A novel preserved partial order for cooperative networks of units with overdamped second order dynamics, and application to tilted Frenkel-Kontorova chains

Claude Baesens* and Robert S. MacKay†
Mathematics Institute, University of Warwick
Coventry CV4 7AL, UK

Abstract

We introduce a novel partial order on the state space for cooperative networks of units with overdamped second order dynamics, which is preserved under the forward dynamics. Among other things, we use it to prove that for chains of particles in a tilted periodic potential with overdamped inertial dynamics, for each class of spatially periodic states if there are no rotationally-ordered equilibria then there is a globally attracting periodically sliding solution.

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*claude@maths.warwick.ac.uk
†mackay@maths.warwick.ac.uk

1 Introduction

The dynamics of various physical systems, e.g. charge density wave materials and Josephson junction arrays, can be modelled as networks of overdamped second order dynamical units with cooperative coupling. "Overdamped" means that the damping term dominates the geometric mean of the inertial and restoring terms (we call the case with no inertial term "gradient dynamics"). "Cooperative" means that the derivative of the forcing by a neighbour with respect to the position of the neighbour is non-negative.
In the gradient case there is a preserved partial order (analogous to the comparison principle for a parabolic PDE) which makes the dynamics "monotone" and forces it to have certain features, see [Smith 1995], [Baesens and MacKay 1998], [Golé 2001] and references therein. Many authors have lamented that the methods do not seem to extend to networks of multi-component (or equivalently higher order) units. Following a success in the context of overdamped nonlinear wave equations [Gallay and Raugel 1997], however, we present here an analogue for networks of overdamped second order units with dynamics of the form

$$m_n \ddot{x}_n + \gamma_n \dot{x}_n = f_n(\mathbf{x}), \qquad m_n, \gamma_n > 0, \tag{1}$$

for n in some index set $\mathcal{S}$, and C^1 functions f_n satisfying

$$\frac{\partial f_n}{\partial x_j} \geq 0 \quad \forall j \neq n \qquad \text{(cooperativity condition)} \tag{2}$$

and

$$-4 m_n \frac{\partial f_n}{\partial x_n} \leq \gamma_n^2 \quad \forall n \qquad \text{(overdamped condition)} \tag{3}$$

in the domain of interest. We take m_n, γ_n independent of $\mathbf{x}$ and $\dot{\mathbf{x}}$ but extensions should be possible. By dividing by γ_n we reduce to the case $\gamma_n = 1$ for all n.
Our principal application here is the case of tilted (generalised) Frenkel-Kontorova chains:

$$m\ddot{x}_n + \dot{x}_n = -h_1(x_n, x_{n+1}) - h_2(x_{n-1}, x_n) + F, \quad n \in \mathbb{Z}, \tag{4}$$

with the masses m and tilt F constant, and function $h \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R})$ satisfying

$$h(x+1, x'+1) = h(x, x') \qquad \text{(periodicity condition)}, \tag{5}$$

$$h_{12} \leq -c < 0 \qquad \text{(cooperativity condition)} \tag{6}$$

(actually (6) is a case of what we will call strict cooperativity). Suffix $i \in \{1, 2\}$ on h denotes the partial derivative with respect to the i-th argument. If $F < 0$ we change h to its reflection $\tilde{h}(x, x') = h(-x', -x)$ and thereby restrict attention to $F \geq 0$.
The archetype class of examples is

$$h(x, x') = \frac{1}{2}(x' - x - a)^2 + V(x)$$

for some $a \in \mathbb{R}$ and C^2 function V with $V(x+1) = V(x)$. In this case the overdamped condition is $0 < m \leq \frac{1}{4 \sup_x (2 + V''(x))}$. For instance, for $V(x) = \frac{k}{4\pi^2} \cos 2\pi x$ the equations of motion are

$$m\ddot{x}_n + \dot{x}_n = x_{n+1} - 2x_n + x_{n-1} + \frac{k}{2\pi} \sin 2\pi x_n + F \tag{7}$$

and the overdamped condition is

$$0 < m \leq \frac{1}{4(2+k)}.$$

For Frenkel-Kontorova chains, we are interested in configurations of bounded spacing $\sup_n |x_{n+1} - x_n| < \infty$. By Aubry-Mather theory, e.g. [Aubry and Le Daeron 1983, Katok and Hasselblatt 1995], when $F = 0$ there are many attracting equilibria of bounded spacing, but they are destroyed for F large enough (see Appendix). So the question arises, what happens when they are destroyed?

In [Baesens and MacKay 1998] we addressed the gradient case ($m = 0$), which is a dynamical system on positions only. Following ideas of [Middleton 1992] we used the fact that the dynamics is monotone, i.e. preserves a partial order, to prove that if there are no equilibria in the class of configurations of spatially periodic type (p, q), i.e. $x_{n+q} = x_n + p$, then there is a globally attracting periodically sliding solution, $x_n(t + T) = x_n(t) + 1$ for some $T > 0$. The relevant partial order on sequences $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ is

$$\mathbf{x} \leq \mathbf{y} \text{ if, for all } n,\ x_n \leq y_n. \tag{8}$$

The gradient case $m = 0$ is an idealised limit, however; physically there is a small but non-negligible inertial term. Numerical results of Floría, Falo and Mazo [Floría and Mazo 1996] suggest that for small positive m the above result remains true (though not for large m, as already made clear by [Aubry and de Seze 1985] and rediscovered by [Carpio and Bonilla 2003]). The main application of our result is to prove this claim by constructing a partial order for this system of twice the dimension, that is preserved for m small.

2 Order-preservation

2.1 The setting

We rewrite (1) (with γ_n reduced to 1) as a system of first order ODEs

$$\begin{cases} \dot{x}_n = v_n \\ \dot{v}_n = \frac{1}{m_n}\left(f_n(\mathbf{x}) - v_n\right) \end{cases} \qquad n \in \mathcal{S}. \tag{9}$$

We define the *state of a unit* to be the pair $\bar{x}_n = (x_n, v_n) \in \mathbb{R}^2$, and the *state of the network* to be the indexed set $\bar{\mathbf{x}} := (\bar{x}_n)_{n\in\mathcal{S}}$. We often use the first equation of (9) to write a state $\bar{\mathbf{x}}$ as $\bar{\mathbf{x}} = (x_n, \dot{x}_n)_{n\in\mathcal{S}}$. For the state space X we take the set X_0 of all states $\bar{\mathbf{x}}$, or various subsets. We assume that conditions hold on X such that equation (9) defines a local flow ϕ on X, i.e. a continuous map ϕ from a neighbourhood of $X \times \{0\}$ in $X \times \mathbb{R}$ to X such that $\phi_{t+s} = \phi_t \circ \phi_s$ for all t, s for which both sides are defined. This holds for X_0 for all finite networks (since the f_n are C^1) and for the restriction of Frenkel-Kontorova chains to the $(\ell_\infty \times \ell_\infty)$-Banach manifold X of states with bounded spacings and velocities: $\sup_n |x_{n+1} - x_n| < \infty$, $\sup_n |v_n| < \infty$ (because h is C^2 and diagonally periodic so the vector field is defined and locally Lipschitz). In such cases, the only way solutions can cease to exist is by finite-time blow-up.

2.2 Motivation for the partial order

To indicate why it is reasonable to expect the flow ϕ to preserve a partial order for small m_n, consider first the case of a single linear damped oscillator

$$m_n\ddot{x}_n + \dot{x}_n + k_n x_n = 0 \tag{10}$$

(with damping normalised to 1, and $m_n, k_n > 0$). In the overdamped regime $4m_nk_n \leq 1$, the phase portrait (with $v_n = \dot{x}_n$) is shown in Figure 1. It can be seen that any cone of the form

$$\left\{(x_n, v_n) \in \mathbb{R}^2 : v_n \geq \lambda_n x_n, x_n \geq \alpha_n v_n\right\} \tag{11}$$

with λ_n between the (negative) slopes $\lambda_n^{\pm} = -\frac{1}{2m_n} \pm \sqrt{\frac{1}{4m_n^2} - \frac{k_n}{m_n}}$ of the eigenvectors and α_n between $-\frac{1}{\lambda_n^-}$ and $\frac{1}{\lambda_n^+}$, is forward invariant (the cone shown is the case $\alpha_n = 0, \lambda_n = -\frac{1}{2m_n}$). Turning now to a network of such units with cooperative linear coupling, the coupling makes no difference to $\dot{x}_n$ and adds non-negative contributions to $\dot{v}_n$ from all units with $x_k \geq 0$ because of the cooperativity condition. Thus, the product cone

$$\{(\mathbf{x}, \mathbf{v}) : v_n \geq \lambda_n x_n, x_n \geq \alpha_n v_n \forall n \in \mathcal{S}\} \tag{12}$$

is forward invariant for any choice of λ_n between $\lambda_n^{\pm}$ and α_n between $-\frac{1}{\lambda_n^-}$ and 0. For simplicity of formulae we will choose $\alpha_n = 0, \lambda_n = -\frac{1}{2m_n}$ though other choices are possible.

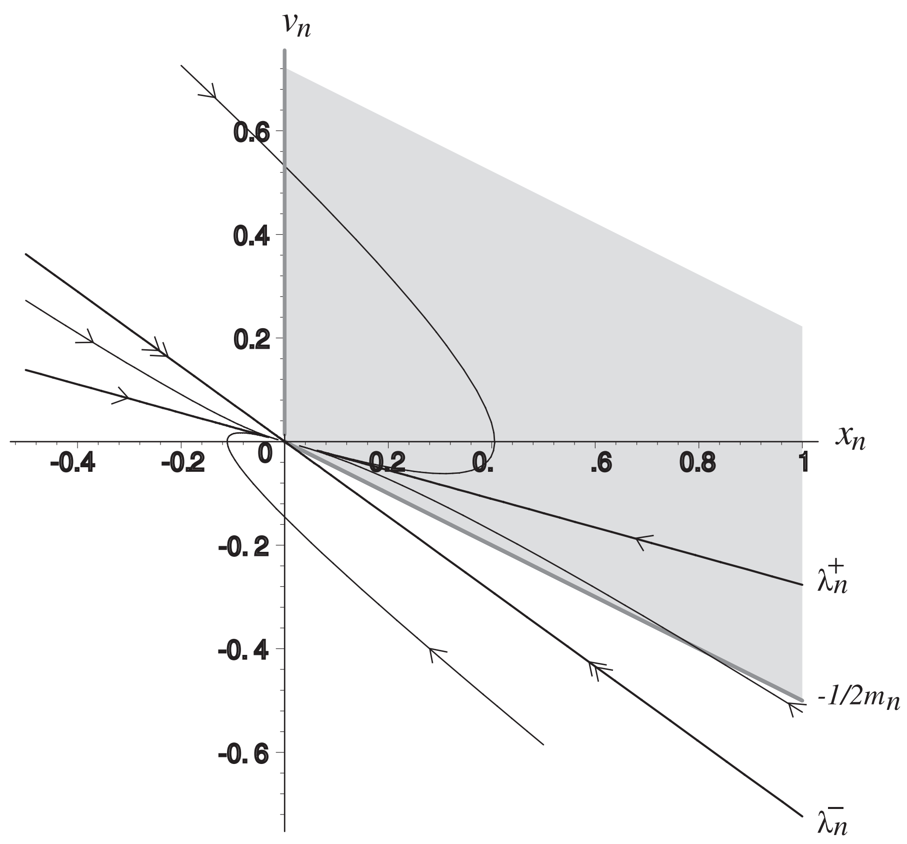

Figure 1: Phase portrait of equation (10) for $m_n = 1, k_n = 0.2$. A forward invariant cone is shown in grey.

2.3 Definitions

Definition 2.1. We define a partial order "less than or equal to" on X by $\bar{\mathbf{x}} \leq \bar{\mathbf{y}}$ if for all $n \in \mathcal{S}$

$$\begin{cases} x_n \leq y_n \\ 2m_n\dot{x}_n + x_n \leq 2m_n\dot{y}_n + y_n \,. \end{cases} \tag{13}$$

For short, we write the partial order as

$$\begin{cases} \mathbf{x} \leq \mathbf{y} \\ 2\mathbf{m}\dot{\mathbf{x}} + \mathbf{x} \leq 2\mathbf{m}\dot{\mathbf{y}} + \mathbf{y}. \end{cases} \tag{14}$$

We use the following standard additional notations

Definition 2.2.

$$\bar{\mathbf{x}} \geq \bar{\mathbf{y}} \quad \text{if} \quad \bar{\mathbf{y}} \leq \bar{\mathbf{x}} \qquad \text{"greater than or equal to"} \tag{15}$$
$$\bar{\mathbf{x}} < \bar{\mathbf{y}} \quad \text{if} \quad \bar{\mathbf{x}} \leq \bar{\mathbf{y}} \text{ and } \bar{\mathbf{x}} \neq \bar{\mathbf{y}} \qquad \text{"less than"} \tag{16}$$
$$\bar{\mathbf{x}} > \bar{\mathbf{y}} \quad \text{if} \quad \bar{\mathbf{y}} < \bar{\mathbf{x}} \qquad \text{"greater than".} \tag{17}$$

It will be convenient to define a relation $\ll$ "strictly less than" on X by

Definition 2.3. For $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in X$

$$\bar{\mathbf{x}} \ll \bar{\mathbf{y}} \quad \text{if} \quad \begin{cases} x_n < y_n \\ 2m_n\dot{x}_n + x_n < 2m_n\dot{y}_n + y_n \end{cases} \quad \text{for all } n \in \mathcal{S}. \tag{18}$$

Definition 2.4. The flow (or local flow) ϕ on X is *monotone* if

$$\bar{\mathbf{x}}, \bar{\mathbf{y}} \in X, \ \bar{\mathbf{x}} \leq \bar{\mathbf{y}} \implies \phi_t\bar{\mathbf{x}} \leq \phi_t\bar{\mathbf{y}} \tag{19}$$

for all $t > 0$ for which both are defined.
The flow (or local flow) ϕ on X is *strictly monotone* if

$$\bar{\mathbf{x}}, \bar{\mathbf{y}} \in X, \ \bar{\mathbf{x}} < \bar{\mathbf{y}} \implies \phi_t\bar{\mathbf{x}} \ll \phi_t\bar{\mathbf{y}} \tag{20}$$

for all $t > 0$ for which both are defined.

Definition 2.5. For a sequence $\mathbf{x} = (x_n)_{n\in\mathcal{S}}$, let

$$\mu(\mathbf{x}) := \sup_n \left(-m_n \frac{\partial f_n}{\partial x_n}(\mathbf{x}) \right) \tag{21}$$

We first prove monotonicity of the *linearisation* of the flow about a path $\mathbf{x}(t)$ for which $\mu(\boldsymbol{x}(t)) \leq \frac{1}{4}$ (e.g. the m_n sufficiently small), and then a little more work and slightly stronger hypotheses give strict monotonicity. Finally we will use homotopy to deduce the same for ϕ, provided $\mu \leq \frac{1}{4}$ in a suitable region.

2.4 Monotonicity of the linearised dynamics

The linearisation of (9) about a path $\bar{\mathbf{x}}(t)$ is

$$\begin{cases} \dot{\xi}_n = v_n \\ \dot{v}_n = \frac{1}{m_n}\left(\sum_k f_{nk}\xi_k - v_n\right). \end{cases} \tag{22}$$

where $f_{nk}(\mathbf{x}) = \frac{\partial f_n}{\partial x_k}(\mathbf{x})$.

Proposition 2.1 (Monotonicity of the linearised dynamics). *The non-negative cone*

$$\begin{cases} \boldsymbol{\xi} \geq \mathbf{0} \\ 2\boldsymbol{m}\boldsymbol{\upsilon} + \boldsymbol{\xi} \geq \mathbf{0} \end{cases} \tag{23}$$

in the partial order (13) *is forward invariant under the linearised dynamics as long as* $\mu(\mathbf{x}(t))$ *remains* $\leq \frac{1}{4}$.

Proof. Suppose (ξ_0, υ_0) is in the cone (23). Let

$$\beta_n := 2m_n\upsilon_n + \xi_n \ . \tag{24}$$

Equation (22) yields

$$\begin{aligned} \dot{\beta}_n &= 2m_n\dot{\upsilon}_n + \upsilon_n \\ &= 2f_{nn}\xi_n + 2\sum_{k\neq n} f_{nk}\xi_k - \upsilon_n \\ &\geq -\frac{1}{2m_n}\beta_n + 2\sum_{k\neq n} f_{nk}\xi_k \qquad\qquad (\text{as } \ -4m_nf_{nn} \leq 1). \end{aligned} \tag{25}$$

So

$$\frac{d}{dt}\left(\beta_n e^{t/2m_n}\right) \ \geq 2\sum_{k\neq n} f_{nk}\xi_k e^{t/2m_n}. \tag{26}$$

By the cooperativity condition (2) the right hand side of (26) is non-negative, hence for $t \geq 0$

$$\beta_n(t) \geq e^{-t/2m_n}\beta_n(0) \geq 0 \ . \tag{27}$$

Also $\xi_n = \upsilon_n = \frac{1}{2m_n}(\beta_n - \xi_n)$ so

$$\frac{d}{dt}\left(\xi_n e^{t/2m_n}\right) = \frac{1}{2m_n}\,\beta_n\, e^{t/2m_n}, \tag{28}$$

hence for $t \geq 0$

$$\xi_n(t) \geq e^{-t/2m_n}\xi_n(0) \geq 0\,. \tag{29}$$

□

Definition 2.6. We say that the system (9) is *strictly cooperative* if for each pair $i, n \in \mathcal{S}$ there is a j-chain $n = n_0, n_1, \cdots, n_j = i$ for some $j \geq 1$, $n_k \neq n_{k+1}\,\forall k$, and for $k = 0, \cdots, j-1$ there is c_k such that

$$\frac{\partial f_{n_k}}{\partial x_{n_{k+1}}}(\boldsymbol{x}) \geq c_k > 0 \qquad \text{for all } \boldsymbol{x} \in X. \tag{30}$$

Proposition 2.2 (Strict monotonicity of the linearised dynamics). *Under the assumption that the system* (9) *is* strictly cooperative, *if* $\bar{\boldsymbol{\xi}}(0)$ *is non-zero in the cone* (23) *then* $\bar{\boldsymbol{\xi}}(t) \gg \bar{\mathbf{0}}$ *for* $t > 0$ *as long as* $\mu(\mathbf{x}(t))$ *remains* $\leq \frac{1}{4}$.

Proof. From equation (26)

$$\beta_n(t) \geq \beta_n(0)e^{-t/2m_n} + 2\sum_{k\neq n}\int_0^t f_{nk}\,\xi_k(t')e^{-(t-t')/2m_n}\,dt' \tag{31}$$

and equation (28) gives

$$\xi_n(t) \geq \left(\xi_n(0) + \frac{t}{2m_n}\beta_n(0)\right) e^{-t/2m_n} + \frac{1}{m_n}\sum_{k\neq n}\int_0^t \int_0^{t'} f_{nk}\, \xi_k(s) e^{-(t-s)/2m_n}\, ds\, dt'. \quad (32)$$

We subdivide the rest of the proof into two cases.
Case 1: At $t = 0$, $\boldsymbol{\xi} > \mathbf{0}$ and $\boldsymbol{\beta} \geq \mathbf{0}$.
It follows that $\xi_i(0) > 0$ for some $i \in \mathcal{S}$; denote it by δ. Then from (29) $\xi_i(t) \geq e^{-t/2m_i}\delta$. By strict cooperativity, $f_{ji} \geq c_{ji} > 0$ for some $j \in \mathcal{S}$, $j \neq i$. Then (31) yields

$$\begin{aligned}
\beta_j(t) &\geq 2c_{ji}\int_0^t \xi_i(t')\, e^{-(t-t')/2m_j}\, dt' \\
&\geq 2c_{ji}\int_0^t \delta\, e^{-t'/2m_i}\, e^{-(t-t')/2m_j}\, dt' \\
&\geq 2c_{ji}\delta\int_0^t e^{-t/2\min\{m_i,m_j\}}\, dt' \\
&= 2c_{ji}\,\delta\, t\, e^{-t/2\min\{m_i,m_j\}} \\
&> 0 \text{ for } t > 0.
\end{aligned} \quad (33)$$

Note that we have used the inequality $A := e^{-s/2m_i}e^{-(t-s)/2m_j} \geq e^{-t/2\min\{m_i,m_j\}}$. To see this, when $t \geq s \geq 0$ and $m_i, m_j > 0$,

$$\begin{cases} m_i \geq m_j \implies e^{-s/2m_i} \geq e^{-s/2m_j} \implies A \geq e^{-t/2m_j} \\ m_j \geq m_i \implies e^{-(t-s)/2m_j} \geq e^{-(t-s)/2m_i} \implies A \geq e^{-t/2m_i}\ . \end{cases} \quad (34)$$

Similarly, (32) yields

$$\begin{aligned}
\xi_j(t) &\geq \frac{c_{ji}}{m_j}\int_0^t \int_0^{t'} \xi_i(s)\, e^{-(t-s)/2m_j}\, ds\, dt' \\
&\geq \frac{c_{ji}}{m_j}\int_0^t \int_0^{t'} \delta\, e^{-s/2m_i}\, e^{-(t-s)/2m_j}\, ds\, dt' \\
&\geq \frac{c_{ji}\,\delta}{m_j}\int_0^t \int_0^{t'} e^{-t/2\min\{m_i,m_j\}}\, ds\, dt' \\
&= \frac{c_{ji}\,\delta}{m_j} e^{-t/2\min\{m_i,m_j\}}\, \frac{t^2}{2!} \\
&> 0 \text{ for } t > 0.
\end{aligned} \quad (35)$$

For a next site k for which $f_{kj} \geq c_{kj} > 0$, we then get

$$\begin{aligned}
\beta_k(t) &\geq 2c_{kj}\int_0^t \xi_j(t')\, e^{-(t-t')/2m_k}\, dt' \\
&\geq 2c_{kj}\int_0^t \frac{c_{ji}\,\delta}{m_j} e^{-t'/2\min\{m_i,m_j\}}\, \frac{t'^2}{2!}\, e^{-(t-t')/2m_k}\, dt' \\
&\geq \frac{2c_{kj}c_{ji}\delta}{m_j}\int_0^t \frac{t'^2}{2!}\, e^{-t/2\min\{m_i,m_j,m_k\}}\, dt' \\
&= \frac{2c_{kj}c_{ji}\delta}{m_j}\,\frac{t^3}{3!}\, e^{-t/2\min\{m_i,m_j,m_k\}} \\
&> 0 \text{ for } t > 0.
\end{aligned} \quad (36)$$

and

$$\begin{aligned}
\xi_k(t) &\geq \frac{c_{kj}}{m_k} \int_0^t \int_0^{t'} \xi_j(s)\, e^{-(t-s)/2m_k}\, ds\, dt' \\
&\geq \frac{c_{kj}}{m_k} \int_0^t \int_0^{t'} \frac{c_{ji}\,\delta}{m_j}\, e^{-s/2\min\{m_i,m_j\}}\, \frac{s^2}{2!}\, e^{-(t-s)/2m_k}\, ds\, dt' \\
&\geq \frac{c_{kj}}{m_k}\frac{c_{ji}}{m_j}\,\delta \int_0^t \int_0^{t'} e^{-t/2\min\{m_i,m_j,m_k\}}\, \frac{s^2}{2!}\, ds\, dt' \\
&= \frac{c_{kj}}{m_k}\frac{c_{ji}}{m_j}\,\delta\, \frac{t^4}{4!} e^{-t/2\min\{m_i,m_j,m_k\}} \\
&> 0 \text{ for } t > 0.
\end{aligned} \tag{37}$$

Continuing in such a way, for a site $n \in \mathcal{S}$ connected to site i by a j-chain $n = n_0, n_1, \cdots, n_j = i$ with $f_{n_{k-1}n_k} \geq c_{n_{k-1}n_k} > 0$, $k = 1, \cdots, j$, we obtain

$$\beta_n(t) \geq 2m_{n_0}\delta \left(\frac{c_{n_0,n_1}}{m_{n_0}}\right)\left(\frac{c_{n_1,n_2}}{m_{n_1}}\right)\cdots\left(\frac{c_{n_{j-1},n_j}}{m_{n_{j-1}}}\right) \frac{t^{2j-1}}{(2j-1)!}\, e^{-t/2m} > 0 \tag{38}$$

$$\xi_n(t) \geq \delta \left(\frac{c_{n_0,n_1}}{m_{n_0}}\right)\left(\frac{c_{n_1,n_2}}{m_{n_1}}\right)\cdots\left(\frac{c_{n_{j-1},n_j}}{m_{n_{j-1}}}\right) \frac{t^{2j}}{(2j)!}\, e^{-t/2m} > 0\,, \tag{39}$$

where $m = \min\{m_{n_k} : k = 0, \cdots, j\}$.
Note that for the site $n = i$, we already obtained $\xi_i(t) \geq e^{-t/2m_i}\,\delta$ and an estimate for $\beta_i(t)$ can be obtained by using a closed j-chain $i = n_0, n_1, \cdots, n_j = i$, with $j \geq 1$.
Case 2 : At $t = 0$, $\boldsymbol{\xi} \geq \mathbf{0}$ and $\boldsymbol{\beta} > \mathbf{0}$.
Then $\beta_i(0) > 0$ for some $i \in \mathcal{S}$; denote it by δ. Then from (27) and (28)

$$\beta_i(t) \geq \delta\, e^{-t/2m_i} \tag{40}$$

$$\xi_i(t) \geq \frac{t\delta}{2m_i} e^{-t/2m_i}\,. \tag{41}$$

For a site $n \in \mathcal{S}$ connected to site i by a j-chain $n = n_0, n_1, \cdots, n_j = i$ with $f_{n_{k-1}n_k} \geq c_{n_{k-1}n_k} > 0$, $k = 1, \cdots, j$, we now obtain

$$\beta_n(t) \geq \delta \left(\frac{c_{n_0,n_1}}{m_{n_1}}\right)\left(\frac{c_{n_1,n_2}}{m_{n_2}}\right)\cdots\left(\frac{c_{n_{j-1},n_j}}{m_{n_j}}\right) \frac{t^{2j}}{(2j)!}\, e^{-t/2m} > 0 \tag{42}$$

$$\xi_n(t) \geq \frac{\delta}{2m_{n_0}} \left(\frac{c_{n_0,n_1}}{m_{n_1}}\right)\left(\frac{c_{n_1,n_2}}{m_{n_2}}\right)\cdots\left(\frac{c_{n_{j-1},n_j}}{m_{n_j}}\right) \frac{t^{2j+1}}{(2j+1)!}\, e^{-t/2m} > 0\,, \tag{43}$$

with $m = \min\{m_{n_k} : k = 0, \cdots, j\}$ again.

□

2.5 Monotonicity of the overdamped semi-flow

Proposition 2.3 (Monotonicity). *Suppose $\bar{\mathbf{x}}(0) \leq \bar{\mathbf{y}}(0)$, then $\bar{\mathbf{x}}(t) \leq \bar{\mathbf{y}}(t)$ for $t \geq 0$, as long as both exist and*

$$\mu(\bar{\mathbf{z}}(t)) \leq \tfrac{1}{4} \text{ for all } \bar{\mathbf{z}}(t) \text{ such that } \bar{\mathbf{x}}(t) \leq \bar{\mathbf{z}}(t) \text{ and } \bar{\mathbf{z}}(t) \leq \bar{\mathbf{y}}(t). \tag{44}$$

Proof. Define the homotopy

$$\bar{\mathbf{x}}_\lambda(0) = (1-\lambda)\,\bar{\mathbf{x}}(0) + \lambda\,\bar{\mathbf{y}}(0), \quad \lambda \in [0,1] \tag{45}$$

and let $\bar{\mathbf{x}}_\lambda(t)$ be the solution of the overdamped flow with initial condition (45), as long as it exists. Let $\bar{\boldsymbol{\xi}}_\lambda(t) = \frac{\partial}{\partial\lambda}\bar{\mathbf{x}}_\lambda(t)$. Then

$$\bar{\boldsymbol{\xi}}_\lambda(0) = \bar{\mathbf{y}}(0) - \bar{\mathbf{x}}(0) \geq \bar{\mathbf{0}} = (\mathbf{0},\mathbf{0})\,. \tag{46}$$

Therefore

$$\bar{\boldsymbol{\xi}}_\lambda(t) \geq \bar{\mathbf{0}} \text{ for } t \geq 0 \text{ as long as } \bar{\mathbf{x}}_\lambda(t) \text{ exists and } \mu(\bar{\mathbf{x}}_\lambda(t)) \leq \tfrac{1}{4}, \tag{47}$$

by Proposition 2.1. Now

$$\bar{\mathbf{x}}_\lambda(t) = \bar{\mathbf{x}}(t) + \int_0^\lambda \bar{\boldsymbol{\xi}}_{\lambda'}(t)\,d\lambda', \tag{48}$$

so is trapped between $\bar{\mathbf{x}}(t)$ and $\bar{\mathbf{y}}(t)$ as long as (47) holds, so the only way for $\bar{\mathbf{x}}_\lambda(t)$ to cease to exist or for $\mu(\bar{\mathbf{x}}_\lambda(t))$ to exceed $\frac{1}{4}$ is for the hypothesis (44) to fail. In particular

$$\bar{\mathbf{y}}(t) = \bar{\mathbf{x}}(t) + \int_0^1 \bar{\boldsymbol{\xi}}_\lambda(t)\,d\lambda \geq \bar{\mathbf{x}}(t), \tag{49}$$

as long as (44) holds.

□

Proposition 2.4 (Strict monotonicity). *If* $\bar{\mathbf{x}}(0) < \bar{\mathbf{y}}(0)$ *then* $\bar{\mathbf{x}}(t) \ll \bar{\mathbf{y}}(t)$ *for* $t > 0$ *as long as both exist and* (44) *holds.*

Proof. Use the same homotopy as in Proposition 2.3 and integrate the result of Proposition 2.2 with respect to λ. □

3 Application to the tilted Frenkel-Kontorova model

We now consider the particular case of systems of the form

$$\begin{cases} \dot{x}_n = v_n \\ \dot{v}_n = \frac{1}{m}(-h_1(x_n,x_{n+1}) - h_2(x_{n-1},x_n) - v_n + F), \qquad n \in \mathbb{Z}, \end{cases} \tag{50}$$

with m, F constant, $m > 0$, $F \geq 0$, and $h \in \mathcal{C}^2(\mathbb{R}^2,\mathbb{R})$ satisfying

$$h(x+1, x'+1) = h(x,x') \qquad \text{(periodicity condition)}, \tag{51}$$

$$h_{12} \leq -c < 0 \qquad \text{(cooperative condition)}. \tag{52}$$

This represents the second order dynamics of a one-dimensional chain of identical masses with nearest-neighbour convex interaction, subject to a tilted periodic potential. The system is strictly cooperative because given site $i > n$ take the chain $n, n+1, \cdots, i$ (similarly for $i < n$) and for $i = n$ take $i, i+1, i$.
We shall restrict to the state space X of configurations with bounded spacings and velocities, $sup_{n\in\mathbb{Z}}|x_n - x_{n-1}| < \infty$ and $sup_{n\in\mathbb{Z}}|v_n| < \infty$, with metric given by supremum norm, for which the system (50) define a local flow ϕ.

The linearisation of (50) about a path $\bar{\boldsymbol{x}}(t)$ reads

$$\begin{cases}\dot{\xi}_n = v_n \\ \dot{v}_n = \frac{1}{m}\left(-h_{21}(x_{n-1},x_n)\xi_{n-1} - (h_{11}(x_n,x_{n+1}) + h_{22}(x_{n-1},x_n))\xi_n - h_{12}(x_n,x_{n+1})\xi_{n+1}\right) ,\end{cases} \tag{53}$$

so the overdamped condition reads

$$\mu(\boldsymbol{x}) = m \sup_n \left(h_{11}(x_n,x_{n+1}) + h_{22}(x_{n-1},x_n)\right) \leq \frac{1}{4} . \tag{54}$$

We assume henceforth that (54) is satisfied in whatever region we deal with.
Note that here $f_{kl} = 0$ if $l \notin \{k+1, k-1\}$ and $f_{k,l\pm 1} \geq c > 0$ so if $\xi_i(0) = \delta > 0$ formulae (38) and (39) simplify to

$$\begin{cases}\beta_{i\pm n}(t) \geq e^{-t/2m}\, 2m\delta\, \left(\frac{c}{m}\right)^{|n|} \frac{t^{2|n|-1}}{(2|n|-1)!} & \text{for } n \neq 0 \\ \beta_i(t) \geq e^{-t/2m}\, 2m\delta\, \left(\frac{c}{m}\right)^2 \frac{t^3}{3!} \end{cases} \tag{55}$$

$$\xi_{i\pm n}(t) \geq e^{-t/2m}\, \delta \left(\frac{c}{m}\right)^{|n|} \frac{t^{2|n|}}{(2|n|)!} \quad \text{for all } n\,, \tag{56}$$

whereas if $\beta_i(0) = \delta > 0$ formulae (42) and (43) simplify to

$$\beta_{i\pm n}(t) \geq e^{-t/2m}\, \delta \left(\frac{c}{m}\right)^{|n|} \frac{t^{2|n|}}{(2|n|)!} \quad \text{for all } n \tag{57}$$

$$\xi_{i\pm n}(t) \geq \frac{\delta}{2m}\, e^{-t/2m} \left(\frac{c}{m}\right)^{|n|} \frac{t^{2|n|+1}}{(2|n|+1)!} \quad \text{for all } n\,. \tag{58}$$

We shall now derive a number of consequences of the monotonicity property of the dynamics.

3.1 Comparison with translates

The local flow ϕ of (50) commutes with the group of *translations* $\{T_{i,j} : (i,j) \in \mathbb{Z}^2\}$ defined by

$$(T_{i,j}\bar{\mathbf{x}})_n = (x_{n-i} + j, v_{n-i}),\ \forall n \in \mathbb{Z}. \tag{59}$$

It follows that if $T_{i,j}\bar{\mathbf{x}}(0) \leq \bar{\mathbf{x}}(0)$ for some i, j then the same holds at all future times for which $\bar{\mathbf{x}}(t)$ is defined.

Definition 3.1. For $w \in \mathbb{N}$ say $\bar{\mathbf{x}}$ has *width* $\leq w$ if $\forall m \in \mathbb{Z}\ \exists n_-(m), n_+(m) \in \mathbb{Z}$ with $0 \leq n_+(m) - n_-(m) \leq w$ such that

$$T_{m,n_-(m)}\bar{\mathbf{x}} \leq \bar{\mathbf{x}} \leq T_{m,n_+(m)}\bar{\mathbf{x}}. \tag{60}$$

Then width $\leq w$ is preserved by the forward dynamics. If $\bar{\mathbf{x}}$ has bounded width then it has a mean spacing ρ, i.e.

$$\frac{x_m - x_n}{m-n} \to \rho \quad \text{as} \quad m - n \to \infty. \tag{61}$$

(In fact $|(x_m - x_n) - (m-n)\,\rho| \leq w$ (cf. [Baesens and MacKay 1998]).) The mean spacing is preserved under the dynamics (because $n_\pm(m)$ are.)

Definition 3.2. Say $\bar{\mathbf{x}}$ is *weakly rotationally-ordered* (wro) if for all $i, j \in \mathbb{Z}$ either

$$T_{i,j}\bar{\mathbf{x}} \leq \bar{\mathbf{x}} \quad \text{or} \quad T_{i,j}\bar{\mathbf{x}} \geq \bar{\mathbf{x}}. \tag{62}$$

(Equivalently, $\bar{\mathbf{x}}$ has width ≤ 1.) Similarly, say $\bar{\mathbf{x}}$ is *(strictly) rotationally-ordered* if for all $i, j \in \mathbb{Z}$ either

$$T_{i,j}\bar{\mathbf{x}} \ll \bar{\mathbf{x}} \quad \text{or} \quad T_{i,j}\bar{\mathbf{x}} = \bar{\mathbf{x}} \quad \text{or} \quad T_{i,j}\bar{\mathbf{x}} \gg \bar{\mathbf{x}}. \tag{63}$$

By strict monotonicity, if $\bar{\mathbf{x}}$ is weakly rotationally-ordered then $\phi_t(\bar{\mathbf{x}})$ is rotationally ordered for $t > 0$. In particular, all weakly rotationally ordered equilibria are rotationally ordered.

Note that $\bar{\mathbf{x}}$ weakly rotationally ordered implies

$$\begin{cases} T_{i,j}\bar{\mathbf{x}} < \bar{\mathbf{x}} & \text{if} \quad j < i\rho \\ T_{i,j}\bar{\mathbf{x}} > \bar{\mathbf{x}} & \text{if} \quad j > i\rho \end{cases} \tag{64}$$

where ρ is its mean spacing.

Recall from Aubry-Mather theory that when the tilt $F = 0$, for every $\rho \in \mathbb{R}$ there exists a rotationally-ordered equilibrium of mean spacing ρ, and for $\rho = \frac{p}{q}$ rational there exists one in

$$Per_{\frac{p}{q}} := \{\bar{\mathbf{x}} \in X : x_{n+q} = x_n + p, v_{n+q} = v_n \ \forall n \in \mathbb{Z}\}. \tag{65}$$

3.2 Global semi-flow

For $N \in \mathbb{N}$ define X_N to be the set of states in X such that

$$|x_n - x_{n-1}| \leq N \tag{66}$$

$$|2m(v_n - v_{n-1}) + (x_n - x_{n-1})| \leq N \tag{67}$$

and let

$$\mu_N = m \sup_{|x_n - x_{n-1}| \leq N, |x_{n+1} - x_n| \leq N} h_{11}(x_n, x_{n+1}) + h_{22}(x_{n-1}, x_n) \tag{68}$$

Note that every $\bar{\mathbf{x}} \in X$ is in some X_N because there exist L_1, L_2, V_1, V_2 such that for all $n \in \mathbb{Z}$, $(x_n - x_{n-1}) \in [L_1, L_2]$ and $v_n \in [V_1, V_2]$; then take N to be the integer above $2m(V_2 - V_1) + \max\{|L_1|, |L_2|\}$.

Theorem 3.1. *The system* (9) *defines a global semi-flow on X_N for $\mu_N \leq \frac{1}{4}$.*

Proof. $\bar{\mathbf{x}} \in X_N$ is equivalent to

$$T_{1,-N}\,\bar{\mathbf{x}} \leq \bar{\mathbf{x}} \leq T_{1,N}\,\bar{\mathbf{x}}. \tag{69}$$

Thus for $\mu_N \leq \frac{1}{4}$ Proposition 2.3 implies that X_N is preserved under the forward flow. The vector field is Lipschitz on X_N since h is $\mathcal{C}^2$ and diagonally periodic, hence the flow is defined for all positive times. $\square$

3.3 Spatially periodic states

Define $UPer_{\frac{p}{q}} = \cup_{k\in\mathbb{N}} Per_{\frac{kp}{kq}}$ and $\bar{\mathbf{1}}$ be the state with $(\bar{\mathbf{1}})_n = (1,0)$ for all $n \in \mathbb{Z}$. We say a map $\bar{\boldsymbol{u}} : \mathbb{R} \to X$ is *periodically sliding* with *period* T if

$$\bar{\mathbf{u}}(t+T) = \bar{\mathbf{u}}(t) + \bar{\mathbf{1}} \quad \forall t. \tag{70}$$

Theorem 3.2. *If there are no rotationally-ordered equilibria in $Per_{\frac{p}{q}}$ then there exists a rotationally-ordered periodically sliding solution $\bar{\mathbf{u}}(t+T) = \bar{\mathbf{u}}(t) + \bar{\mathbf{1}}$ with $T > 0$ in $Per_{\frac{p}{q}}$, with $\dot{\bar{\mathbf{u}}} \gg \bar{\mathbf{0}}$ and $T_{m,n}\bar{\mathbf{u}} = \bar{\mathbf{u}}((n - m\frac{p}{q})T)$, which attracts all of $UPer_{\frac{p}{q}}$ with phase, i.e. for every $\bar{\mathbf{x}} \in UPer_{\frac{p}{q}}$ there exists $\tau \in \mathbb{R}$ such that $\|\bar{\mathbf{x}}(t+\tau) - \bar{\mathbf{u}}(t)\| \to 0$ as $t \to \infty$.*

Proof. Define

$$Per^{+,wro}_{\frac{p}{q}} = \left\{ \bar{\mathbf{x}} \in Per_{\frac{p}{q}} : \bar{\mathbf{x}} \text{ is wro and } \dot{\bar{\mathbf{x}}} \geq \bar{\mathbf{0}} \right\} \tag{71}$$

It is non-empty because the Aubry-Mather minimal energy equilibrium for $F = 0$ in $Per_{\frac{p}{q}}$ is weakly rotationally-ordered and has $\dot{\bar{\mathbf{x}}} = F\bar{\mathbf{1}} \geq \bar{\mathbf{0}}$. It is forward invariant by monotonicity. The quotient space $Q := Per^{+,wro}_{\frac{p}{q}}/\langle T_{0,1}\rangle$ is compact. Therefore, by Birkhoff's recurrence theorem (for a proof not requiring Zorn's lemma see hint 3.3.4 in [Katok and Hasselblatt 1995]), Q contains a recurrent point. Let $\bar{\mathbf{u}}$ be a representative.

We now show that the orbit of $\bar{\mathbf{u}}$ is a periodically sliding solution. The orbit $\bar{\mathbf{u}}(t)$ satisfies $\dot{\bar{\mathbf{u}}}(t) \geq 0$, so is unbounded, else $(\sup_{t>0} \mathbf{u}(t), \mathbf{0})$ is a weakly rotationally-ordered equilibrium in $Per_{\frac{p}{q}}$ contradicting the hypothesis of no equilibria. Since $\bar{\mathbf{u}}(t)$ remains weakly rotationally ordered it follows that $\mathbf{u}(t) \to \infty$ (uniformly) as $t \to \infty$. Let

$$\begin{cases} \tau_+ = \inf\{\tau \geq 0 : \bar{\mathbf{u}}(\tau) \geq \bar{\mathbf{u}}(t) + \bar{\mathbf{1}}\} \\ \tau_- = \sup\{\tau \geq 0 : \bar{\mathbf{u}}(\tau) \leq \bar{\mathbf{u}}(t) + \bar{\mathbf{1}}\}. \end{cases} \tag{72}$$

Then $0 < \tau_- \leq \tau_+ < \infty$. Suppose $\tau_- < \tau_+$. Then, by strict monotonicity, given $s > 0$, there exists $\tau'_\pm \in \mathbb{R}$ such that $\tau_- < \tau'_- < \tau'_+ < \tau_+$ and

$$\bar{\mathbf{u}}(\tau'_- + s) \leq \bar{\mathbf{u}}(s) + \bar{\mathbf{1}} \leq \bar{\mathbf{u}}(\tau'_+ + s).$$

Note that $\bar{\mathbf{u}}(\tau'_-) \nleq \bar{\mathbf{u}} + \bar{\mathbf{1}} \nleq \bar{\mathbf{u}}(\tau'_+)$. There exists a recurrence time $S > s$ such that $\bar{\mathbf{u}}(S) + \bar{\mathbf{1}}$, $\bar{\mathbf{u}}(\tau'_- + S)$, $\bar{\mathbf{u}}(\tau'_+ + S)$ are close enough to $\bar{\mathbf{u}} + (N+1)\bar{\mathbf{1}}$, $\bar{\mathbf{u}}(\tau'_-) + N\bar{\mathbf{1}}$, $\bar{\mathbf{u}}(\tau'_+) + N\bar{\mathbf{1}}$ respectively that

$$\bar{\mathbf{u}}(\tau'_- + S) \nleq \bar{\mathbf{u}}(S) + \bar{\mathbf{1}} \nleq \bar{\mathbf{u}}(\tau'_+ + S). \tag{73}$$

But this contradicts monotonicity from time s to S. Hence $\tau_- = \tau_+$. Denoting the common value by T we have $\bar{\mathbf{u}}(T) = \bar{\mathbf{u}}(0) + \bar{\mathbf{1}}$ and hence the desired sliding periodicity.

Next we show that $T_{m,n}\bar{\mathbf{u}} = \bar{\mathbf{u}}\left((n - m\frac{p}{q})T\right)$. For $n > m\frac{p}{q}$, the previous argument applied to $T_{m,n}\bar{\mathbf{u}}$ instead of $\bar{\mathbf{u}} + \bar{\mathbf{1}}$, but with $\tau_\pm$ permitted negative since we can extend $\bar{\mathbf{u}}$ to negative times by periodicity, shows that $T_{m,n}\bar{\mathbf{u}} = \bar{\mathbf{u}}(\tau)$ for some τ. Thus

$$\bar{\mathbf{u}}\left((nq - mp)T\right) = \bar{\mathbf{u}} + (nq - mp)\bar{\mathbf{1}} = T^q_{m,n}\bar{\mathbf{u}} = \bar{\mathbf{u}}(q\tau). \tag{74}$$

But $\dot{\bar{\mathbf{u}}} \gg \bar{\mathbf{0}}$ so $\tau = (n - m\frac{p}{q})T$.

Finally, we show that the orbit of $\bar{\mathbf{u}}$ attracts all of $UPer_{\frac{p}{q}}$ with phase. Let $\bar{\mathbf{x}} \in UPer_{\frac{p}{q}}$. Then there exists $k \in \mathbb{N}$ such that $\bar{\mathbf{x}} \in Per_{\frac{kp}{kq}}$. Let

$$\begin{cases} \tau_+(t;\bar{\mathbf{x}}) = \inf\{\tau \in \mathbb{R} : \bar{\mathbf{u}}(\tau + t) \geq \bar{\mathbf{x}}(t)\} \\ \tau_-(t;\bar{\mathbf{x}}) = \sup\{\tau \in \mathbb{R} : \bar{\mathbf{u}}(\tau + t) \leq \bar{\mathbf{x}}(t)\}. \end{cases} \tag{75}$$

Then $-\infty < \tau_-(t;\bar{\mathbf{x}}) \leq \tau_+(t;\bar{\mathbf{x}}) < \infty$. If $\tau_-(0;\bar{\mathbf{x}}) = \tau_+(0;\bar{\mathbf{x}}) = \tau$ then $\bar{\mathbf{x}} = \bar{\mathbf{u}}(\tau)$ and hence $\bar{\mathbf{x}}(t) = \bar{\mathbf{u}}(t+\tau)$ for all $t \geq 0$. Else by strict monotonicity, τ_- is increasing and τ_+ is decreasing. Let $\tau_-^\infty = \sup_{t\geq 0} \tau_-(t;\bar{\mathbf{x}})$ and $\tau_+^\infty = \inf_{t\geq 0} \tau_+(t;\bar{\mathbf{x}})$. Then $\tau_-^\infty \leq \tau_+^\infty$. If $\tau_-^\infty < \tau_+^\infty$ then let $\bar{\mathbf{y}}$ be a limit point of the sequence $\bar{\mathbf{x}}_k = (\bar{\mathbf{x}}(kT) - k\bar{\mathbf{1}})_{k\in\mathbb{N}}$ (which exists because it is bounded between $\bar{\mathbf{u}}(\tau_\pm(0;\bar{\mathbf{x}}))$ and $Per_{\frac{kp}{kq}}$ is finite-dimensional). It follows that $\tau_\pm(0;\bar{\mathbf{y}}) = \tau_\pm^\infty$. But by strict monotonicity $\tau_-^\infty < \tau_-(t;\bar{\mathbf{y}}) \leq \tau_+(t;\bar{\mathbf{y}}) < \tau_+^\infty$ for $t > 0$. Choose a $t > 0$ and take k large enough that $\bar{\mathbf{x}}_k$ is close enough to $\bar{\mathbf{y}}$ that $\tau_-^\infty < \tau_-(t;\bar{\mathbf{x}}_k) \leq \tau_+(t;\bar{\mathbf{x}}_k) < \tau_+^\infty$. Hence $\tau_-^\infty < \tau_-(t+kT;\bar{\mathbf{x}}) \leq \tau_+(t+kT;\bar{\mathbf{x}}_k) < \tau_+^\infty$ contradicting the definition of $\tau_\pm^\infty$. Thus $\tau_-^\infty = \tau_+^\infty$. Denoting the common value by τ we deduce that $\|\bar{\mathbf{x}}(t+\tau) - \bar{\mathbf{u}}(t)\| \to 0$ as $t \to \infty$. □

To complete the analysis of the spatially periodic case we address the dynamics in the situation where there exist equilibria in $UPer_{\frac{p}{q}}$. Note from Theorem 1 that in this case there must be a rotationally ordered equilibrium in $Per_{\frac{p}{q}}$. On $Per_{\frac{p}{q}}$ define

$$W_{p.q}(\bar{\mathbf{x}}) = \sum_{n=0}^{q-1} h(x_n, x_{n+1}) + \tfrac{1}{2}\dot{x}_n^2 \tag{76}$$

Theorem 3.3. *If there exists a rotationally-ordered equilibrium in $Per_{\frac{p}{q}}$ then for all $\bar{\mathbf{x}} \in UPer_{\frac{p}{q}}$ the ω-limit set $\omega(\bar{\mathbf{x}})$ is an unordered connected subset of the set of equilibria in $Per_{\frac{kp}{kq}}$, some k, with constant $W_{(kp,kq)}$.*

Remark: Generically $\omega(\bar{\mathbf{x}})$ is a single equilibrium.

Proof. The initial condition $\bar{\mathbf{x}}$ is bounded by certain translates of the equilibrium. W is a Lyapunov function. Apply La Salle's principle [La Salle 1967] to deduce that there exists $E \in \mathbb{R}$ such that $\omega(\bar{\mathbf{x}})$ is a connected subset of the set of equilibria in $Per_{\frac{kp}{kq}}$ with $W_{kp,kq} = E$. That $\omega(\bar{\mathbf{x}})$ is unordered follows from [Smith 1995]. □

4 Extensions and open problems

1. The preserved partial order can be extended to networks of more general type than second order mechanical units, e.g. the units can be any dimensional systems of first order ODEs such that the linearized forward dynamics of each preserves some cone and the coupling respects the product cone. It would be interesting to find a worthwhile application of this extension.

2. Finite tilted Frenkel-Kontorova chains can be treated similarly (as we did in the gradient case in [Baesens and MacKay 1998]).

3. What is the fate of spatially non-periodic configurations of rational mean spacing for tilted Frenkel-Kontorova chains? e.g. discommensurations, i.e. $u_n \to v_n$ as $n \to \infty$ and $\to v_{n+j} - i$ as $n \to -\infty$ for some $\mathbf{v}$ of type (p, q) with $iq - jp = \pm 1$ (these are called discommensurations of type $(p, q)\pm$). For instance
   (a) if there are no type (p, q) equilibria does every configuration of mean spacing p/q and bounded spacing converge pointwise to the periodically sliding type (p, q) solution, with phase?
   (b) if there are type (p, q) equilibria but no equilibrium discommensurations of type $(p, q)+$ is there a periodically sliding type $(p, q)+$ discommensuration? What is its basin of attraction?

   These questions are still open in the gradient case too.

4. Extend Theorem 3.2 to weakly rotationally ordered configurations with irrational mean spacing. Note this needs doing in the gradient case too, because in 1999 we found an error in our proof of the result claimed in [Baesens and MacKay 1998]: to obtain a compact forward invariant set there, we switched to product topology but then implicitly assumed that the resulting non-wandering orbit is also non-wandering in uniform topology, which is not automatic. We hope to fix this error soon.

   What is the basin of attraction of the sliding solution with irrational mean spacing if it exists?

5. Can one show that the sliding velocity $v(\rho)$ (defined to be $1/T$, with T the sliding period) is continuous at irrationals (assuming the solution to question 4 is found) and $\lim_{\rho \nearrow \frac{p}{q}\pm} v(\rho)$ exist and equal $v\left(\frac{p}{q}\pm\right)$ (assuming solution to question 3b) and $v\left(\frac{p}{q}\pm\right) \geq v\left(\frac{p}{q}\right)$ with generic strict inequality if $v\left(\frac{p}{q}\pm\right) \neq 0$? (cf numerics of [Floría and Mazo 1996]).

Appendix

Proposition 4.1. *For a tilted Frenkel-Kontorova chain, given $b > 0$ there exists $F(b)$ such that for $F > F(b)$ there are no equilibria with spacing $\sup_n |x_n - x_{n-1}| \leq b$.*

Proof. If there is an equilibrium then $h_2(x_{n-1}, x_n) + h_1(x_n, x_{n+1}) = F$. Since h is $\mathcal{C}^1$ and diagonally periodic there exist $K_1(b), K_2(b) > 0$ such that $|h_2(x_{n-1}, x_n)| \leq K_2$ and $|h_1(x_n, x_{n+1})| \leq K_1$ for sequences of spacing $\leq b$. Thus $F \leq K_1 + K_2$. Take $F(b) = K_1 + K_2$. □

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