Coupled map lattices with phase transition

G. Gielis[1]
King's College Research Centre and
Statistical Laboratory
Cambridge CB2 1ST
United Kingdom

R.S. MacKay[2]
The Nonlinear Centre
DAMTP
Cambridge CB3 9EW
United Kingdom

Abstract: Examples of coupled map lattices are presented for which we prove the existence of phase transitions, in the senses of non-unique "natural" measure or Gibbs state. They are constructed from probabilistic cellular automata.

Keywords: coupled map lattice, space–time chaos, non–trivial collective behaviour, phase transition, spatially extended dynamics, probabilistic cellular automata.
PACS codes: 05.45.+b, 05.70.Fh. AMS codes: 82C26, 58F11.

[1]e–mail: Guy.Gielis@kings.cam.ac.uk
[2]e–mail: R.S.MacKay@damtp.cam.ac.uk

1. Introduction

A big question in the theory of chaotic spatially extended dynamical systems, raised already in [Hak] and in more mathematical form by [BS], is whether there can be a phase transition. Some numerical investigations suggest that there can, e.g. [K, CM, MH], but these claims are not easy to justify mathematically. The papers [BC, JP] give rigorously analysed examples which they call phase transitions, but we feel that their examples do not justify the term, for reasons that we give below. Sakaguchi [Sak] comes much closer but his analysis did not totally convince us.

The point of our paper is to propose and discuss a class of examples in which we prove occurrence of phase transitions in what we feel is an indisputable sense (indeed, in two senses for some of our examples). They are constructed from probabilistic cellular automata.

The term 'phase transition' is used in many different ways (for a nice review, see §2.6.5 of [EFS]), so it is essential first to make precise how we will use it. Some people mean a qualitative change in the behaviour as some external parameters vary, for instance loss of symmetry, divergence of correlation length or formation of an infinite cluster. Although such phenomena can be associated with a phase transition, they would not be adequate as definitions. Indeed, already in ordinary (finite dimensional) dynamical systems, there are many well understood qualitative changes, called 'bifurcations' (e.g. [AAIS]), and we do not want to include them. Thus we think that a change in the behaviour at the topological level, as reported in [BC], does not merit to be called a phase transition.

Instead, following ideas from statistical mechanics, we want 'phase transition' (in the context of deterministic dynamical systems) to refer to some purely measure–theoretic properties: a qualitative change in the statistical behaviour of typical solutions in the basin of a single topologically mixing attractor. We insist on a single topologically mixing attractor because we do not want to include the effects of different orbits going to different attractors, nor those of going to an attractor with several cyclically permuting components but in different phases. Furthermore, we do not regard invariant measures on non–attracting sets as of central interest; this excludes the example of [JP] which is a product of horseshoes; in any case, their example of phase transition is for a interaction potential which they do not prove comes from any coupled map lattice.

Thus, following [BS] and others, we will mean by 'phase transition' the non–uniqueness of a 'relevant' type of probability measure for an indecomposable system. The idea behind this is that different typical small variations in parameters of the system are likely to select measures close to different ones of the non-unique case. In statistical mechanics it would be called a 'first order' phase transition. It could be interesting to investigate also the analogues of 'higher order' or 'continuous' phase transitions, but at least in our examples, they only appear as limit points of first order phase transitions; we concentrate on the notion of non-unique measure.

We will follow a double track by considering two different, but related notions of 'relevant' measure. Both are inspired by the concept of SRB (Sinai-Ruelle-Bowen) measure for finite dimensional systems, but they focus on different aspects of this measure. The first one focusses on its absolutely continuous character, while the second one retains the idea of describing the space–time configurations by Gibbs measures. The second definition includes the interesting phenomenon of non–trivial collective behaviour, cf. [Hak, K, CM] (Kaneko initially called it "hidden coherence"), where there is apparently a unique 'nice' invariant measure, but nevertheless non–uniqueness of 'nice' measures on the space–time configurations, because a two-cycle or more complicated dynamic behaviour occurs in time. The case of time-periodic behaviour of measures is called "asymptotic periodicity" by [LM]. For a recent paper on non-trivial collective behaviour, see [SK].

Our examples of systems exhibiting non–unique phase are of a special form: they are piecewise linear coupled map lattices, with a Markov partition, and the coupling is not of the standard form of discrete diffusion – rather the local map depends only on the symbolic state of its neighbours. Also we restrict attention to networks of one–dimensional expanding maps. Nevertheless we believe that this way of constructing examples sheds useful light on the subject of phase transition in spatially extended dynamical systems, and we hope that it will open up the way to analyse much more general examples. A first extension, for instance, might be to make examples of globally coupled (or mean field coupled) maps with non–trivial collective behaviour.

In the next section the two notions of phase transition are defined and explained in more detail. In sections 3 and 4 we introduce a class of interacting maps that show a non–unique phase in both senses as soon as the interaction is sufficiently strong. To understand these maps better, we also investigate their behaviour for small couplings and we give several examples. Finally, the proofs are collected in section 5.

2. Two notions of phase transition

As described in the introduction, we will mean by "phase transition" non-uniqueness of a probability measure from some distinguished class of measures selected by the system. Following statistical mechanics terminology, we will refer to these measures as "phases" (cf. ice, water, steam as possible phases of H_2O). So the principal aim of this section is to define an appropriate notion of phase for coupled map lattices. In fact, we define two notions of phase, each of which has its merits: "marginally absolutely continuous invariant measures" and "Gibbs measures".

First we set the scene. We assign to every site i of a network S (we will often take $S = \mathbb{Z}^d$ for some dimension d, but this restriction is not necessary), a local state space M_i, which is a copy of a compact differentiable manifold M with a metric d_M (we will often take M to be the interval $[0, 1]$ or the unit circle $\mathbb{R}/\mathbb{Z}$). This defines a global state space $\Omega = M^S$, which we endow with the supremum metric (in order to make Ω a manifold). On this space we consider piecewise $C^{1+\alpha}$ (for some $\alpha \in (0, 1]$), expanding maps T:

$$T : \Omega \longrightarrow \Omega.$$

We will not give a general definition of "expanding" in this context, but our maps have Markov partitions and the expanding condition is then that there is at most one orbit for a given space-time symbol configuration.

We propose two possible characteristics in a family of maps T that we will call a phase transition. Our approach towards the idea of a phase transition for an infinite dimensional deterministic dynamics is basically along the lines of the theory of statistical physics of particle (spin) systems on a lattice: a phase transition occurs only in the thermodynamic limit and it is expressed by the non–uniqueness of some type of probability measure. This last condition means that we consider only the analogue of first order phase transitions.

Since we focus on phenomena that do not appear in finite dimensional maps, we do not want non–unique phase to arise simply because the dynamical system has more than one attractor or an attractor with a periodic (or other) factor (like 2–band attractors for 1–dimensional maps). Therefore, we must restrict attention to an indecomposable component of the dynamics. We choose a definition of indecomposability that is roughly analogous to topological mixing but uses the metric and also takes spatial structure into account. One advantage of our definition is that it applies equally well to non–autonomous or non-spatially translation invariant systems as to autonomous translation-invariant ones.

Let

$$\boldsymbol{\Omega_T} = \{\boldsymbol{x} = (x_n)_{n\in\mathbb{Z}} : x_{n+1} = Tx_n\}$$

be the set of solutions on the space-time lattice $S \times \mathbb{Z}$, and choose a metric D on $S \times \mathbb{Z}$. For finite subsets $\boldsymbol{A}$ and $\boldsymbol{B}$ of $S \times \mathbb{Z}$, the *separation*

$$D(\boldsymbol{A}, \boldsymbol{B}) = \min\{D(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \boldsymbol{A}, \boldsymbol{b} \in \boldsymbol{B}\}.$$

Definition 1 *We say that the system (T, Ω) is indecomposable if for all $\epsilon > 0$ and $K > 0$ there exists $D_0(\epsilon, K) \in \mathbb{R}$ such that for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\Omega_T}$ and subsets $\boldsymbol{A} \subset S \times \mathbb{Z}$ and $\boldsymbol{B} \subset S \times \mathbb{Z}$ of diameters $\leq K$ and separation $D(\boldsymbol{A}, \boldsymbol{B}) \geq D_0$, there exists a $\boldsymbol{z} = (z_n)_{n\in\mathbb{Z}} \in \boldsymbol{\Omega_T}$ that satisfies*

$$d_M(x_m(i), z_m(i)) \leq \epsilon \quad \textit{for every} \quad (i, m) \in \boldsymbol{A},$$

and

$$d_M(y_m(i), z_m(i)) \leq \epsilon \quad \textit{for every} \quad (i, m) \in \boldsymbol{B}.$$

This is a spatially extended analogue of the "specification property" of dynamical systems theory (e.g. [KH]). A similar property was introduced in [BuS] to study subshifts on $\mathbb{Z}^d$. There it is called "strongly irreducible".
An invariant subset $\Omega' \subseteq \Omega$ for $T^q, q \geq 1$ is called an indecomposable part if the subsystem $(T^q|_{\Omega'}, \Omega')$ is indecomposable.
Both of the phenomena that will be dubbed "phase transition" refer to the existence of more than one relevant measure associated to an indecomposable part of T, the first notion applying to measures on Ω and the second to measures on the set of space-time solutions $\boldsymbol{\Omega_T}$. Both groups of measures are obtained by extending definitions for SRB measures introduced for finite dimensional systems. As in the finite dimensional case, one can find examples of maps for which the definitions are equivalent, but this will not be true in general.

2.1. Non-uniqueness of marginally absolutely continuous invariant measures.

The first family of measures that are considered keep the smooth character of the SRB measure: we say that a measure m on Ω is *marginally absolutely continuous* if, for any finite volume $\Lambda \subset S$, the projection m_Λ to M^Λ is absolutely continuous with respect to (any) Lebesgue measure on M^Λ. This is the physically appropriate class of measures for coupled expanding maps; for the generalisation to coupled hyperbolic maps, not considered here, one requires the measure to have absolutely continuous conditional measure along the unstable leaves when restricted to finite sets Λ. The appropriate generalisation of SRB measures to this spatially extended context is marginally absolutely continuous invariant measures. Following the physics literature, we will refer to them as *natural* measures. This generalisation of the SRB measure already appeared in several publications (see for instance [BEIJK, BK1, BK2, FR, MM]) and leads to analyses via the Perron–Frobenius operator. In several of these papers, conditions on the interaction are sought for which there is a unique natural measure. Although the only results obtained up till now deal with an even more restrictive set of measures – extra conditions on the density function are needed, like Hölder continuity or analyticity – they make it legitimate to call the complementary event, that is the existence of more than one invariant marginally absolutely continuous measure, a phase transition.

Definition 2 *We say that a map T is at a type-N phase transition*[1] *if it has more than one natural measure (i.e. invariant measure with absolutely continuous marginals) on an indecomposable part of the dynamics.*

A natural way to prove a phase transition along the lines of this definition is to push forwards different initial measures and investigate the corresponding asymptotic behaviour. As an interesting by-product this approach can yield information about the speed of relaxation towards the invariant measures.

2.2. Non-uniqueness of Gibbs measures.

The second notion of phase transition has the advantage that it detects non-trivial collective behaviour, even if there is no loss of uniqueness of natural measure. It relies on the existence of a symbolic representation for the dynamics.

Such a representation relates the evolution of continuous variables arranged in a d–dimensional network to behaviour of discrete variables in a $d+1$–dimensional grid. The idea is to study this representation via (equilibrium) Gibbs measures for the $d+1$-dimensional system.

The idea of transferring Gibbs states to the context of dynamical systems goes back to [Sin] and [Ru]. A solution of a dynamical system on a space M can be regarded as a configuration of points of M over the time axis.

[1] N stands for "natural".

Then the question is what probability distributions on the space of such configurations are relevant.
For discrete-time systems $f : M \longrightarrow M$ with a uniformly hyperbolic subset K, space–time configurations $\boldsymbol{x} = (x_n)_{n\in\mathbb{Z}}$ and a symbolic representation $\boldsymbol{\xi} = (\xi_n)_{n\in\mathbb{Z}}$, the answer is the Gibbs measures for the (formal) Hamiltonian

$$H(\boldsymbol{\xi}) = \sum_{n\in\mathbb{Z}} \log|\det Df^u(x(\boldsymbol{\xi})_n)|, \tag{1}$$

where Df^u is the derivative restricted to the unstable subspace.
To define such a Gibbs measure, one first shows that there exists a family of real–valued functions $(\Phi_{\boldsymbol{A}})_{\boldsymbol{A}\subset\mathbb{Z}}$, the interaction potential, in which for every $\boldsymbol{A}$, $\Phi_{\boldsymbol{A}}$ only depends on $\boldsymbol{\xi}_{\boldsymbol{A}} = (\xi_n)_{n\in\boldsymbol{A}}$ and for all configurations $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$, verifying $\boldsymbol{\xi} = \boldsymbol{\xi}'$ on $\boldsymbol{\Lambda}^c = \mathbb{Z}\setminus\boldsymbol{\Lambda}$, $\boldsymbol{\Lambda}$ finite,

$$\begin{aligned}\sum_{n\in\mathbb{Z}}(\log|\det Df^u(x(\boldsymbol{\xi})_n)| - \log|\det Df^u(x(\boldsymbol{\xi}')_n)|) \\ = H_{\boldsymbol{\Lambda}}(\boldsymbol{\xi}) - H_{\boldsymbol{\Lambda}}(\boldsymbol{\xi}')\end{aligned} \tag{2}$$

with

$$H_{\boldsymbol{\Lambda}}(\boldsymbol{\xi}) = \sum_{\boldsymbol{A}:\boldsymbol{A}\cap\boldsymbol{\Lambda}\neq\emptyset} \Phi_{\boldsymbol{A}}(\boldsymbol{\xi}). \tag{3}$$

A probability measure $\boldsymbol{\mu}$ on the space of symbolic configurations is called a Gibbs measure for the (formal) Hamiltonian (1), if the conditional probability to find a configuration $\boldsymbol{\xi}$, given that the configuration is $\boldsymbol{\eta}$ off $\boldsymbol{\Lambda}$, is

$$\boldsymbol{\mu}(\boldsymbol{\xi}|\boldsymbol{\eta} \text{ on } \boldsymbol{\Lambda}^c) = Z(\boldsymbol{\Lambda},\boldsymbol{\eta})^{-1}\exp(-H_{\boldsymbol{\Lambda}}(\boldsymbol{\xi}))I\{\xi_n = \eta_n, n\in\boldsymbol{\Lambda}^c\}. \tag{4}$$

$Z(\boldsymbol{\Lambda},\boldsymbol{\eta})$ is the normalisation factor and $I\{\cdot\}$ is the indicator function that gives one when its argument is true and zero otherwise.
Notice that the interaction potential is not unique. What is important is that in the case of uniform hyperbolicity there exist a version in which $\max_{\boldsymbol{\xi}}|\Phi_{\boldsymbol{A}}(\boldsymbol{\xi})|$ decreases exponentially as a function of $\max\{|m-n| : m,n\in\boldsymbol{A}\}$. Since the time–axis is one dimensional, this turns out to be sufficient to prove that there is precisely one Gibbs measure for each topologically mixing component of K. The marginal probability distribution of x_n for a single time n is an invariant measure for f, which in the case of an attracting topological mixing component describes the long–time distribution of Lebesgue-typical orbits in its basin of attraction. It is called the SRB measure on the attractor (after Sinai, Ruelle and Bowen).

For periodic components of K, the situation is slightly different; call the period p. Then there are p extremal Gibbs measures, corresponding to the p possible phases of cycling through the p subcomponents, and their convex combinations. Averaging the marginals over one period p gives a unique natural measure on the attractor, which is again called SRB.

For spatially extended dynamical systems the notion of SRB measure is generalised in this direction; the study of the unique invariant measure (among a family of 'good' measures) often reduces to investigating the unique Gibbs measure for a relevant potential. This was first done in [BS] and later extended in [BK2, JP]. Here we employ this idea the other way around, saying that the map T shows a phase transition as soon as there is more than one Gibbs measure with respect to this potential.

Although we feel that this is a natural approach, it has the defect that the definition of Gibbs measures relies on the existence of a symbolic description. A more fundamental point of view, avoiding this shortcoming, could be to extend the notion of equilibrium measures (a measure μ that realises the supremum of $h_\mu - \mu(\log|\det Df^u|)$ over invariant probability measures (see [KH], for instance), and ask for the existence of more than one equilibrium measure, but we did not yet pursue this approach. In this paper we restrict ourselves to assuming just those properties for the map T that allow us to formulate the notions we need and construct examples of phase transition, and we do not engage in the more general discussion about conditions on the dynamical law that ensure these properties.

First of all we assume the existence of symbolic dynamics, that is a semi–conjugacy Π to the map T on Ω from a subshift τ on a symbolic configuration space $\boldsymbol{\Omega^s_T}$, i.e. for $\boldsymbol{\xi} \in \boldsymbol{\Omega^s_T}$:

$$\Pi(\tau\boldsymbol{\xi}) = T\Pi(\boldsymbol{\xi}).$$

This means that there is a Markov partition $(\Omega_\xi)_{\xi\in\Omega^s}$, with index set $\Omega^s = A^S$ for some finite alphabet A, e.g. $\{1,\dots,N\}$, that allows us to assign to every path $\boldsymbol{x} = (x_n)_{n\in\mathbb{Z}} \in \boldsymbol{\Omega_T}$ a code in $\boldsymbol{\xi} = (\xi_n)_{n\in\mathbb{Z}}$ such that $T^n x_m \in \Omega_{\xi_{m+n}}$ for all $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. $\boldsymbol{\Omega^s_T}$ is the set of all such codes and Π maps a code to a corresponding space–time state in $\boldsymbol{\Omega_T}$. The code can be ambiguous at the boundaries of partition elements, which is why we allow Π to be a semi-conjugacy rather than requiring a full conjugacy.

Consider now for each $(i,n) \in S \times \mathbb{Z}$ the following function on $\boldsymbol{\Omega^s_T}$:

$$H_{(i,n)}(\boldsymbol{\xi}) = \mathrm{Tr}(\log[DT|_{E^u}])_{i,i}(\Pi(\boldsymbol{\xi}^{\boldsymbol{n}})), \tag{5}$$

with $\boldsymbol{\xi}^{\boldsymbol{n}} = (\xi_{m+n})_{m\in\mathbb{Z}}$.
For expanding maps, the unstable subspace E^u is the whole tangent space $T\Omega$, so we shall drop it from further discussion. In this expression, DT is the derivative on $T\Omega$ and, when invertible, its logarithm is to be interpreted by continuation with respect to a parameter s from a reference case $s = 0$, using

$$\frac{d}{ds}\log DT_s = DT_s^{-1}\frac{d}{ds}DT_s.$$

Uncoupled maps $T = \times_i T_i$ provide convenient reference cases: for these the definition of $\log DT$ reduces to that for its diagonal blocks $\log DT_i$. If the local maps are 1–dimensional then $\log DT_i$ can be defined to be just the number $\log |DT_i|$. If the local maps are higher dimensional, one way to define $\log DT_i$ is to take its singular value decomposition UDV with U, V orthogonal (with respect to some inner product) and $D = diag(d_j)$ positive diagonal and let $\log DT_i = ULV$, where $L = diag(\log d_j)$. The trace is independent of any choices. The notation $(\log[DT])_{i,i}$ denotes the diagonal block corresponding to site $i \in S$. For coupled 1–dimensional maps the blocks are 1–dimensional and so the interaction function is simply the diagonal entry. The connection between (1) and (5) is that $\log\det = \mathrm{Tr}\log$.

Analogous to (1) we refer to the (not necessarily convergent) sum

$$H = \sum_{(i,n)\in S\times\mathbb{Z}} H_{(i,n)} \tag{6}$$

as the (formal) Hamiltonian for the map T. We also define Gibbs measure along the lines of (2), (3) and (4), replacing $\mathbb{Z}$ by $S \times \mathbb{Z}$.
This idea appears already in [BK2] and [JP] and was independently proposed by MacKay in February 1995 ([Ma]). The only difference from his proposal is that [BK2, JP] go further, writing $DT = B(I - W)$ where B is the uncoupled case, and then expanding $\log DT$ as $\log B - \sum_{k=1}^{\infty} W^k/k$. We refer to [BK2] for an example of how a localisation procedure can be used to derive an interaction $(\Phi_{\boldsymbol{A}})_{\boldsymbol{A}\subset S\times\mathbb{Z}}$ from (6).

Definition 3 *We say that a map T is at a type-G phase transition*[2] *if there is more than one Gibbs measure for the (formal) Hamiltonian*

$$H = \sum_{(i,n)\in S\times\mathbb{Z}} H_{(i,n)} \tag{7}$$

[2] G stands for "Gibbs".

defined on the symbolic configuration space $\boldsymbol{\Omega^s_T} = \boldsymbol{\Omega^s_T}(\Omega)$ *corresponding to an indecomposable part* Ω *of the dynamics.*

Remark The Hamiltonian (6) is somewhat unusual for statistical mechanics because even in the uncoupled case it usually has non-trivial interaction in the time direction. This makes the weak coupling analysis already non-trivial, as discussed by [BK2, JP].

3. A deterministic and a stochastic dynamical system

In this section we present a large family of maps that, depending on the values of its parameters, does or does not show a type–N or type–G phase transition. The deterministic maps are constructed such that their action on marginally absolutely continuous measures is naturally described using a particular sort of stochastic dynamics: probabilistic cellular automata (PCA). The key idea is to consider maps for which the supporting PCA is well (enough) understood and to relate the phase diagrams of the two systems. We find that the number of invariant measures for the PCA is not larger than the number of natural measures for the map, i.e. a non-unique invariant measure for the stochastic evolution implies a type–N phase transition. For Gibbs measures the situation is similar: the space–time configurations of the PCA are Gibbs for the Hamiltonian defined by (5).

To properly define the maps (paragraph 3.1) and the supporting PCA (paragraph 3.2) and to state the results in section 4, we first give some more notation.

From now on, we set $S = \mathbb{Z}^d$ for some positive integer d. We take as state space $\Omega = [0,1]^{\mathbb{Z}^d}$ by assigning a continuous variable $x(i) \in [0,1]$ to every site i of the regular lattice $\mathbb{Z}^d$. Similarly, we obtain the symbol configuration space $\Omega^s = \{-1,+1\}^{\mathbb{Z}^d}$ when we attach a discrete variable $\sigma(i) \in \{-1,+1\}$ to each site $i \in \mathbb{Z}^d$. For any finite $\Lambda \subset \mathbb{Z}^d$ and every $x \in \Omega$ we set $x_\Lambda = (x(i))_{i\in\Lambda}$ and $\Omega_\Lambda = \{x_\Lambda : x \in \Omega\}$. The discrete analogues are $\sigma_\Lambda = (\sigma(i))_{i\in\Lambda}$ and $\Omega^s_\Lambda = \{\sigma_\Lambda : \sigma \in \Omega^s\}$. By Λ^c, we mean the set $\mathbb{Z}^d \setminus \Lambda$. $|\Lambda|$ is the number of sites in Λ.

For every two configurations $\sigma \in \Omega^s$ and $\eta \in \Omega^s$, $\sigma_\Lambda \times \eta_{\Lambda^c}$ is the configuration that coincides with σ inside Λ and with η outside.

The map $\pi_0 : [0,1] \longrightarrow \{-1,+1\}$ discretizes the interval $[0,1]$:

$$\begin{aligned} \pi_0 z &= -1 \quad \text{if} \quad z \in [0, \tfrac{1}{2}] \\ &= +1 \quad \text{if} \quad z \in (\tfrac{1}{2}, 1]. \end{aligned}$$

Using this map, we obtain for every state $x \in \Omega$ a configuration $\pi_0 x \in \Omega^s$ by setting $\pi_0 x(i) = \pi_0[x(i)]$ at every site i of the lattice.
For a bounded function f on Ω, $\|f\|$ is the supremum norm and Λ_f denotes the smallest set of sites in $\mathbb{Z}^d$ such that f is independent of the state in Λ_f^c. If this set is finite, then f is called local.
We endow both Ω and Ω^s with the product topology. This is different from what we required in the previous section where we assumed the uniform topology for Ω. That was to ensure that Ω is a manifold on which the function (5) is properly defined. In this section we will focus on the statistical properties of the maps and these will be described in the product topology. This is not a contradiction; the definition of (5) on the one hand and its application to describe measures on Ω^s on the other hand, are two separate issues.
Finally, we denote by $\mathcal{P}(\Omega)$ the set of marginally absolutely continuous Borel probability measures on Ω (as defined in section 2.1) and by $\mathcal{P}(\Omega^s)$ the set of all Borel probability measures on Ω^s.

3.1. A system of coupled maps

For every $\eta \in \Omega^s$, we consider two numbers, $0 < p(-1|\eta) < 1$ and $p(+1|\eta) = 1 - p(-1|\eta)$. Starting from these numbers we introduce a map

$$T_0(\cdot, \eta) : z \in [0,1] \longrightarrow T_0(z, \eta) \in [0,1] \tag{8}$$

as follows:

$$\begin{aligned} T_0(z,\eta) \; &= [p(-1|\eta^{0,-1})]^{-1} z \\ &\qquad\qquad \text{if } \; 0 \le z \le \tfrac{1}{2} p(-1|\eta^{0,-1}), \\ &= \tfrac{1}{2} + [p(+1|\eta^{0,-1})]^{-1} (z - \tfrac{1}{2} p(-1|\eta^{0,-1})) \\ &\qquad\qquad \text{if } \; \tfrac{1}{2} p(-1|\eta^{0,-1}) \le z \le \tfrac{1}{2}, \\ &= [p(-1|\eta^{0,+1})]^{-1} (z - \tfrac{1}{2}) \\ &\qquad\qquad \text{if } \; \tfrac{1}{2} < z \le \tfrac{1}{2} + \tfrac{1}{2} p(-1|\eta^{0,+1}), \\ &= \tfrac{1}{2} + [p(+1|\eta^{0,+1})]^{-1} (z - \tfrac{1}{2} - \tfrac{1}{2} p(-1|\eta^{0,+1})) \\ &\qquad\qquad \text{if } \; \tfrac{1}{2} + \tfrac{1}{2} p(-1|\eta^{0,+1}) \le z \le 1. \end{aligned} \tag{9}$$

Here, the configuration $\eta^{k,a} \in \Omega^s$, for $k \in \mathbb{Z}^d, a \in \{-1,+1\}$, is given by

$$\begin{aligned} \eta^{k,a}(l) \; &= \eta(l) \qquad \text{if } \; l \neq k, \\ &= a \qquad\quad\; \text{if } \; l = k. \end{aligned} \tag{10}$$

This means that $T_0(\cdot, \eta)$ is independent of $\eta(0)$. Moreover, we will assume that the maps are completely determined by the values $\eta(\alpha e_i)$, with $e_i, i =$

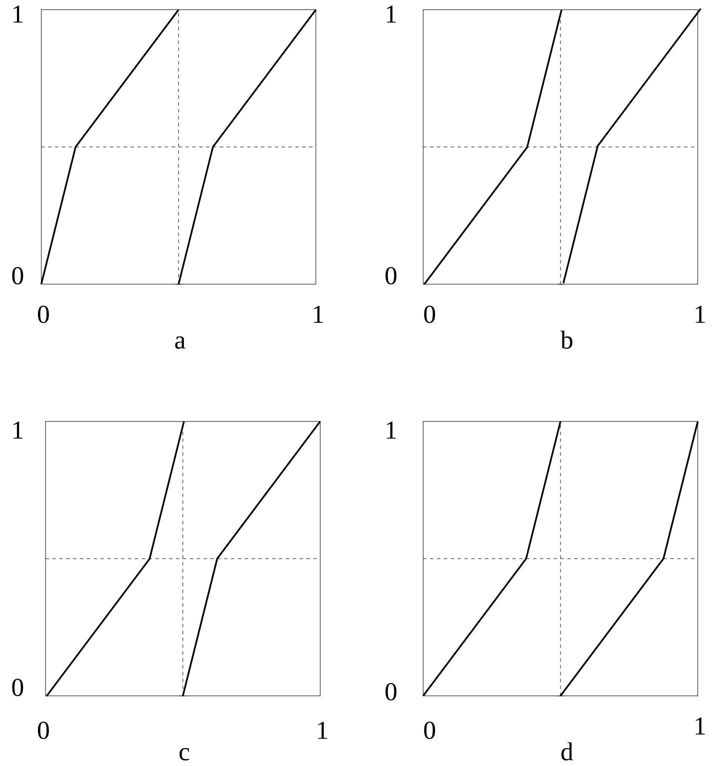

Figure 1: The map $T_0(\cdot,\eta)$ corresponding to the PCA of Toom for $\epsilon = 1/2$ when: a. $\eta(e_1) = \eta(e_2) = +1$, b. $\eta(e_1) = +1, \eta(e_2) = -1$, c. $\eta(e_1) = -1, \eta(e_2) = +1$, d. $\eta(e_1) = \eta(e_2) = -1$. Toom's PCA will be introduced as Example 1 of Section 4.

$1, \ldots, d$ the unit vectors and $\alpha = \pm 1$. Consequently there are no more than 2^{2d} different maps.
An example is sketched in Figure 1.

Using this single site map we define, for a finite $\Lambda \subset \mathbb{Z}^d$, the map

$$T_\Lambda(\cdot,\eta) : x \in \Omega_\Lambda \longrightarrow T_\Lambda(x,\eta) \in \Omega_\Lambda$$

by saying that for each $i \in \Lambda$

$$T_\Lambda(x,\eta)(i) = T_i(x(i),\eta)$$

with, for $z \in [0,1]$,

$$T_i(z,\eta) = T_0(z,\tau_i\eta).$$

Here, $\tau_i \eta$ is the translated configuration given by $\tau_i \eta(k) = \eta(k+i)$.
Introducing boundary conditions ζ, we first set for $x \in \Omega_\Lambda, \zeta \in \Omega^s$,

$$T_{\Lambda,\zeta} x = T_\Lambda(x, \pi_0 x_\Lambda \times \zeta_{\Lambda^c}),$$

and then define the infinite volume map T as the following limit; for every state $x \in \Omega$ and $i \in \mathbb{Z}^d$

$$Tx(i) = \lim_{\Lambda \uparrow \mathbb{Z}^d} T_{\Lambda,\zeta} x_\Lambda(i).$$

As there is only a coupling between neighbouring sites, the resulting map is independent of the choice of ζ.
Here, and also later on, $\lim_{\Lambda \uparrow \mathbb{Z}^d} F(\Lambda)$ denotes the limit $\lim_{n \uparrow \infty} F(\Lambda_n)$ for the cubes $\Lambda_n = \{k = (k_1, \ldots, k_d) \in \mathbb{Z}^d : \max_i |k_i| \leq n\}, n = 1, 2, \ldots$. The particular meaning of $F = F(\Lambda)$ can change from case to case.

Next, we introduce the inverse branches

$$T_\Lambda^{-1}(\cdot, \eta) : x \in \Omega_\Lambda \longrightarrow T_\Lambda^{-1}(x, \eta) \in \Omega_\Lambda$$

by saying that $y = T_\Lambda^{-1}(x, \eta)$ if $x(i) = T_i(y(i), \eta)$ and $\pi_0 y(i) = \eta(i)$ for each $i \in \Lambda$. As a consequence, $\pi_0[T_\Lambda^{-1}(x, \eta)(i)] = \eta(i)$ for all $i \in \Lambda$, independently of $x \in \Omega_\Lambda$.

Finally, we say that a measure $m \in \mathcal{P}(\Omega)$ is *invariant* for the map T if for every bounded local and measurable function f on Ω

$$\int_\Omega m(dx) f(Tx) = \int_\Omega m(dx) f(x). \tag{11}$$

Remark It is interesting to notice that when

$$\max_{\eta, \eta' : \eta(0) = \eta'(0)} |p(+1|\eta) - p(+1|\eta')| \tag{12}$$

is small, the interaction between neighbouring sites decreases: the state $y(i) = Tz(i)$ on a site $i \in \mathbb{Z}^d$ hardly differs for different values $z(j)$ at $j, j \neq i$. On the other hand, when (12) is close to 1, the state on neighbouring sites might strongly influence the evolution. Regardless of the strength of the coupling, however, $y(i) = Tz(i)$ is only a function of $z(j), j \neq i$ through $\pi_0 z(j)$, making this a less sensitive interaction than the diffusive coupling often considered in coupled map lattices, in particular rendering T discontinuous.

3.2. Probabilistic cellular automata

To study the above deterministic map, we introduce a stochastic dynamics in the form of a probabilistic cellular automaton (PCA). We call this the *supporting* PCA.

A PCA is a parallel update discrete time Markov process $\sigma_n, n = 0, 1, \dots$ on Ω^s that is defined by a family of transition probabilities $(p(\alpha|\eta))_{\alpha=\pm1,\eta\in\Omega^s}$. These are such that for any finite volume $\Lambda \subset \mathbb{Z}^d$, the probability $p(\sigma(k), k \in \Lambda|\eta)$ to find an updated configuration σ on Λ given that the previous configuration was $\eta \in \Omega^s$, is simply the following product

$$p(\sigma(k), k \in \Lambda|\eta) = \prod_{k\in\Lambda} p(\sigma(k)|\eta). \tag{13}$$

In this paper we will assume that the probability $p(\sigma(i)|\eta)$ to find a configuration $\sigma(i)$ at the site $i = (i_1, \dots, i_d)$ only depends on $\eta(j)$ when $\sum_k |i_k - j_k| = 1$. Imposing boundary condition $\zeta \in \Omega^s$ on Λ^c, we define the one step transition probability

$$P_{\Lambda,\zeta}(\sigma|\eta) = \prod_{k\in\Lambda} p(\sigma(k)|\eta_\Lambda \times \zeta_{\Lambda^c})$$

and, by induction, the n–step transition probability

$$P^n_{\Lambda,\zeta}(\sigma|\eta) = \sum_{\xi\in\Omega_\Lambda} P^{n-1}_{\Lambda,\zeta}(\sigma|\xi) P_{\Lambda,\zeta}(\xi|\eta).$$

Finally, we denote by $P^n(\cdot|\eta)$ the measure on Ω^s that is obtained as the (weak) limit $\lim_{\Lambda\uparrow\mathbb{Z}^d} P^n_{\Lambda,\zeta}(\cdot|\eta)$ for any ζ, *i.e.*

$$P^n(\sigma(i), i \in \tilde\Lambda|\eta) = \lim_{\Lambda\uparrow\mathbb{Z}^d} P^n_{\Lambda,\zeta}(\sigma(i), i \in \tilde\Lambda|\eta)$$

for every finite $\tilde\Lambda \subset \mathbb{Z}^d$.

A measure $\mu \in \mathcal{P}(\Omega^s)$ is called *invariant* for the PCA if for every finite $\Lambda \subset \mathbb{Z}^d$,

$$\int_{\Omega^s} \mu(d\xi) P(\sigma(i), i \in \Lambda|\xi) = \mu(\sigma(i), i \in \Lambda). \tag{14}$$

The PCA is called *ergodic* if it has a unique invariant measure μ and for all $\nu \in \mathcal{P}(\Omega^s)$,

$$\int_{\Omega^s} \nu(d\xi) P^n(\sigma(i), i \in \Lambda|\xi) \longrightarrow \mu(\sigma(i), i \in \Lambda) \quad \text{as} \quad n \uparrow \infty. \tag{15}$$

Remark One should not confuse this use of the term "ergodic", which is conventional in the context of stochastic systems, with its use in deterministic dynamical systems, where it means weak Cesaro convergence. Property (15) is closer to the dynamical systems notion of "mixing".
Finally, we call the dynamics *exponentially ergodic* if there exist constants $\lambda > 0$ and $C < \infty$ so that

$$\sup_{\sigma,\eta} |P^n(\sigma(k), k \in \Lambda|\eta) - \mu(\sigma(k), k \in \Lambda)| \leq C|\Lambda|\mathrm{e}^{-\lambda n}.$$

4. Results and discussion

For every measure $\mu \in \mathcal{P}(\Omega^s)$ we define the measure $m^\mu \in \mathcal{P}(\Omega)$ as the (only) marginally absolutely continuous Borel probability measure with marginals m^μ_Λ that, for finite subsets $\Lambda \subset \mathbb{Z}^d$, are described by the density functions

$$h_\Lambda(x) = 2^{|\Lambda|} \sum_{\xi \in \Omega^s_\Lambda} \mu_\Lambda(\xi) I\{\pi_0 x_\Lambda = \xi\}. \tag{16}$$

Theorem 1 *If $\mu \in \mathcal{P}(\Omega^s)$ is invariant for the supporting PCA, then $m^\mu \in \mathcal{P}(\Omega)$ is invariant for the map T.*

This almost immediately implies the following result.

Theorem 2 *If there is more than one invariant measure for the supporting PCA, then the map T has both non-unique natural measure and non-unique Gibbs measure.*

Remark For the maps considered here, (5) simply becomes

$$H_{(i,n)}(\boldsymbol{\xi}) = -\log p(\xi_{n+1}(i)|\xi_n).$$

One may wonder whether the reverse of Theorem 2 is also true; does the uniqueness of the invariant measure for the PCA imply that there is only one Gibbs measure and only one natural measure for the deterministic dynamics? For natural measures, we can prove such a result only if we limit the set of measures to those for which there exist constants $\delta > 0$ and $C < \infty$ so that for every finite $\Lambda \subset \mathbb{Z}^d$ and all states $x \in \Omega$ and $y \in \Omega$ verifying $\pi_0 x = \pi_0 y$, the density function $h_\Lambda = \frac{dm_\Lambda}{dx_\Lambda}$ satisfies:

$$|\log h_\Lambda(x) - \log h_\Lambda(y)| \leq C \sum_{i \in \Lambda} [|x(i) - y(i)|]^\delta. \tag{17}$$

Let $\mathcal{P}^*(\Omega)$ be the set of all such measures. It is interesting to note that for each measure $\mu \in \mathcal{P}(\Omega^s)$, $m^\mu \in \mathcal{P}^*(\Omega)$. This means that the presence or the absence of a phase transition is a mere consequence of the strength of the interaction.

Theorem 3 *Suppose that the supporting PCA is ergodic and let be μ the unique invariant measure. Then m^μ is the unique natural measure for the map T in $\mathcal{P}^*(\Omega)$.*
Moreover, when the supporting PCA is exponentially ergodic, then there exist constants $\lambda = \lambda(\delta) > 0$ and $C = C(\lambda) < \infty$ such that for every measurable bounded local function f,

$$|\int_\Omega m(dx) f(T^n x) - \int_\Omega m^\mu(dx) f(x)| \leq C|\Lambda_f| \|f\| e^{-\lambda n}, \tag{18}$$

uniformly in $m \in \mathcal{P}^*(\Omega)$.

For Gibbs measures, we can at present give only a weaker result, that only holds for space–translation invariant systems.

Theorem 4 *If the supporting PCA is ergodic and space-translation invariant, then there is only one space–time translation invariant Gibbs measure for the interaction (5), and there are no periodic Gibbs measures other than this.*

Recall [GKLM] that a Gibbs measure on $\mathbb{Z}^{d+1}$ is periodic if it is invariant under a subgroup of $\mathbb{Z}^{d+1}$ of finite index.

Example 1. The PCA of Toom on $\mathbb{Z}^2$
Let $0 < \epsilon < 1$. The transition probabilities of Toom's PCA are given by the following perturbation of a majority rule

$$\begin{aligned} p(\sigma(0) = +1|\eta) \quad &= 1 - \epsilon/2 \qquad \text{if} \quad \operatorname{sgn}(\eta(0) + \eta(e_1) + \eta(e_2)) = +1 \\ &= \epsilon/2 \qquad\qquad \text{otherwise.} \end{aligned} \tag{19}$$

Here, ϵ is the noise of the system and e_1 and e_2 are the unit vectors in $\mathbb{Z}^2$. An important feature of this dynamics is the qualitative difference in behaviour for different levels of the noise. Indeed, when ϵ is near to 1, there is only one invariant measure for the dynamics, but as soon as ϵ is small, there are at least two measures in $\mathcal{P}(\Omega^s)$ that are not affected by the dynamics. Since these properties of the Toom model play a key role in our arguments, we formulate them more precisely in Theorem 5. For further details and for the corresponding proofs we refer to [LMS] and [To].

Theorem 5
i. When $0 < \epsilon < 1$ *is sufficiently large, the PCA of Toom is exponentially ergodic.*
ii. Let η^+ *be the all* $+1$ *configuration, i.e.* $\eta^+(k) = +1, k \in \mathbb{Z}^2$. *When* $0 < \epsilon < 1$ *is sufficiently small, there exists a constant* $c^+ = c^+(\epsilon) > \frac{1}{2}$ *so that*

$$P^n(\sigma(0) = +1|\eta^+) > c^+,$$

uniformly in n.
iii. In the limit $n \uparrow \infty$, *ii. implies that there is a measure* $\mu^+ \in \mathcal{P}(\Omega^s)$ *which is invariant for the dynamics and*

$$\mu^+(\sigma(0) = +1) > c^+ \tag{20}$$

for the same constant c^+ *as in ii.*
iv. A similar result holds when we start the dynamics in the all -1 *configuration. This yields an invariant measure* μ^- *for which*

$$\mu^-(\sigma(0) = +1) < c^-,$$

for $c^- = c^-(\epsilon) = 1 - c^+(\epsilon) < \frac{1}{2}$.

The maps that are derived from Toom's PCA were shown in Figure 1. A large noise parameter ϵ implies that all the slopes are near to 2. Theorem 3 and Theorem 5 *i.* imply that in this case there is both a unique natural measure and a unique translation–invariant Gibbs measure. Any other marginally absolutely continuous measure in $\mathcal{P}^*(\Omega)$ is pushed forward exponentially fast to the natural one.
The second part of Theorem 5 ensures that for all low enough values of the parameter ϵ, the map shows both a type–N and type–G phase transition.

Example 2: Toom with an anti–majority rule.
If instead of the perturbed majority rule we choose the following perturbation of an anti–majority rule:

$$\begin{aligned} p(\sigma(0) = -1|\eta) &= 1 - \epsilon/2 \quad \text{if } \operatorname{sgn}(\eta(0) + \eta(e_1) + \eta(e_2)) = +1 \\ &= \epsilon/2 \quad \text{otherwise.} \end{aligned} \tag{21}$$

then we obtain a PCA equivalent to the Toom case, up to recoding the symbol configurations at odd times. Applying this recoding to the different

Gibbs measures of Toom, we find that for ϵ small enough, the coupled map lattice (CML) is at a type-G phase transition. The two Gibbs measures that we construct for the Toom model, however (see the proof of Theorem 2), give rise to two Gibbs measures for the map (21) that have the same projection onto space. Thus this CML gives a rigorous example of non-trivial collective behaviour. Note that we are not able to prove that there is not also a type-N phase transition, because Theorem 5 does not specify the full list of Gibbs states for ϵ small.

Example 3: The Kinetic Ising model.
Denote by $\mathbb{Z}^d_{\rm even}$ the set of sites $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$ for which $\sum_{k=1}^d |i_k|$ is even. $\mathbb{Z}^d_{\rm odd}$ contains the sites for which this sum is odd. We consider the following non-autonomous rule. On even time steps only the variables at the sites of the sublattice $\mathbb{Z}^d_{\rm even}$ are updated and this happens according to the following rule

$$p(\sigma_{n+1}(i)|\sigma_n) = \frac{1}{2}[1 + \sigma_{n+1}(i) \tanh(J \sum_{j:j\sim i} \sigma_n(j))], \tag{22}$$

for some positive coupling $J > 0$. The sum is over all neighbouring sites j of i, i.e. those sites j that verify $\sum_{k=1}^d |i_k - j_k| = 1$. The remaining variables are just copied. On odd time steps, the same rule is applied, but for the variables on sites in $\mathbb{Z}^d_{\rm odd}$.
The (equilibrium) Gibbs measures with respect to the d–dimensional Ising interaction $\Phi(\sigma) = (-J\sigma(i)\sigma(j))_{i\sim j}, \sigma \in \Omega^s$ are unaffected by this dynamical rule. (See for instance [LMS].) This implies that for dimensions $d \geq 2$ there are at least two invariant measures for the PCA as soon as $J > 0$ is large enough. To avoid misunderstanding, we underline that the (spatial) equilibrium measure for the d–dimensional Ising model should not be mixed up with the Gibbs measures that describe the symbolic space–time configurations on the $d + 1$ dimensional space–time lattice.
Apart from some slight modifications (due to the non-autonomous character), the same conclusions hold as for the Toom model. In [Sak], Sakaguchi already proposed a CML that generated symbolic configurations that could be described by this PCA.

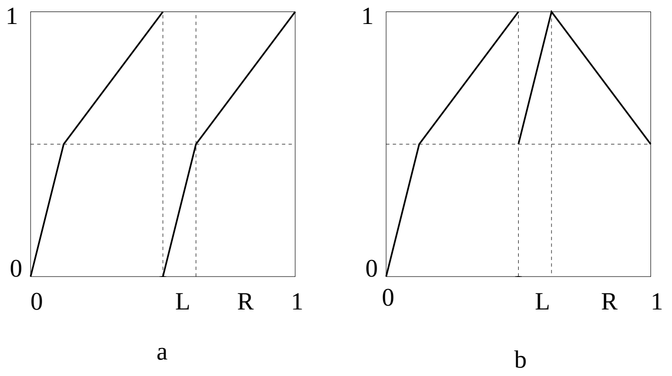

Figure 2: The maps $T_0(\cdot,\eta)$ derived from Stavskaya's PCA for $\epsilon = 3/4$ after we refined the interval [1/2,1] when; a. $\eta(1) = -1$, b. $\eta(1) = +1$.

Example 4: Stavskaya.
Let $0 < \epsilon < 1$, then Stavskaya's PCA defines a stochastic dynamics on $\{-1,+1\}^{\mathbb{Z}}$ via the following rule:

$$
\begin{aligned}
p(\sigma_{n+1}(i) = +1|\sigma_n) \; &= 1 \quad \text{if} \quad \sigma_n(i) = \sigma_n(i+1) = +1 \\
&= \epsilon \quad \text{otherwise.}
\end{aligned} \tag{23}
$$

When ϵ is near 1 the only invariant measure is concentrated on the all +1 configuration, but for sufficiently small values of ϵ there is also an invariant measure that allows configurations with a positive density of -1 [To].
This PCA does not meet the conditions we have set so far; the forbidden transition, from two +1 parents to a -1 offspring, gives rise to a one site map which is no longer expanding. Therefore we refine the symbol code +1 into two symbols: L for the interval $[\frac{1}{2}, 1-\frac{\epsilon}{2}]$ and R for $[1-\frac{\epsilon}{2}, 1]$, and consider the maps shown in Figure 2. These maps are expanding and the marginally absolutely continuous measures (16) that are obtained from the invariant measures of Stavskaya are invariant. Since the corresponding CML is indecomposable, there is both a type–N and type–G phase transition. Notice that several elements in $\{-1, L, R\}^{\mathbb{Z}^2}$ are not in $\boldsymbol{\Omega}_{\boldsymbol{T}}^{\boldsymbol{s}}$. The proofs in the next section do not immediately apply to the case of Stavskaya, but can all be modified to verify the above statements. This is an interesting example because non-uniqueness of its phase does not arise by symmetry breaking.

5. Proofs

To illustrate the effect of applying the maps and the idea behind the proofs, especially the proofs of Proposition 2, of Theorem 1 and of Theorem 3, we first consider an easier system that only consists of one map. The underlying stochastic process is not a PCA, but a simple Markov Chain with only two possible states, -1 and $+1$. The transition between those states is described by the probabilities $p(a|b) = p(b|a) < 1$ for $a = \pm 1, b = \pm 1$. The stationary distribution μ assigns equal weight, $i.e.1/2$, to each of the states. The corresponding map T resembles Figure 1.b., but with slopes $[p(-1|+1)]^{-1}$ and $[p(+1|+1)]^{-1}$.

In a natural way, this map defines the following series of partitions $\mathcal{P}^{(n)}, n = 0, 1, \ldots$ of the interval $[0,1]$. For $n = 0$, the $\mathcal{P}^{(0)}$ consists of the sets $I^0_{-1} = [0, 1/2]$ and $I^0_{+1} = (1/2, 1]$. For $n \geq 1$, there are 2^{n+1} disjoint sets $I^n_{a_0,\ldots,a_n}$ in $\mathcal{P}^{(n)}$, which are indexed by $a_i = \pm 1, i = 0, \ldots, n$.
These sets satisfy

$$I^n_{a_0,\ldots,a_{n-1},-1} \cup I^n_{a_0,\ldots,a_{n-1},+1} = I^{n-1}_{a_0,\ldots,a_{n-1}},$$

and $T^n z \in I^0_{a_n}$ as soon as $z \in I^n_{a_0,a_1\ldots,a_n}$.
This means that the sets of $\mathcal{P}^{(n)}$ are obtained by dividing each interval $I^{n-1}_{a_0,\ldots,a_{n-1}} \in \mathcal{P}^{(n-1)}$ in two parts. The first piece, $I^n_{a_0,\ldots,a_{n-1},-1}$ will be mapped on I^0_{-1} after applying T n times, while T^n maps the second part, $I^n_{a_0,\ldots,a_{n-1},+1}$, onto I^0_{+1}.

Denoting the length of the interval I by $l(I)$, we easily see from the picture of the map that

$$l(I^{n+1}_{a_0,\ldots,a_{n+1}}) = p(a_{n+1}|a_n) l(I^n_{a_0,\ldots,a_n}) \tag{24}$$

for all choices of $a_i, i = 0, \ldots, n+1$.
We use these partitions to describe the outcome of applying the map to a continuous probability distribution m with probability density $h(z), z \in [0,1]$. For instance, the probability that $T^n z$ is in I^0_{-1} is given by

$$\sum_{a_0,\ldots,a_n : a_n = -1} \int dz h(z) I\{z \in I^n_{a_0,\ldots,a_{n-1},-1}\}.$$

This observation is interesting in particular when we combine it with (24) and then apply it to a density function $h(z)$ that is constant on all the intervals $I^n_{a_0,a_1\ldots,a_n}$. Suppose that $h(z) = h^n_{a_0,a_1\ldots,a_n}$ when $z \in I^n_{a_0,a_1\ldots,a_n}$ for

2^{n+1} constants $h^n_{a_0,a_1\dots,a_n}$, chosen such that h is a probability density. If we compute again the above probability, but now for $n+1$ steps of the dynamics, we obtain

$$\begin{aligned}\sum_{a_0,\dots,a_n} \int dz h(z) I\{z \in I^{n+1}_{a_0,\dots,a_n,-1}\} \\ = \sum_{b=\pm1} \sum_{a_0,\dots,a_{n-1}} h^n_{a_0,\dots,a_{n-1},b} l(I^{n+1}_{a_0,\dots,a_{n-1},b,-1}) \\ = \sum_{b=\pm1} p(-1|b) \sum_{a_0,\dots,a_{n-1}} h^n_{a_0,\dots,a_{n-1},b} l(I^n_{a_0,\dots,a_{n-1},b}).\end{aligned}$$

In other words, as soon as $h(z)$ is piecewise constant on the different intervals of one of the above partitions determined by the map, the statistical properties of the system at later time steps can be described by the supporting Markov Chain acting on an initial probability distribution that assigns weight $\sum_{a_0,\dots,a_{n-1}} h^n_{a_0,\dots,a_{n-1},b} l(I^n_{a_0,\dots,a_{n-1},b})$ to the state b.
Consequently, if we apply the dynamics with an initial distribution with a density function of the form

$$h(z) = h^0_{-1} I\{z \in I^0_{-1}\} + h^0_{+1} I\{z \in I^0_{+1}\},$$

then, the probability of being in I^0_{-1} or in I^0_{+1} at later time steps, is completely described by the stochastic process. Moreover, when we take $h^0_{-1} = 2\mu(-1) = 1$, then this probability is invariant in time. Theorem 1 is the analogue statement for coupled maps.

The Markov Chain that we consider here converges exponentially fast to the invariant probability law μ. In terms of the language we used to describe the PCA, it is exponentially ergodic. Hence, it provides us with a toy model to illustrate the argument that leads to Theorem 3. Suppose that the map is applied to a probability distribution m with density h. In a first step we will replace h by a probability density $h^{(n)}$ of the form

$$h^{(n)}(x) = \sum_{a_0,\dots,a_n} h^n_{a_0,\dots,a_n} I\{x \in I^n_{a_0,\dots,a_n}\}, \tag{25}$$

with

$$h^n_{a_0,\dots,a_n} = \frac{1}{l(I^n_{a_0,\dots,a_n})} \int dx h(x) I\{x \in I^n_{a_0,\dots,a_n}\}.$$

When we apply the map $2n$ times to this distribution, the above argument indicates that the outcome is described by the supporting Markov Chain

after it has run for n steps, when started in some particular initial probability distribution on $\{-1,+1\}$ derived from $h^{(n)}$. This means that the convergence towards the invariant measure after applying T $2n$ times can be studied by the convergence of the corresponding Markov Chain after n steps.
In a second step, we estimate the error that is made by considering $h^{(n)}$ instead of h in terms of the difference between $h(x)$ and $h^n_{a_0,\ldots,a_n}$ for each $x \in I^n_{a_0,a_1\ldots,a_n}$ and for all possible choices of the indices. According to (24) the length $l(a_0,\ldots,a_n)$ satisfies

$$l(a_0,\ldots,a_n) \le \max_{a,b} \frac{1}{2}[p(a|b)]^n$$

so that $h^n_{a_0,\ldots,a_n}$ converges to h, provided that h is a sufficiently smooth function. This illustrates why Theorem 3 only deals with a restricted class of density functions.

Before we apply a similar reasoning to coupled maps, we first verify that the whole state space Ω is an indecomposable part for all maps T defined along the lines of Section 3.1.

Proposition 1 $(T,[0,1]^{\mathbb{Z}^d})$ *is indecomposable for all the maps T defined in Section 3.*

Proof
Let $\mathbf{\Lambda} \subset \mathbb{Z}^{d+1}$ be the $d+1$ dimensional cube:

$$\mathbf{\Lambda} = \{(i,n) : 0 \le n \le K, 0 \le i_k \le K \quad \text{for} \quad k = 1,\ldots,d\} \tag{26}$$

and let $\mathbf{\Lambda}_{(c,m)}$ be the translation of this cube by $(c,m) \in \mathbb{Z}^d \times \mathbb{Z}$.
It is sufficient to prove that, given $\epsilon > 0$, there exists D_0 such that if $|m| + \sum_{i=1}^d |c_i| > D_0$, the conditions for indecomposability are satisfied for $\mathbf{\Lambda}$ and $\mathbf{\Lambda}_{(c,m)}$. Moreover, we can assume that $m \ge 0$.
First we show that there exists D_1 such that when $m > D_1$ we can find for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{\Omega}_{\boldsymbol{T}}, c \in \mathbb{Z}^d$, a space–time state $\boldsymbol{z}$ that verifies

$$\max\{|x_n(i) - z_n(i)| : (i,n) \in \mathbf{\Lambda}\} \le \epsilon \tag{27}$$

and

$$\max\{|y_n(i) - z_n(i)| : (i,n) \in \mathbf{\Lambda}_{(c,m)}\} \le \epsilon. \tag{28}$$

This follows from the following three observations.

1. For any $\boldsymbol{x} \in \boldsymbol{\Omega_T}$ and all $\epsilon > 0$ there exists $\boldsymbol{x^{(0)}} \in \boldsymbol{\Omega_T}$ such that

$$
\begin{aligned}
{x^{(0)}}_0(i) \quad &= x_0(i) \quad \text{if} \quad x_0(i) \neq 0, \\
&= a \qquad\ \ \text{if} \quad x_0(i) = 0,
\end{aligned}
\tag{29}
$$

with $a \leq \min_{s=\pm 1, \eta \in \Omega^s} [p(s|\eta)]^K \epsilon$. Since for all the local maps, $z \in (0,1]$ has two preimages in $(0,1]$, this state is indeed in $\boldsymbol{\Omega_T}$. Notice also that because it is not possible to escape from 0, nor to reach 0 from another value, (29) guarantees that ${x^{(0)}}_n(i) \neq 0$ for all $(i,n) \in \mathbb{Z}^d \times \mathbb{Z}$.
Moreover, since ${x^{(0)}}_0(i) = a$ implies that $T^k {x^{(0)}}_0(i) \leq \epsilon (< 1/2)$ for $k = 0, \ldots, K$ and since the interaction with neighbouring sites only depends on the symbolic configuration of the state, $T^k {x^{(0)}}_0(i) = T^k x_0(i), k = 0, \ldots, K$ on all sites where ${x^{(0)}}_0(i) = x_0(i)$.
2. For every $y \in (0,1]^{\mathbb{Z}^d}$ (0 is excluded), and every set $\{\eta_0, \ldots, \eta_m\} \subset \Omega^s, m \in \mathbb{Z}^+$ there is a unique $z \in \Omega$ so that $\pi_0(T^k z) = \eta_k, k = 0, 1, ..., m$ and $T^{m+1} z = y$. This is not true for states $y \in \Omega$ that contain a site i with $y(i) = 0$. As we already mentioned previously, the only preimage of 0 is 0, which excludes ancestors with a symbolic configuration for which $\eta(i) = +1$.
3. If $x \in \Omega$ and $y \in \Omega$ are such that $\pi_0(T^k x) = \pi_0(T^k y), k = 0, 1, \ldots, m$, then

$$
|x(i) - y(i)| \leq \frac{1}{2} \max_{s=\pm 1, \xi \in \Omega^s} [p(s|\xi)]^m.
$$

for all $i \in \mathbb{Z}^d$. This can be obtained by induction since, for $x, y \in \Omega$, $|x(i) - y(i)| \leq 1/2$ as soon as $\pi_0 x(i) = \pi_0 y(i)$ and since the combination of both $\pi_0 x = \pi_0 y$ and $\pi_0 T x = \pi_0 T y$ implies that $|x(i) - y(i)| = p(\pi_0 T x(i) | \pi_0 x) |Tx(i) - Ty(i)|$ for all $i \in \mathbb{Z}^d$.
By combining these three observations, we deduce that there is a D_1 so that for $m > D_1$, there exists $\boldsymbol{z}$ that verifies (27) and (28). Indeed, given $\epsilon, c, \boldsymbol{x}, \boldsymbol{y}$ we first consider for every m a space–time state $\boldsymbol{y^{(m)}}$ such that

$$
\begin{aligned}
{y^{(m)}}_m(i) \quad &= y_m(i) \quad \text{if} \quad y_m(i) \neq 0, \\
&= a \qquad\ \ \ \text{if} \quad y_m(i) = 0,
\end{aligned}
\tag{30}
$$

with a as in observation 1. Then, we take $D_1 > K$ such that

$$
\frac{1}{2} \max_{s=\pm 1, \xi \in \Omega^s} [p(s|\xi)]^{D_1 - K} \leq \epsilon. \tag{31}
$$

Next, the second observation ensures, for every m, the existence of a state $z \in \Omega$ such that $\pi_0(T^k z) = \pi_0(x_k), k = 0, 1, ..., m-1$ and $T^m z = {y^{(m)}}_m$. This means that

$$T^n z(i) = {y^{(m)}}_n(i) \text{ for all } \quad (i,n) \in \mathbf{\Lambda}_{(c,m)},$$

or, according to observation 1,

$$|T^n z(i) - y_n(i)| \leq \epsilon \text{ for all } \quad (i,n) \in \mathbf{\Lambda}_{(c,m)}.$$

Finally, taking $m > D_1$, (31) and observation 3 imply that

$$\max\{|T^k z(i) - x_k(i)| : (i,k) \in \mathbf{\Lambda}\} \leq \frac{1}{2} \max_{s=\pm 1, \xi \in \Omega^s} [p(s|\xi)]^{m-K+1} \leq \epsilon.$$

To complete the proof of the proposition, we still need to consider those pairs of cubes $\mathbf{\Lambda}$, $\mathbf{\Lambda}_{(c,m)}$ for which $0 \leq m \leq D_1$. We have to find a D_2 so that if $\sum_{i=1}^d |c_i| > D_2$ there is a space-time state that is near to the given states in both cubes.
Take z_0 coinciding with x_0 in $\{i : -K \leq i_k \leq 2K, k = 1, \ldots, d\} \subset \mathbb{Z}^d$ and with any preimage of y_m in $\{i : c_k - (K + D_1) \leq i_k \leq c_k + 2K + D_1, k = 1, \ldots, d\} \subset \mathbb{Z}^d$. Because of the nearest neighbour interaction, we see that as soon as $\max_{k=1,\ldots,d} |c_k| \geq 4K + D_1$, the corresponding $\boldsymbol{z}$ coincides with the prescribed states $\boldsymbol{x} \in \mathbf{\Lambda}$ and $\boldsymbol{y} \in \mathbf{\Lambda}_{(c,m)}$. ■

In the following proposition we relate the supporting PCA to the map T. Therefore we introduce for every finite $\Lambda \subset \mathbb{Z}^d$ a family of density functions $h_\Lambda^{(m)}$, $m \in \mathbb{Z}^+$, that are all derived from a given density function h_Λ. For $\zeta \in \Omega^s$ we define

$$\begin{aligned} h^{(m)}_{\Lambda,\zeta}(x) = \sum_{\substack{\eta^k \in \Omega^s_\Lambda \\ k=0,\ldots,m}} & \Big\{ 2^{|\Lambda|} [\prod_{k=0}^{m-1} P_{\Lambda,\zeta}(\eta^{k+1}|\eta^k)]^{-1} \prod_{k=0}^{m} I\{\pi_0 T^k_{\Lambda,\zeta} x_\Lambda = \eta^k\} \\ & \times \textstyle\int_{\Omega_\Lambda} dy h_\Lambda(y) \prod_{k=0}^{m} I\{\pi_0 T^k_{\Lambda,\zeta} y_\Lambda = \eta^k\} \Big\}. \end{aligned} \tag{32}$$

When $m = 0$, we set the product over the empty set equal to 1.

To see that this is indeed the density function for a probability measure, integrate $h^{(m)}_{\Lambda,\zeta}(x)$ over Ω and use that

$$\begin{aligned} & \textstyle\int_{\Omega_\Lambda} dy I\{\pi_0 y_\Lambda = \eta^0, \pi_0 T_{\Lambda,\zeta} y_\Lambda = \eta^1, \ldots, \pi_0 T^m_{\Lambda,\zeta} y_\Lambda = \eta^m\} \\ & = 2^{-|\Lambda|} \prod_{k=0}^{m-1} P_{\Lambda,\zeta}(\eta^{k+1}|\eta^k). \end{aligned}$$

That the latter equality is true follows from the first lines of the proof of Proposition 2 when the function f is replaced by 1. These density functions are the analogues of (25) as were introduced for one map.

For these density functions the following proposition can be verified:

Proposition 2 *Suppose that $n > m$. For every local function f and for all finite $\Lambda \subset \mathbb{Z}^d$ satisfying $\Lambda \supseteq \Lambda_f$, the following equality holds*

$$\int_{\Omega_\Lambda} dx f(T^n_{\Lambda,\zeta}x)h^{(m)}_{\Lambda,\zeta}(x) = \\ 2^{|\Lambda|} \sum_{\substack{\xi\in\Omega^s_\Lambda \\ \eta\in\Omega^s_\Lambda}} [\int_{\Omega_\Lambda} dy I\{\pi_0 y_\Lambda = \xi\} f(y)] P^{n-m}_{\Lambda,\zeta}(\xi|\eta)[\int_{\Omega_\Lambda} dy h_\Lambda(y) I\{\pi_0 T^m_{\Lambda,\zeta} y_\Lambda = \eta\}]. \tag{33}$$

This means that $\int_{\Omega_\Lambda} dx f(T^n_{\Lambda,\zeta}x)h^{(m)}_{\Lambda,\zeta}(x)$ equals the expected value of the function $\int_{\Omega_\Lambda} dy I\{\pi_0 y_\Lambda = \cdot\} f(y)$ on Ω^s_Λ after $n-m$ steps of the supporting PCA started with an initial probability distribution that assigns weight $\int_{\Omega_\Lambda} dy h_\Lambda(y) I\{\pi_0 T^m_{\Lambda,\zeta} y_\Lambda = \eta\}$ to the configuration $\eta, \eta \in \Omega^s_\Lambda$.

Proof
For any h_Λ,

$$\int_{\Omega_\Lambda} dx f(T_{\Lambda,\zeta}x)h_\Lambda(x) = \sum_{\xi\in\Omega^s_\Lambda} \int_{\Omega_\Lambda} dx I\{\pi_0 x_\Lambda = \xi\} f(T_\Lambda(x, \xi_\Lambda \times \zeta_{\Lambda^c}))h_\Lambda(x) \tag{34}$$

If we set $y = T_\Lambda(x, \xi_\Lambda \times \zeta_{\Lambda^c})$, then (34) becomes

$$\begin{aligned} &\sum_{\xi\in\Omega^s_\Lambda} \int_{\Omega_\Lambda} dy I\{\pi_0 T^{-1}_\Lambda(y, \xi_\Lambda \times \zeta_{\Lambda^c}) = \xi\} f(y) h_\Lambda(T^{-1}_\Lambda(y, \xi_\Lambda \times \zeta_{\Lambda^c})) \\ &\qquad \times [Det(\tfrac{\partial T_i}{\partial x_j}(\cdot, \xi_\Lambda \times \zeta_{\Lambda^c}) \big|_{T^{-1}_\Lambda(y,\xi_\Lambda\times\zeta_{\Lambda^c})})_{i\in\Lambda, j\in\Lambda}]^{-1} \\ &= \sum_{\xi\in\Omega^s_\Lambda} \int_{\Omega_\Lambda} dy f(y) h_\Lambda(T^{-1}_\Lambda(y, \xi_\Lambda \times \zeta_{\Lambda^c})) \prod_{i\in\Lambda} p(\pi_0 y(i)|\xi_\Lambda \times \zeta_{\Lambda^c}) \\ &= \sum_{\substack{\xi^0\in\Omega^s_\Lambda \\ \xi^1\in\Omega^s_\Lambda}} \int_{\Omega_\Lambda} dy I\{\pi_0 y_\Lambda = \xi^0\} f(y) h_\Lambda(T^{-1}_\Lambda(y, \xi^1_\Lambda \times \zeta_{\Lambda^c})) P_{\Lambda,\zeta}(\xi^0|\xi^1). \end{aligned} \tag{35}$$

Similarly, we find that

$$
\begin{aligned}
&\int_{\Omega_\Lambda} dx f(T^n_{\Lambda,\zeta}x)h_\Lambda(x) \\
&\quad = \sum_{\xi^k\in\Omega^s_\Lambda, k=0,\dots,n} \int_{\Omega_\Lambda} dy I\{\pi_0 y_\Lambda = \xi^0\} f(y) \\
&\qquad \times h_\Lambda(T_\Lambda^{-1}(\cdot,\xi^n_\Lambda\times\zeta_{\Lambda^c})\circ\ldots\circ T_\Lambda^{-1}(\cdot,\xi^1_\Lambda\times\zeta_{\Lambda^c})(y)) \\
&\qquad \times \prod_{k=0}^{n-1} P_{\Lambda,\zeta}(\xi^k|\xi^{k+1}).
\end{aligned} \tag{36}
$$

When we replace in (36) $h_\Lambda(x)$ by $h^{(m)}_{\Lambda,\zeta}(x)$ and use that, for $k < n$

$$
\pi_0 T^k_{\Lambda,\zeta}[T_\Lambda^{-1}(\cdot,\xi^n_\Lambda\times\zeta_{\Lambda^c})\circ\ldots\circ T_\Lambda^{-1}(\cdot,\xi^1_\Lambda\times\zeta_{\Lambda^c})(y)] = \xi^{n-k}_\Lambda, \tag{37}
$$

then we obtain that

$$
\begin{aligned}
&\int_{\Omega_\Lambda} dx f(T^n_{\Lambda,\zeta}x)h^{(m)}_{\Lambda,\zeta}(x) \\
&\quad = 2^{|\Lambda|}\sum_{\xi^k\in\Omega^s_\Lambda,\eta^l\in\Omega^s_\Lambda,k=0,\dots,n-m-1,l=0,\dots,m} \int_{\Omega_\Lambda} dy I\{\pi_0 y_\Lambda = \xi^0\} f(y) \\
&\qquad \times \int_{\Omega_\Lambda} dz h_\Lambda(z) \prod_{k=0}^{m} I\{\pi_0 T^k_{\Lambda,\zeta} z_\Lambda = \eta^k\} \\
&\qquad \times \prod_{k=0}^{n-m-2} P_{\Lambda,\zeta}(\xi^k|\xi^{k+1}) P_{\Lambda,\zeta}(\xi^{n-m-1}|\eta^m) \\
&\quad = 2^{|\Lambda|}\sum_{\xi^0\in\Omega^s_\Lambda,\eta^m\in\Omega^s_\Lambda} \int_{\Omega_\Lambda} dy I\{\pi_0 y_\Lambda = \xi^0\} f(y) \\
&\qquad \times \int_{\Omega_\Lambda} dz h_\Lambda(z) I\{\pi_0 T^m_{\Lambda,\zeta} z_\Lambda = \eta^m\} P^{n-m}_{\Lambda,\zeta}(\xi^0|\eta^m).
\end{aligned} \tag{38}
$$

■

Proof of Theorem 1
Notice that for the density function (16), $h^{(0)}_{\Lambda,\zeta}(x) = h_\Lambda(x)$ independent of ζ. Taking Λ large enough (depending on Λ_f), Proposition 2 implies that for all choices of $\zeta\in\Omega^s$,

$$
\begin{aligned}
&\int_\Omega m^\mu(dx) f(Tx) \\
&\quad = \int_{\Omega_\Lambda} dx f(T_{\Lambda,\zeta}x) h^{(0)}_{\Lambda,\zeta}(x) \\
&\quad = 2^{|\Lambda|}\sum_{\substack{\xi\in\Omega^s_\Lambda\\ \eta\in\Omega^s_\Lambda}} [\textstyle\int_{\Omega_\Lambda} dy I\{\pi_0 y_\Lambda = \xi\} f(y)] P_{\Lambda,\zeta}(\xi|\eta) [\textstyle\int_{\Omega_\Lambda} dy h_\Lambda(y) I\{\pi_0 y_\Lambda = \eta\}] \\
&\quad = 2^{|\Lambda|}\sum_{\substack{\xi\in\Omega^s_\Lambda\\ \eta\in\Omega^s_\Lambda}} [\textstyle\int_{\Omega_\Lambda} dy I\{\pi_0 y_\Lambda = \xi\} f(y)] P_{\Lambda,\zeta}(\xi|\eta) \mu_\Lambda(\eta) \\
&\quad = \int_\Omega m^\mu(dx) f(x).
\end{aligned} \tag{39}
$$

■

Proof of Theorem 2.
That T exhibits a type-N phase transition as soon as the corresponding PCA has more than one invariant measure is immediate from Theorem 1. Indeed, every invariant (Borel) measure of the PCA in $P(\Omega^s)$ defines a different collection of density functions $h_\Lambda, \Lambda \subset \mathbb{Z}^d$ and hence another natural measure for T.
To prove that the map also shows a type–G phase transition we first notice that $\boldsymbol{\Omega_T^s} = \{-1,+1\}^{\mathbb{Z}^d\times\mathbb{Z}}$. We have to prove that there is more than one Gibbs measure for the (formal) Hamiltonian $H = \sum_{(i,n)\in\mathbb{Z}^d\times\mathbb{Z}} H_{(i,n)}$ with, for $\boldsymbol{\xi} \in \boldsymbol{\Omega_T^s}$,

$$H_{(i,n)}(\boldsymbol{\xi}) = -\log p(\xi_{n+1}(i)|\xi_n). \tag{40}$$

Therefore we consider for every invariant measure μ of the PCA a measure $\boldsymbol{\mu}_\mu^{-N}, N \in \mathbb{Z}^+$ on the configurations in the set $(\boldsymbol{\Omega_T^s})_{\{(i,n):n\geq -N\}}$. This space time measure $\boldsymbol{\mu}_\mu^{-N}$ is obtained by starting the PCA at time $-N$ with as initial measure the measure μ.
It is known (see for instance in [LMS]) that the measure $\boldsymbol{\mu}$ that is obtained as $\lim_{N\uparrow\infty} \boldsymbol{\mu}_\mu^{-N}$ is Gibbs for (40). The projection of this measure on any time layer is again the measure μ. So, for every invariant measure we get a distinct Gibbs measure, which means that the map T indeed shows a type–G phase transition. ∎

Proof of Theorem 3.
Recall that $\Lambda_n, n \in \mathbb{Z}^+$ is the d–dimensional cube $\Lambda_n = \{k = (k_1,\ldots,k_d) : \max_i |k_i| \leq n\}$. Without losing generality we can assume that Λ_f contains the origin of $\mathbb{Z}^d$. Take $\Lambda = \Lambda_{2(l+n)}$, with l so that $\Lambda_f \subset \Lambda_l$. Denote by h_Λ the density function that describes m_Λ. Then, for any $\zeta \in \Omega^s$,

$$\begin{aligned}
&|\textstyle\int_\Omega m(dx) f(T^{2n}x) - 2^{|\Lambda_f|}\int_{\Omega^s}\mu(d\xi)\int_\Omega dx f(x) I\{\pi_0 x_{\Lambda_f} = \xi_{\Lambda_f}\}| \\
&\quad = |\textstyle\int_{\Omega_\Lambda} dx h_\Lambda(x) f(T^{2n}_{\Lambda,\zeta}x) - 2^{|\Lambda_f|}\int_{\Omega^s}\mu(d\xi)\int_\Omega dx f(x) I\{\pi_0 x_{\Lambda_f} = \xi_{\Lambda_f}\}| \\
&\quad \leq |\textstyle\int_{\Omega_\Lambda} dx h^{(n)}_{\Lambda,\zeta}(x) f(T^{2n}_{\Lambda,\zeta}x) - \int_{\Omega_\Lambda} dx h_\Lambda(x) f(T^{2n}_{\Lambda,\zeta}x)| \\
&\qquad\quad + |\textstyle\int_{\Omega_\Lambda} dx h^{(n)}_{\Lambda,\zeta}(x) f(T^{2n}_{\Lambda,\zeta}x) \\
&\qquad\qquad\qquad -2^{|\Lambda_f|}\textstyle\int_{\Omega^s}\mu(d\xi)\int_\Omega dx f(x) I\{\pi_0 x_{\Lambda_f} = \xi_{\Lambda_f}\}| \\
&\quad \leq \|f\| \textstyle\int_{\Omega_\Lambda} dx |h^{(n)}_{\Lambda,\zeta}(x) - h_\Lambda(x)| \\
&\qquad\quad + |2^{|\Lambda|} \sum_{\xi\in\Omega^s_\Lambda, \eta\in\Omega^s_\Lambda} [\textstyle\int_{\Omega_\Lambda} dy I\{\pi_0 y_\Lambda = \xi\} f(y)] \\
&\qquad\qquad\qquad\qquad \times P^n_{\Lambda,\zeta}(\xi|\eta)[\textstyle\int_{\Omega_\Lambda} dy h_\Lambda(y) I\{\pi_0 T^n_{\Lambda,\zeta} y_\Lambda = \eta\}] \\
&\qquad\qquad\qquad -2^{|\Lambda_f|}\textstyle\int_{\Omega^s}\mu(d\xi)\int_\Omega dx f(x) I\{\pi_0 x_{\Lambda_f} = \xi_{\Lambda_f}\}|.
\end{aligned} \tag{41}$$

In the last inequality we used Proposition 2.
Since

$$\textstyle\int_{\Omega_\Lambda} dy I\{\pi_0 y_\Lambda = \eta^0, \pi_0 T_{\Lambda,\zeta} y_\Lambda = \eta^1, \ldots, \pi_0 T^m_{\Lambda,\zeta} y_\Lambda = \eta^m\} \\ = 2^{-|\Lambda|} \prod_{k=0}^{m-1} P_{\Lambda,\zeta}(\eta^{k+1}|\eta^k),$$

we estimate the first term as follows:

$$\begin{aligned} &\textstyle\int_{\Omega_\Lambda} dx |h^{(n)}_{\Lambda,\zeta}(x) - h_\Lambda(x)| \\ &\quad\le \sum_{\substack{\eta^k \in \Omega^s_\Lambda \\ k=0,\ldots,n}} \Big\{ 2^{|\Lambda|} [\prod_{k=0}^{n-1} P_{\Lambda,\zeta}(\eta^{k+1}|\eta^k)]^{-1} \textstyle\int_{\Omega_\Lambda} dx \prod_{i=0}^n I\{\pi_0 T^i_{\Lambda,\zeta} x_\Lambda = \eta^i\} \\ &\qquad\quad \times \textstyle\int_{\Omega_\Lambda} dy \prod_{i=0}^n I\{\pi_0 T^i_{\Lambda,\zeta} y_\Lambda = \eta^i\} |h_\Lambda(x) - h_\Lambda(y)| \Big\}. \end{aligned} \tag{42}$$

Then we use that

$$\begin{aligned} &|h_\Lambda(x) - h_\Lambda(y)| = h_\Lambda(y) |1 - \tfrac{h_\Lambda(x)}{h_\Lambda(y)}| \\ &\quad \le h_\Lambda(y) \sup_{u,v} \left\{ \left|\log\left(\tfrac{h_\Lambda(u)}{h_\Lambda(v)}\right)\right| \left(\exp\left|\log\left(\tfrac{h_\Lambda(u)}{h_\Lambda(v)}\right)\right|\right) \right\} \\ &\quad \le h_\Lambda(y) \sup_{u,v} \left\{ C(\textstyle\sum_{i\in\Lambda} |u(i) - v(i)|^\delta) \exp\left(C \sum_{i\in\Lambda} |u(i)-v(i)|^\delta\right)\right\}, \end{aligned} \tag{43}$$

with $\delta > 0$ and $C < \infty$ as in (17). The maximum is over all u, v such that

$$\pi_0 T^i_{\Lambda,\zeta} u_\Lambda = \pi_0 T^i_{\Lambda,\zeta} v_\Lambda, \quad \text{for} \quad i = 0, \ldots, n.$$

Hence, (43) is not larger than

$$\frac{C|\Lambda|}{2^\delta} h_\Lambda(y) (\max_{\alpha,\eta} p(\alpha|\eta))^{\delta n} \exp\left(\frac{C|\Lambda|}{2^\delta} (\max_{\alpha,\eta} p(\alpha|\eta))^{\delta n}\right). \tag{44}$$

If we insert (44) in (42), and recall that we supposed Λ to be $\Lambda_{2(l+n)}$, we have proved that the first term in (41) decreases exponentially fast for increasing n.
When the PCA is ergodic, the second term in (41) also vanishes as $n \to \infty$. For an exponentially ergodic supporting PCA this happens exponentially fast. More precisely, there exist constants $\lambda > 0$ and $C < \infty$ so that the term is bounded by

$$\begin{aligned} &2^{|\Lambda_f|} \sum_{\xi \in \Omega^s_{\Lambda_f}} [\textstyle\int_\Omega dx f(x) I\{\pi_0 x_{\Lambda_f} = \xi_{\Lambda_f}\}] \\ &\quad \times |\mu(\xi(k), k \in \Lambda_f) - \sum_{\eta\in\Omega_\Lambda} P^n(\xi(k), k \in \Lambda_f | \eta) \textstyle\int_{\Omega_\Lambda} dy h(y) I\{\pi_0 T^m_{\Lambda,\zeta} y_\Lambda = \eta\}| \\ &\le C|\Lambda_f| \|f\| \mathrm{e}^{-\lambda n}. \end{aligned} \tag{45}$$

■

Proof of Theorem 4
This is immediate from the results in [GKLM] and [GKS]. In [GKLM] the authors prove that non–uniqueness of the (space–time) translation invariant Gibbs measure for (40) implies a non–unique invariant measure for the PCA. In [GKS] is shown that every periodic measure that is Gibbs with respect to (40) is a space–time measure for the PCA. ■

Acknowledgement

RSM is grateful to Stephen Cornell for suggesting to construct CML from PCA rather than the equilibrium statistical mechanics spin models first tried.

References

[AAIS] Arnol'd, V.I., Afrajmovich, V.S., Il'yashenko. Yu. S., Shil'nikov, L.P.: Bifurcation theory. In: Arnol'd V.I. (ed.), *Dynamical systems V.* Springer, 1994

[BEIJK] Baladi, V., Esposti, M. D., Isola, S., Järvenpää, E., Kupiainen, A.: The spectrum of weakly coupled map lattices. Journal de Mathématiques Pures et Appliquées **77**, 539-584 (1998)

[BK1] Bricmont, J., Kupiainen, A.: Coupled Analytic Maps. Nonlinearity **8**, 379–396 (1995)

[BK2] Bricmont, J., Kupiainen, A.: High temperature expansions and dynamical systems. Commun. Math. Phys. **178**, 703–732 (1996)

[BC] Bunimovich, L. A., Carlen, E.A.: On the problem of stability in lattice dynamical systems. J. Diff. Eq. **123**, 213–229 (1995)

[BS] Bunimovich, L. A., Sinai, Ya.G.: Spacetime chaos in coupled map lattices. Nonlinearity **1**, 491–516 (1988)

[BuS] Burton, R., Steif, J.E.: Non–uniqueness of measures of maximal entropy for subshifts of finite type. Ergod. Theor. Dynam. Syst. **14**, 213–235 (1994)

[CM] Chaté, H., Manneville, P.: Collective behaviors in spatially extended systems with local interactions and synchronous updating. Prog. Theor. Phys. **87**, 1–60 (1992)

[EFS] van Enter, A.C.D., Fernandez, R., Sokal, A.D.: Regularity properties and pathologies of position-space renormalization group transformation: Scope and limitation of Gibbsian theory. J. Stat. Phys. **72**, 679-1167 (1995)

[FR] Fischer, T., Rugh, H.H.: Transfer operators for coupled analytic maps. Ergod. Theor. Dynam. Syst. **20**, 109–143 (2000)

[GKLM] Goldstein, S., Kuik, J., Lebowitz, J.L., Maes, C.: From PCA's to equilibrium measures and back. Commun. Math. Phys. **125**, 71–79 (1989)

[GKS] Goldstein, S., Kuik, J., Schlijper, A. G.: Entropy and global Markov properties. Commun. Math. Phys. **126**, 469–482 (1990)

[Hak] Haken, H.: Cooperative phenomena in systems far from thermal equilibrium and in nonphysical systems. Rev. Mod. Phys. **47**, 67–121 (1975)

[JP] Jiang, M., Pesin, Ya.B.: Equilibrium measures for coupled map lattices: existence, uniqueness and finite dimensional approximations. Commun. Math. Phys. **193**, 677–711 (1998)

[K] Kaneko, K. : Globally coupled chaos violates the law of large numbers. Phys. Rev. Lett. **65**, 1391-1394 (1990)

[KH] Katok, A., Hasselblatt, B.: *Introduction to the Modern Theory of Dynamical Systems.* Cambridge University Press, 1995

[LMS] Lebowitz, J.L., Maes, C., Speer, E.: Statistical mechanics of probabilistic cellular automata. J. Stat. Phys. **59**, 117–170 (1990)

[LM] Losson, J., MacKey, M.C.: Thermodynamic properties of coupled map lattices. In: van Strien, S. , Verduyn Lunel, S.M. (eds), *Stochastic and spatial structures of dynamical systems.* North–Holland, 1996

[Ma] MacKay, R.S.: Unpublished notes, Dijon, February 1995

[MM] Maes, C., Van Moffaert, A.: Stochastic stability of weakly coupled maps. Nonlinearity **10**, 715–730 (1997)

[MH] Miller, J., Huse, D.A.: Macroscopic equilibrium from microscopic irreversibility in a chaotic coupled–map lattice. Phys. Rev. E **48**, 2528–2535 (1993)

[Ru] Ruelle, D.: *Thermodynamic Formalism.* Addison–Wesley, 1978

[Sak] Sakaguchi, H.: Phase transitions in coupled Bernoulli maps. Prog. Theor. Phys. **80**, 7–12 (1988)

[SK] Shibata, T., Kaneko, K.: Tongue-Like Bifurcation Structures of The Mean-Field Dynamics in a Network of Chaotic Elements. Physica D **124**, 177–200 (1998)

[Sin] Sinai, Ya.G.: Gibbs measures in ergodic theory. Russian Math Surveys **28**, 21–64 (1972)

[To] Toom, A.L.: Stable and attractive trajectories in multicomponent systems. In: Dobrushin, R.L., Sinai, Ya.G. (eds.), *Multicomponent random systems.* Dekker, New York, 1980