ multiple of the Hamiltonian vector field, for any $n \in \mathbb{N}$, and in the analytic case by optimising over n one should be able to achieve $\mathcal{O}(\epsilon e^{-c/\tau^\alpha})$ for some c,α. Note that we must expect to retain a $\mathcal{O}(\epsilon\tau)$ multiple of the Hamiltonian vector field because generically a geometric phase change ("Berry phase") results from adiabatic movement round non-trivial loops. Strategies to achieve such results were developed by Nekhoroshev and Neishtadt; for a recent application of the method to a closely related problem, see [34].

We gave an extension to the case where the amplitudes of some excited oscillators are allowed to go to zero, and a tentative explanation of observations of travelling breathers as approximate solutions in certain regimes.

We presented the analysis in the context of coupled 1 degree of freedom oscillators, but the ideas extend easily to coupled oscillators with more than one degree of freedom: they apply to any choices of normally non-degenerate anharmonic 1-parameter family of periodic orbits for each excited oscillator.

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