Chaotic Properties of Subshifts Generated by a Non-Periodic Recurrent Orbit

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Xin-Chu Fu[1], Yibin Fu[2], Jinqiao Duan[3], and Robert S. MacKay[4]

1. *Department of Mathematics and Statistics*
University of Surrey, Guildford, GU2 5XH, U. K.
2. *Department of Mathematics, Keele University*
Keele, Staffordshire, ST5 5BG, U. K.
3. *Department of Mathematical Sciences, Clemson University*
Clemson, South Carolina 29634, U. S. A.
4. *Nonlinear Centre, Department of Applied Mathematics and Theoretical Physics*
Cambridge University, Silver Street, Cambridge CB3 9EW, U. K.

Abstract

The chaotic properties of some subshift maps are investigated. These subshifts are the orbit closures of certain non-periodic recurrent points of a shift map. We first provide a review of basic concepts for dynamics of continuous maps in metric spaces. These concepts include nonwandering point, recurrent point, eventually periodic point, scrambled set, sensitive dependence on initial conditions, Robinson chaos, and topological entropy. Next we review the notion of shift maps and subshifts. Then we show that the one-sided subshifts generated by a non-periodic recurrent point are chaotic in the sense of Robinson. Moreover, we show that such a subshift has an infinite scrambled set if it has a periodic point. Finally, we give some examples and discuss the topological entropy of these subshifts, and present two open problems on the dynamics of subshifts.

Key words: discrete dynamical systems, continuous maps, subshifts, chaotic dynamics, recurrent points, nonwandering points, orbit closures

Short Running Title: Chaotic Properties of Subshifts

1 Introduction

Periodic and chaotic behaviour are the two poles of dynamical behaviour of a nonlinear system. These two poles, however, are closely related. An early paper discussing this topic is Li and Yorke's work "Period three implies chaos" [21]. They proved that for a one-dimensional continuous map on an interval, if it has a periodic point with period 3, then it is chaotic in a strong sense that they define. Related and similar work was also published even earlier by Sharkovskii [29]. There are other similar results for higher dimensional continuous maps; see, e.g. [23, 22, 7, 5].

Discrete dynamical systems have been used as simplified prototypical models for some engineering and biological processes, e.g. [24, 9, 6, 16, 32]. Through the Poincaré maps, discrete dynamical systems have also been used to show "chaos" in continuous dynamical systems, e.g. [14, 33].

In this article, we first review basic concepts for dynamics of continuous maps on metric spaces (Section 2). These concepts include nonwandering point, recurrent point, eventually periodic point, scrambled set, sensitive dependence on initial conditions, Robinson chaos, and topological entropy. Then we recall the definition of shift maps and their closed invariant subsets, called subshifts (Section 3). Our primary interest is in one-sided subshift maps generated by a non-periodic recurrent point. We demonstrate that these subshifts are chaotic in the sense of Robinson (Section 4). Moreover we show that such a subshift has an infinite scrambled set if it has a periodic point (Section 5). Finally, we give some examples and discuss their topological entropy, and present two open problems on the dynamics of subshifts (Section 6).

2 Principal concepts

In this section we review the principal concepts in the theory of the dynamics of continuous maps on metric spaces, and introduce related notations.

Let $F : M \to M$ be a continuous map, defined on a metric space M with metric ρ. A point $x \in M$ is called *nonwandering* if for any neighbourhood U of x, there exists $n > 0$ such that $F^n(U) \cap U \neq \emptyset$. A point $x \in M$ is called a *periodic point* of F, if $F^n(x) = x$ for some $n > 0$. The minimal such n is called the *period* of this periodic point. A point y is called an *eventually periodic point* of F if there exists $n \geq 0$ such that $F^n(y)$ is a periodic point of F. A point $x \in M$ is called a *recurrent point* of F, if for any neighbourhood U of x, there exists $n > 0$ such that $F^n(x) \in U$. The *ω-limit set* of a point x under F is

$$\omega(x, F) = \{y \in M : \exists n_i \to \infty, F^{n_i}(x) \to y\}. \tag{1}$$

In the following, we denote by $\Omega(M, F)$, $P(M, F)$, $EP(M, F)$ and $R(M, F)$ the sets of nonwandering points, periodic points, eventually periodic points and recurrent points, of F on M, respectively.

There are many definitions of chaos (e.g. see the discussions in [19, 26]). We introduce three principal ones:

1. **Infinite scrambled set**: Two points a, b in M form a *chaotic pair* [34] for the map F, if a and b are nonwandering and non-periodic points, and the following conditions hold:
$$\text{(i)} \quad \limsup_{n\to+\infty} \rho(F^n(a), F^n(b)) > 0,$$
$$\text{(ii)} \quad \liminf_{n\to+\infty} \rho(F^n(a), F^n(b)) = 0.$$
A subset $S \subseteq M$ is called a *scrambled set* [30], if for any $a, b \in S$, with $a \neq b$, then (a, b) is a chaotic pair. Our first notion of chaos is possession of an infinite scrambled set. This is closely related to that used in [21], but they did not require chaotic pairs to be non-wandering; instead they required an uncountable scrambled set and also a relation to $P(M, F)$ which we will not give here. Our feeling is that Li and Yorke's definition of chaos was of historical importance but is not general enough.

2. **Robinson chaos**: A map F is said to have *sensitive dependence on initial conditions* (SDIC) on M if there exists $\delta > 0$ such that, for any $x \in M$ and any neighbourhood U of x, there exists $y \in U$ and $n > 0$ such that $\rho(F^n(x), F^n(y)) > \delta$. A map F is said to be *topologically transitive* on M if for any two open sets $U, V \subset M$, there exists an integer $n > 0$ such that $F^n(U) \cap V \neq \emptyset$. An equivalent definition is that there is a dense orbit. This condition might not apply directly to F but often applies to the restriction of F to some closed invariant subset. A map is called *chaotic in the sense of Robinson* [26] if it has sensitive dependence on initial conditions and is topologically transitive. This definition is based on one of Devaney [8], who also required the periodic points to be dense, but as argued by Robinson, this does not seem to be intrinsic to the phenomenon of chaos, so we leave it out. A curious twist, incidentally, is that Banks et al [4] realised that topological transitivity plus density of periodic points imply SDIC, so that SDIC can be left out of Devaney's definition with making any change to his concept. Finally, note that Wiggins [33] defines chaos to mean SDIC plus M has more than one orbit.

3. **Positive topological entropy**: Topological entropy [1, 3, 15] was first defined by C. E. Shannon [28] in 1948 and called by him *noiseless channel capacity.* Engineers prefer this term while mathematicians like the term *topological entropy.* It is an important quantity measuring the rate of growth of the complexity of the orbit structure of a dynamical system with respect to time. In the case where M is compact, which suffices for this paper, it is most simply defined as follows. Given $\epsilon > 0$, say two orbit segments $(x_0, \ldots, x_n)$ and $(y_0, \ldots, y_n)$ of length $n > 0$ are *ϵ-distinguishable* if there exists $j \in \{0, \ldots, n\}$ such that $d(x_j, y_j) > \epsilon$. Let

$N(n, \epsilon)$ be the maximum size of a set of ϵ-distinguishable orbit segments of length n (which is finite by compactness). Then the *topological entropy* of F is

$$ent(F) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \epsilon). \tag{2}$$

We say F has *topological chaos* if its topological entropy is positive.

3 Shift Maps and Subshifts

We now recall the definitions of shift maps and subshifts, and define the class of subshifts to be studied in this paper.

Let X be a metric space with metric d. The basic example to bear in mind is $X = \{0, \dots, N-1\}$ (often denoted just by N) for some integer $N > 1$ with the discrete metric:

$$d(m, n) = \begin{cases} 0, & m = n \\ 1, & m \neq n. \end{cases}$$

Denote by Σ^X the space consisting of one-sided sequences in the metric space X (it can alternatively be written as $X^{\mathbf{Z}_+}$). So $x \in \Sigma^X$ may be denoted by $x = (x_0, x_1, \dots, x_i, \dots)$, $x_i \in X$, $i \geq 0$. Let Σ^X be endowed with the product topology. Then Σ^X is metrizable, and the metric on Σ^X can be chosen to be

$$\rho(x, y) = \sum_{i=0}^{+\infty} \frac{1}{2^i} \frac{d(x_i, y_i)}{1 + d(x_i, y_i)}, \qquad x = (x_0, x_1, \dots), \;\; y = (y_0, y_1, \dots) \in \Sigma^X.$$

The *shift map* σ: $\Sigma^X \to \Sigma^X$ is defined by $(\sigma(x))_i = x_{i+1}$, $i = 0, 1, \cdots$. Since

$$\rho(\sigma(x), \sigma(y)) \leq 2\rho(x, y),$$

σ is continuous. If X has more than one element, the shift map is chaotic in all three senses that we have defined. A two-sided shift map can be defined similarly, acting by the same formula but on doubly infinite sequences of elements of X. We shall consider only one-sided shifts in this paper.

As an aside, when X is a vector space, Σ^X becomes a vector space in the natural way. If X is a nontrivial vector space, Σ^X is infinite dimensional. Fu and Duan [11] have shown that the shift map σ induces an infinite dimensional *linear chaotic* discrete-time system. Furthermore, this shift map has been used to demonstrate the chaotic behaviour of a quantum harmonic oscillator by Duan et al [9].

A closed σ-invariant subset of Σ_X is called a *subshift*. Restricting attention now to the case of a set X with the discrete topology, a subshift is said to be *of finite type* if it can be completely specified by giving a finite list of excluded words, i.e. finite strings of elements of X that do not occur in any of the sequences of the subshift. The topological entropy of a shift or subshift with X discrete is equal to the exponential growth rate of the number of different blocks of length n occurring in the symbol sequences. For a shift this is just $\log N$, where N is the size of X. For subshifts of finite type, it reduces to computing the largest eigenvalue of an associated transition matrix, and is positive if the subshift does not consist only of periodic and eventually periodic points. In this case, the subshift of finite type is also chaotic in the other two senses. We call shifts and subshifts *symbolic dynamics systems*. Symbolic dynamics is a powerful tool to study more general dynamical systems, because the latter often contain invariant subsets on which the dynamics is equivalent to a shift map or subshift. Readers are referred to a recent book [20] for more background and details on symbolic dynamics.

For the rest of the paper, we will concentrate on subshifts that arise by taking the closure of a single orbit of a shift. These are not necessarily subshifts of finite type, and this is why their study is interesting and non-trivial. The study of orbit closures goes back at least to Morse, who studied the closure of the famous Morse sequence under the shift, which yields what we now call an "adding machine". We refer to [13] for an early source discussing orbit closures. Denote by $\Sigma(a)$ the closure of the orbit starting at point $a \in \Sigma^X$, i.e.,

$$\Sigma(a) = cl(\{\sigma^n(a) : \ n \geq 0\}).$$

It is easy to show that $\sigma(\Sigma(a)) \subseteq \Sigma(a)$, and it is closed by construction. So $(\Sigma(a), \sigma)$ is a subshift. We call $(\Sigma(a), \sigma)$ the subshift generated by the orbit starting at point a, or, in short, by the point a.

In the following, we discuss dynamical behaviour of the subshift $(\Sigma(a), \sigma)$.

4 Properties of Subshift $(\Sigma(a), \sigma)$

We now discuss the dynamical properties of the subshift $(\Sigma(a), \sigma)$ generated by a point $a \in \Sigma^X$. Initially, X is an arbitrary metric space, but we shall rapidly specialise to the case $X = N$. The results of this section are elementary but we spell them out for pedagogical purposes.

Suppose $x \in \Sigma^X$ is recurrent but non-periodic, i.e., $x \in R(\Sigma^X, \sigma) - P(\Sigma^X, \sigma)$. Since $x \in R(\Sigma^X, \sigma)$, there exist $\{n_i\}$, $n_i \to +\infty$ as $i \to +\infty$, such that when i is big enough, $\sigma^{n_i}(x)$ enters any neighbourhood of x. Since $x \notin P(\Sigma^X, \sigma)$, $\sigma^{n_i}(x) \neq x$, $\forall i \geq 1$. So n_i can be chosen such that $\sigma^{n_i}(x)$, $i = 1, 2, \ldots$, are all different, and thus $\{\sigma^n(x), n \geq 0\}$ is a countably infinite set. This implies that $x \notin EP(\Sigma^X, \sigma)$. So we have the following result

Proposition 4.1 *Non-periodic recurrent points of σ are not eventually periodic points, i.e., $(R(\Sigma^X, \sigma) - P(\Sigma^X, \sigma)) \cap EP(\Sigma^X, \sigma) = \emptyset$.*

Now suppose $a \in R(\Sigma^N, \sigma) - P(\Sigma^N, \sigma)$. Because $\Sigma(a)$ is a closed subset of the compact space Σ^N, $\Sigma(a)$ is compact. Also, since Σ^N is totally disconnected, its subset

$\Sigma(a)$ is also totally disconnected. From $a \in R(\Sigma^N, \sigma)$, we have $\sigma^n(a) \in R(\Sigma^N, \sigma)$, $\forall n \geq 0$. By Proposition 4.1, $\sigma^n(a) \notin P(\Sigma^N, \sigma)$. Therefore, $\sigma^n(a) \in R(\Sigma^N, \sigma) - P(\Sigma^N, \sigma)$, $\forall n \geq 0$. So there exist $\{n_i\}, n_i \to +\infty$ as $i \to +\infty$, such that $\sigma^{n_i}(\sigma^n(a))$, $i = 1, 2, \ldots$ are all different, and $\sigma^{n_i}(\sigma^n(a))$ enters any neighbourhood of $\sigma^n(a)$. These imply $\sigma^n(a) \in (\{\sigma^m(a), m \geq 0\})'$, $\forall n \geq 0$, where for a subset S of a topological space, S' denotes its set of limit points. So $\{\sigma^n(a), n \geq 0\} \subseteq (\Sigma(a))'$. So $(\Sigma(a))' = \Sigma(a)$, i.e., $\Sigma(a)$ is perfect, and $\Sigma(a)$ is homeomorphic to a Cantor set [33]. We summarize this result as

Proposition 4.2 *If a is a non-periodic recurrent point for σ, i.e., $a \in R(\Sigma^N, \sigma) - P(\Sigma^N, \sigma)$, then $\Sigma(a)$ is homeomorphic to the Cantor set.*

In particular, if $a \in R(\Sigma^N, \sigma) - P(\Sigma^N, \sigma)$, then $\Sigma(a)$ is uncountable. Moreover, for $a \in R(\Sigma^N, \sigma)$, we have $\Sigma(a) = \omega(a, \sigma)$ (the ω-limit set of the orbit from a). The converse is also true. So $a \in R(\Sigma^N, \sigma)$ if and only if $\Sigma(a) = \omega(a, \sigma)$.

We further have the following conclusion

Proposition 4.3 *If a is a non-periodic recurrent point for σ, i.e., $a \in R(\Sigma^N, \sigma) - P(\Sigma^N, \sigma)$, then σ has sensitive dependence on initial conditions on $\Sigma(a)$.*

This can be shown as follows. Take $\delta_0 = 1/2$. For $x \in \Sigma(a)$, let V be an arbitrary neighbourhood of x in $\Sigma(a)$. If there exists $y \in V$, such that $y \neq x$, then taking $k = \min\{n \geq 0 : x_n \neq y_n\}$, we have

$$\rho(\sigma^k(x), \sigma^k(y)) = d(x_k, y_k) + \cdots \geq 1 > \delta_0.$$

So we only need to show the existence of y.

If there exists n such that $x = \sigma^n(a)$, then from Proposition 4.2, $x \in R(\Sigma^N, \sigma) - P(\Sigma^N, \sigma)$. Therefore, there exist $\{n_i\} \to \infty$, such that $\sigma^{n_i}(x)$, $i = 1, 2, \ldots$, are all different, and $\lim_{i \to +\infty} \sigma^{n_i}(x) = x$. So we can take $y = \sigma^{n_i}(x)$ (i large enough). Alternatively, if $\sigma^n(a) \neq x$ for all $n \in \mathbf{Z}_+$ then there exist $\{m_i\}$, $m_i \to \infty$, such that $x = \lim_{i \to +\infty} \sigma^{m_i}(a)$, and hence we can take $y = \sigma^{m_i}(a)$ for i large enough.

This completes the proof of Proposition 4.3. The important ingredient was that the shift on one-sided sequences is *expanding*, meaning that there exists $\delta > 0$ such that for all pairs of distinct points x, y there exists $n \geq 0$ such that $\rho(\sigma^n(x), \sigma^n(y)) \geq \delta$. The result does not apply to the two-sided shift.

We now discuss the image and nonwandering set of the subshift σ. Let a be a recurrent point, i.e., $a \in R(\Sigma^X, \sigma)$. It is easy to verify that $\sigma(\Sigma(a)) = \Sigma(a)$. Since the orbit of a is dense in $\Sigma(a)$, σ is topologically transitive on $\Sigma(a)$. Thus every point is nonwandering. We thus obtain the following conclusion.

Proposition 4.4 *Assume that a is a recurrent point, i.e., $a \in R(\Sigma^X, \sigma)$. Then $\sigma(\Sigma(a)) = \Sigma(a)$, $\Omega(\sigma|_{\Sigma(a)}) = \Sigma(a)$.*

For $a \in R(\Sigma^N, \sigma) - P(\Sigma^N, \sigma)$, Proposition 4.3 claims that σ has sensitive dependence on initial conditions on $\Sigma(a)$. As noted in the proof of Proposition 4.4, σ is topologically transitive on $\Sigma(a)$. So we have

Theorem 4.5 *If a is a non-periodic recurrent point for σ, i.e., $a \in R(\Sigma^N, \sigma) - P(\Sigma^N, \sigma)$, then the subshift $(\Sigma(a), \sigma)$ is chaotic in the sense of Robinson.*

5 Further Properties of Subshift $(\Sigma(a), \sigma)$ When It Has a Periodic Point

The above discussion shows some interesting properties of subshifts generated by non-periodic recurrent points. In particular, Theorem 4.5 declares that all these subshifts are chaotic in the sense of Robinson. In this section we will show that if $\Sigma(a)$ contains a periodic point, then there exist infinitely many chaotic pairs in $\Sigma(a)$; in particular, if $\Sigma(a)$ contains a fixed point, then there exists an infinite scrambled set which is dense in $\Sigma(a)$.

Theorem 5.1 *Assume that a is a non-periodic recurrent point and that $\Sigma(a)$ contains a periodic point of σ, i.e., $a \in R(\Sigma^N, \sigma) - P(\Sigma^N, \sigma)$ and $\Sigma(a) \cap P(\Sigma^N, \sigma) \neq \emptyset$. Then there exists an infinite set $S \subset \Sigma(a)$ in which any two different points are a chaotic pair for the subshift σ, i.e., S is an infinite scrambled set for the subshift σ.*

Proof: Take $x_0 \in \Sigma(a) \cap P(\Sigma^N, \sigma)$ and denote by k the period of x_0. Define $S = \{\sigma^{kn}(a),\ n \geq 0\}$. From Proposition 4.4, $S \subseteq \Omega(\sigma|_{\Sigma(a)})$. By Proposition 4.1, $S \cap P(\Sigma^N, \sigma) = \emptyset$. That is, every point in S is a nonwandering and non-periodic point.

Whenever $m_1 \neq m_2$, there exists a subsequence $\{n_i\}$ such that $(\sigma^{km_1}(a))_{n_i} \neq (\sigma^{km_2}(a))_{n_i}$, where subscript n_i denotes the n_i-entry. For suppose that this claim is not true. Then there would exist $N > 0$, such that $\sigma^N(\sigma^{km_1}(a)) = \sigma^N(\sigma^{km_2}(a))$. Without loss of generality, we assume that $m_1 < m_2$. Thus

$$\sigma^{N+km_1}(a) = \sigma^{N+km_2}(a) = \sigma^{k(m_2-m_1)}(\sigma^{N+km_1}(a)).$$

So $a \in EP(\Sigma^N, a)$. That is a contradiction.

Therefore, S is an infinite set, and there exist $\{n_i\}$, $n_i \to +\infty$ as $i \to +\infty$, such that

$$\rho\left(\sigma^{n_i}(\sigma^{km_1}(a)),\ \sigma^{n_i}(\sigma^{km_2}(a))\right) \geq 1,$$

i.e.,

$$\limsup_{n \to +\infty} \rho(\sigma^n(x), \sigma^n(y)) \geq 1 > 0, \quad \forall x, y \in S,\ \ x \neq y.$$

This proves condition (i) in the definition of a chaotic pair in Section 1.

Since $x_0 \in \Sigma(a) = \omega(a, \sigma)$, there exists a subsequence $\{n_i\}$ such that

$$\lim_{i \to +\infty} \rho(\sigma^{n_i}(a), x_0) = 0,$$

i.e., $\forall M > 0$, $\exists N > 0$, such that whenever $i \geq N$, we have

$$\rho(\sigma^{n_i}(a), x_0) < \frac{1}{2^{Mk}}.$$

Hence the first $Mk+1$ entries of $\sigma^{n_i}(a)$ and x_0 agree. Therefore, $\forall 0 \leq m < M$, whenever $i \geq N$, we have

$$\rho(\sigma^{n_i}(\sigma^{km}(a)), x_0) \leq \frac{1}{2^{(M-m)k}}.$$

This implies that for $\forall m \geq 0$, we have

$$\lim_{i \to +\infty} \rho(\sigma^{n_i}(\sigma^{km}(a)), x_0) = 0.$$

Moreover, because

$$\rho\left(\sigma^{n_i}(\sigma^{km_1}(a)), \sigma^{n_i}(\sigma^{km_2}(a))\right) \leq \rho\left(\sigma^{n_i}(\sigma^{km_1}(a)), x_0\right) + \rho\left(\sigma^{n_i}(\sigma^{km_2}(a)), x_0\right),$$

we have

$$\lim_{i \to +\infty} \rho\left(\sigma^{n_i}(\sigma^{km_1}(a)), \sigma^{n_i}(\sigma^{km_2}(a))\right) = 0, \quad \forall m_1, m_2 \geq 0,$$

i.e.,

$$\liminf_{n \to +\infty} \rho(\sigma^n(x), \ \rho(\sigma^n(y)) = 0, \quad \forall x, y \in S, \quad x \neq y.$$

This proves condition (ii) in the definition of a chaotic pair in Section 1. So any pair of distinct points x, y in S is a chaotic pair. This completes the proof of Theorem 5.1.

We remark that if σ has a fixed point (a periodic point with period $k = 1$), the chaotic pair set S is even more special. Recall the construction of S in the proof of Theorem 5.1, $S = \{\sigma^{kn}(a), \ n \geq 0\}$, where k is the period of some periodic point in $\Sigma(a) \cap P(\Sigma^N, \sigma)$. If $\Sigma(a) \cap P(\Sigma^N, \sigma)$ contains a fixed point of σ, then we can take $k = 1$, so $S = \{\sigma^n(a), \ n \geq 0\}$. Therefore S is dense in $\Sigma(a)$ and $\sigma(S) \subseteq S$. So we have

Theorem 5.2 *Assume that a is a non-periodic recurrent point and that $\Sigma(a)$ contains a fixed point (a periodic point with period 1) of σ. Then the subshift σ has a dense invariant scrambled set $S \subseteq \Sigma(a)$.*

6 Examples, Discussions and Open Problems

The subshifts of the form $(\Sigma(a), \sigma)$ are common and fundamental, because the full shift (Σ^N, σ) and any subshift of finite type (Σ_A, σ_A) with an irreducible transition matrix A can be generated by a point in $R(\Sigma^N, \sigma) - P(\Sigma^N, \sigma)$. In particular, any topologically transitive subshift can be written in the form of $(\Sigma(a), \sigma)$, and by definition a chaotic subshift should be topologically transitive. Also it is obvious that any subshift contains subshifts of the form $(\Sigma(a), \sigma)$.

Moreover, many subshifts of infinite type can be generated by points in the subset $R(\Sigma^N,\sigma)-P(\Sigma^N,\sigma)$. The following are three examples.

Example 6.1. ([12]) Given sequences s_i of 0 and 1 as follows: $s_0=00110$, $s_1=t_0s_0$, $s_2=t_1s_0s_0s_1$, $\ldots$, $s_n=t\overbrace{s_0\cdots s_0}^{n}s_1s_2\cdots s_{n-1}(n\geq 2)$, $\ldots$, where $t_0=00000$, $t_1=11111$, $t=t_0$ when n is odd, and $t=t_1$ when n is even, take $a=(s_0s_1s_2\cdots s_n\cdots)\in\Sigma^2$, then $a\in R(\Sigma^2,\sigma)-P(\Sigma^2,\sigma)$. $(\Sigma(a),\sigma)$ is a subshift of infinite type. $\Sigma(a)$ is homeomorphic to the Cantor set; $\Omega(\sigma|_{\Sigma(a)})=\Sigma(a)$; $(\Sigma(a),\sigma)$ is chaotic in the sense of Robinson. Because the 5-period point $(s_0s_0\cdots s_0\cdots)\in\Sigma(a)$, $S_1=\{\sigma^{5n}(a),n\geq 0\}$ is an infinite scrambled set (from Theorem 5.1).

Example 6.2. ([12]) Take sequences $s_0=00110$, $t_1=11111$. Form a symbol sequence $b\in\Sigma^2$ by concatenating all words of finite length formed from s_0 and t_1. Then $b\in R(\Sigma^2,\sigma)-P(\Sigma^2,\sigma)$. $(\Sigma(b),\sigma)$ is a subshift of infinite type. $\Sigma(b)$ is homeomorphic to the Cantor set; $\Omega(\sigma|_{\Sigma(b)})=\Sigma(b)$; $(\Sigma(b),\sigma)$ is chaotic in the sense of Robinson. Because the fixed point $(11\cdots 1\cdots)\in\Sigma(b)$, $S_2=\{\sigma^n(b),n\geq 0\}$ is a dense invariant scrambled set (according to Theorems 5.1 and 5.2).

Example 6.3. ([34]) Let $P=i_0i_1\cdots i_{n-1}$ and $Q=j_0j_1\cdots j_{m-1}$ be words of 0 and 1 with lengths $|P|=n$ and $|Q|=m$. Take $P_0=10$, $Q_0=00$, $P_1=P_0Q_0$, $Q_1=0\cdots 0$, $|Q_1|=|P_0Q_0P_1|\times 2=|P_1P_1|\times 2$. For $n>1$, let $P_n=P_0Q_0P_1Q_1\cdots P_{n-1}Q_{n-1}=P_{n-1}P_{n-1}Q_{n-1}$, $Q_{n-1}=0\cdots 0$, $|Q_{n-1}|=|P_0Q_0\cdots P_{n-1}|\times n=|P_{n-1}P_{n-1}|\times n$. Take $c=(P_0Q_0P_1Q_1\cdots P_nQ_n\cdots)\in\Sigma^2$. Then $c\in R(\Sigma^2,\sigma)-P(\Sigma^2,\sigma)$, and $(\Sigma(c),\sigma)$ is a subshift of infinite type. $\Sigma(c)$ is homeomorphic to the Cantor set; $\Omega(\sigma|_{\Sigma(c)})=\Sigma(c)$; $(\Sigma(c),\sigma)$ is chaotic in the sense of Robinson. Because $\Sigma(c)\cap P(\Sigma^2,\sigma)=\{e\}$, where $e=(0\cdots 0\cdots)\in F(\Sigma^2,\sigma)$, $S_3=\{\sigma^n(c),n\geq 0\}$ is a dense invariant scrambled set.

Now we discuss the topological entropy of the above subshifts. For the subshifts constructed in Examples 6.1-3, we can calculate their topological entropies [12]:

$$\mathrm{ent}(\sigma|_{\Sigma(a)})=\mathrm{ent}(\sigma|_{\Sigma(b)})=\frac{1}{5}\log 2>0,\quad \mathrm{ent}(\sigma|_{\Sigma(c)})=0.$$

It is somewhat surprising that $(\Sigma(c),\sigma)$ has zero entropy, despite having both Robinson chaos and an infinite scrambled set. One may ask if there exists a subshift of infinite type with positive topological entropy but no infinite scrambled set nor Robinson chaos. Of course, there are many examples of the latter, because $\mathrm{ent}(\sigma|_\Sigma)>0$ does not imply that (Σ,σ) is topologically transitive, but we could ask instead for a closed invariant subset with Robinson chaos. For one- and two-dimensional smooth maps, [17] shows that positive entropy implies existence of a subshift of finite type, but we do not know the answer in the general context.

Finally, we present two open problems which are closely related to the topics of this paper:

Problem 1. Suppose $a\in R(\Sigma^N,\sigma)-P(\Sigma^N,\sigma)$, $\Sigma(a)\cap P(\Sigma^N,\sigma)\neq\emptyset$. Take $p\in\Sigma(a)\cap P(\Sigma^N,\sigma)$, and denote its period by k. Does the set $cl\{\sigma^{kn}(a),n\geq 0\}$ contain

an uncountable scrambled set?
Problem 2. Can Theorem 5.1 be strengthened to give an uncountable scrambled set?

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