Approximate renormalization with codimension-one fixed point for the break-up of some three-frequency tori

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Abstract

An approximate renormalization transformation is constructed for Hamiltonian systems with three degrees of freedom to study the break-up of invariant tori with a three-dimensional frequency vector from the cubic field $Q(\tau)$, where $\tau^3+\tau^2-2\tau-1=0$. This renormalization has two fixed points : an attracting one and a hyperbolic one with codimension-one stable manifold. The associated critical exponents that characterize the universality class for the break-up of these invariant tori are computed.

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I. INTRODUCTION

In this paper, we define and study an approximate renormalization scheme for Hamiltonians with three degrees of freedom, to shed light on the break-up of invariant tori with the frequency vector $\boldsymbol{\omega}_0 = (\tau^2+\tau, \tau, 1)$, where $\tau = 2\cos(2\pi/7) \approx 1.2469796$ is the root of modulus larger than 1 of the polynomial :

$$\tau^3 + \tau^2 - 2\tau - 1 = 0.$$

This frequency vector is conjectured to be the closest analogue in three dimensions of the golden vector from two dimensions (see discussion in Ref. [1]). A renormalization scheme on a class of dynamical systems is a map which transforms any of the systems into another one that is equivalent to the original one up to changes of variables which focus on smaller phase space scales and longer time scales. The expectation, well supported for two degrees of freedom (e.g. [2, 3]) but not yet more, is that suitably designed renormalization schemes for Hamiltonian systems, such as [4] and a precursor sketched in [5], possess a "critical set", i.e. an invariant set with codimension-one stable manifold (the "critical surface") which separates systems that under repeated renormalization go to an attracting "integrable set" from ones which go to infinity, and that these systems have respectively a smooth or no invariant torus of a given frequency vector. The break-up of invariant tori in one-parameter families crossing the critical surface transversely would then be governed by the unstable manifold of the critical set, leading to quantitatively the same asymptotic behaviour in all such families, a phenomenon known as "universality".

To gain a handle on the renormalization dynamics for three degree of freedom systems, it is convenient to study finite-dimensional approximations (see Ref. [6]). An approximate renormalization scheme for $\boldsymbol{\omega}_0$ was already defined and studied in Ref. [1]. Although this was an important first attempt, significant linear terms in the actions were neglected, and the renormalization dynamics obtained was not structurally stable. The first objection is serious because they produce terms of comparable size to those retained. The second objection is serious, as acknowledged in Ref. [1], because the qualitative dynamics can not be expected to persist to exact renormalization schemes.

The same two problems occurred for a previous approximate renormalization scheme, for a frequency vector based on the spiral mean [7], but were partially corrected in Refs. [8, 9]. In addition to allowing linear terms in the actions, a key feature of Ref. [9] was to change the choice of normalization for the "inverse mass matrix" to permit it to become degenerate. The corrected renormalization dynamics is still structurally unstable, but it has the special form of a skew-product over a certain map on the projective plane with a repelling focus and an attracting irrationally rotating circle, a feature which has a chance of persisting to exact renormalization schemes for this frequency vector because the forcing dynamics is purely number-theoretic. The corrected renormalisation is still unsatisfactory, however, because it appears that the critical set obtained can best be described as the graph of an eight-valued function over a circle with a dense set of singularities, which is inconsistent with structurally stable dynamics over an irrational rotation, and the approximations are not valid in the singularities.

Here, we make the analogous corrections to the approximate renormalization of Ref. [1] for frequency vector $\boldsymbol{\omega}_0$. Remarkably, we find fully structurally stable dynamics and a (codimension-one) critical fixed point. If this picture persists to the exact renormalization schemes then the unstable manifold of this fixed point will govern the break-up of invariant tori of this frequency vector for all one-parameter families crossing its stable manifold transversely. Hence the prospect of searching for such a fixed point becomes both easier and more valuable.

Our approximate renormalization is defined for the following family of Hamiltonians which are quadratic in the actions $\boldsymbol{A} = (A_1, A_2, A_3)$:

$$H(\boldsymbol{A}, \boldsymbol{\varphi}) = H_0(\boldsymbol{A}) + V(\boldsymbol{A}, \boldsymbol{\varphi}), \tag{I.1}$$

where the Hamiltonian H_0 is quadratic :

$$H_0(\boldsymbol{A}) = \boldsymbol{\omega}_0 \cdot \boldsymbol{A} + \frac{1}{2}(\boldsymbol{\Omega} \cdot \boldsymbol{A})^2, \tag{I.2}$$

and the perturbation is described by two scalar functions of the angles :

$$V(\boldsymbol{A}, \boldsymbol{\varphi}) = g(\boldsymbol{\varphi})\boldsymbol{\Omega} \cdot \boldsymbol{A} + f(\boldsymbol{\varphi}). \tag{I.3}$$

Although this is a limited class of Hamiltonians, in particular the frequency map for H_0 is highly degenerate, we show at the end of Sec. II that a more general class of Hamiltonians converges to it under renormalization. For H_0, there are invariant tori with frequency vector $\boldsymbol{\omega}_0$ and energy E at those $\boldsymbol{A}$ such that $\boldsymbol{\Omega} \cdot \boldsymbol{A} = 0$ and $\boldsymbol{\omega}_0 \cdot \boldsymbol{A} = E$. Despite the degeneracy of the frequency map for H_0, a special version of the KAM theorem for Hamiltonians (I.1) states that, since $\boldsymbol{\omega}_0$ satisfies a diophantine condition, for sufficiently small and smooth perturbation V, these invariant tori persist [10]. For many large perturbations, however, there can be expected to be no continuations of these tori (to prove this would require an extension of converse KAM theory [11] to the degenerate case (I.1)).

Our renormalization is a combination of a linear coordinate change that shifts the resonances, removal of perturbation terms far from resonance by a nonlinear coordinate change, followed by a truncation to remove unwanted perturbation terms, and a rescaling of time and the actions to enforce some normalization conditions. The truncation step follows the approach of Escande and Doveil [6, 12] for systems with two degrees of freedom, namely, we perform two approximations in this renormalization :

(1) A three-resonance approximation : we keep only the three main Fourier modes of the perturbation V at each step of the transformation.

(2) A mean-value quadratic approximation : we neglect the dependence of the quadratic term in the actions on the angles.

It is important to note, however, that no truncation of higher than quadratic terms in the actions is applied, because we choose our nonlinear coordinate change in such a way that none are generated.

The frequency vector $\boldsymbol{\omega}_0$ is an eigenvector of the matrix $\tilde{N}$ with the eigenvalue $(\tau+1)^{-1} \approx 0.445$, where $\tilde{N}$ is the transpose of the following matrix N with integer coefficients and determinant -1 :

$$N = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{I.4}$$

From N one can generate a sequence of periodic orbits of H_0 (with frequency vectors $\{\boldsymbol{\Omega}_k\}$) approximating motion with frequency vector $\boldsymbol{\omega}_0$, i.e. $\boldsymbol{\varphi}(t) = \boldsymbol{\omega}_0 t + \boldsymbol{\varphi}_0 \bmod 2\pi$:

$$\boldsymbol{\Omega}_k = (p_k/r_k, q_k/r_k, 1) \underset{k\to\infty}{\longrightarrow} \boldsymbol{\omega}_0,$$

where p_k, q_k and r_k are determined by the following recursion relations :

$$\begin{aligned} p_{k+1} &= p_k + 2q_k + r_k, \\ q_{k+1} &= p_k, \\ r_{k+1} &= q_k + r_k, \end{aligned}$$

and $p_0 = 3$, $q_0 = 1$, $r_0 = 1$, i.e. $\boldsymbol{\Omega}_k = \tilde{N}^{-k}\boldsymbol{\Omega}_0$. These relations define a sequence of simultaneous rational approximations $(p_k/r_k, q_k/r_k)$ to the pair $(\tau^2+\tau, \tau)$.

Similarly, a sequence of resonances can be generated by the matrix N from the vector $\boldsymbol{\nu}_1 = (1, 0, 0)$:

$$\boldsymbol{\nu}_k = N^{k-1}\boldsymbol{\nu}_1.$$

The small denominators decrease geometrically to zero with the ratio $(\tau+1)^{-1} \approx 0.445$:

$$\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_k = \tau(\tau+1)^{2-k}.$$

The modes $\boldsymbol{\nu}_k$ are in resonance with the periodic motion with frequency vector $\boldsymbol{\Omega}_k$ (with period r_k) :

$$\boldsymbol{\Omega}_k \cdot \boldsymbol{\nu}_{k+4} = 0.$$

The matrix N can be decomposed in the following way :

$$N = LPL^2, \tag{I.5}$$

where

$$L = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \tag{I.6}$$

and

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{I.7}$$

We notice that $\det L = -\det P = 1$ and that P is a permutation of two elements of the basis; in particular, it satisfies $P^2 = 1$. This decomposition follows from the construction of the Farey sequence for an incommensurate vector $\boldsymbol{\omega} \in \mathbb{R}^3$ [13, 7, 1]. With this decomposition of the matrix N, we construct the renormalization from two operators (one associated with L and the other with P) in a similar way as Ref. [14] did for Hamiltonian systems with two degrees of freedom to investigate the break-up of invariant tori with arbitrary winding ratio. The approximate renormalization transformation we define for $\boldsymbol{\omega}_0$ can be generalized to other frequency vectors by composing these two operators in the order given the Farey decomposition of the frequency vector. For instance, for the spiral mean vector [8, 9], the renormalization is just the operator associated with L.

II. DEFINITION OF THE RENORMALIZATION TRANSFORMATION

The renormalization transformation $\mathcal{R}$ we define for frequency vector $\boldsymbol{\omega}_0$ is composed of two operators : one associated with the matrix P and the other with L. We denote them by $\mathcal{R}_P$ and $\mathcal{R}_L$. The transformation $\mathcal{R}$ for the particular frequency vector $\boldsymbol{\omega}_0$ is defined by

$$\mathcal{R} = \mathcal{R}_L \circ \mathcal{R}_L \circ \mathcal{R}_P \circ \mathcal{R}_L.$$

A Definition of $\mathcal{R}_P$

The renormalization operator $\mathcal{R}_P$ acts on the following Hamiltonians :

$$H(\boldsymbol{A},\boldsymbol{\varphi}) = H_0(\boldsymbol{A}) + h(\boldsymbol{\Omega}\cdot\boldsymbol{A},\boldsymbol{\varphi}), \tag{II.1}$$

where

$$H_0(\boldsymbol{A}) = \boldsymbol{\omega}\cdot\boldsymbol{A} + \frac{1}{2}(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2,$$
$$h(\boldsymbol{\Omega}\cdot\boldsymbol{A},\boldsymbol{\varphi}) = \sum_{i=1}^{3} h_i(\boldsymbol{A})\cos\varphi_i = \sum_{i=1}^{3}(f_i + g_i\boldsymbol{\Omega}\cdot\boldsymbol{A})\cos\varphi_i.$$

It contains a shift of the Fourier modes and a rescaling of time and of the actions. The shift of the Fourier modes is constructed such that it exchanges the mode $(1,0,0)$ and $(0,0,1)$ without changing the mode $(0,1,0)$. More precisely, we require that the new angles $\boldsymbol{\varphi}'$ satisfy :

$$\cos\varphi_1' = \cos\varphi_3,$$
$$\cos\varphi_2' = \cos\varphi_2,$$
$$\cos\varphi_3' = \cos\varphi_1.$$

This is performed by the following linear canonical transformation :

$$(\boldsymbol{A},\boldsymbol{\varphi}) \mapsto (\boldsymbol{A}',\boldsymbol{\varphi}') = (P\boldsymbol{A}, P\boldsymbol{\varphi}),$$

where we recall that P is symmetric and orthogonal. The vectors $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ are changed into $P\boldsymbol{\omega}$ and $P\boldsymbol{\Omega}$. We impose that the images $\boldsymbol{\omega}'$ and $\boldsymbol{\Omega}'$ of the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ by the renormalization $\mathcal{R}_P$ satisfy the following normalization conditions : $\boldsymbol{\Omega}'$ must be of (euclidean) norm one, and the third component of $\boldsymbol{\omega}'$ must be equal to one. Since P is orthogonal, $P\boldsymbol{\Omega}$ is again of norm one, and thus $\boldsymbol{\Omega}' = P\boldsymbol{\Omega}$. The third component of $P\boldsymbol{\omega}$ is equal to ω_1. We rescale the time by a factor ω_1, i.e. we multiply the Hamiltonian by $1/\omega_1$. Then $\boldsymbol{\omega}'$ is given by $\boldsymbol{\omega}' = \boldsymbol{\omega}/\omega_1 = (\omega_3/\omega_1, \omega_2/\omega_1, 1)$.

The quadratic part of the Hamiltonian H_0 becomes $(\boldsymbol{\Omega}'\cdot\boldsymbol{A})^2/(2\omega_1)$. In order to put this quadratic term into the form $(\boldsymbol{\Omega}'\cdot\boldsymbol{A})^2/2$, we rescale the actions by a factor $\lambda_P = 1/\omega_1$, i.e. we change the Hamiltonian H into $\lambda_P H(\boldsymbol{A}/\lambda_P, \boldsymbol{\varphi})$.

In summary, a Hamiltonian H given by Eq. (II.1) is mapped into

$$H'(\boldsymbol{A},\boldsymbol{\varphi}) = \boldsymbol{\omega}'\cdot\boldsymbol{A} + \frac{1}{2}(\boldsymbol{\Omega}'\cdot\boldsymbol{A})^2 + \sum_{i=1}^{3}(f_i' + g_i'\boldsymbol{\Omega}'\cdot\boldsymbol{A})\cos\varphi_i,$$

where $\boldsymbol{\omega}' = (\omega_3/\omega_1, \omega_2/\omega_1, 1)$, $\boldsymbol{\Omega}' = (\Omega_3, \Omega_2, \Omega_1)$, and

$$f_1' = f_3/\omega_1^2,$$

$$f_2' = f_2/\omega_1^2,$$

$$f_3' = f_1/\omega_1^2,$$

$$g_1' = g_3/\omega_1,$$

$$g_2' = g_2/\omega_1,$$

$$g_3' = g_1/\omega_1.$$

The renormalization operator $\mathcal{R}_P$ is equivalent to the 10-dimensional map given by the above equations (we recall that we impose a normalization condition on $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$) :

$$(\boldsymbol{\omega}, \boldsymbol{\Omega}, f_i, g_i; i = 1,2,3) \mapsto (\boldsymbol{\omega}', \boldsymbol{\Omega}', f_i', g_i'; i = 1,2,3).$$

B Definition of $\mathcal{R}_L$

The renormalization operator $\mathcal{R}_L$ is associated to the matrix L and acts on the family of Hamiltonians (II.1). It contains an elimination of the mode $\boldsymbol{\nu}_1$ of the perturbation, and a rescaling procedure (shift of the resonances, rescaling of time and of the actions) such that the image of a Hamiltonian H given by Eq. (II.1) is of the same general form as H and describes the system on a smaller scale in phase space and at a longer time scale. This operator is similar to the approximate renormalization constructed for the spiral mean torus in Refs. [8, 9].

We eliminate the mode $\boldsymbol{\nu}_1$ of the scalar functions f and g, by a near-identity canonical transformation. We perform a Lie transformation generated by a function $S(\boldsymbol{A},\boldsymbol{\varphi})$. The image of a Hamiltonian H is given by :

$$H' = \exp(\hat{S})H = H + \{S,H\} + \frac{1}{2}\{S,\{S,H\}\} + \cdots\,, \tag{II.2}$$

where $\{\,,\,\}$ denotes the Poisson bracket :

$$\{f,g\} = \frac{\partial f}{\partial \boldsymbol{\varphi}} \cdot \frac{\partial g}{\partial \boldsymbol{A}} - \frac{\partial g}{\partial \boldsymbol{\varphi}} \cdot \frac{\partial f}{\partial \boldsymbol{A}},$$

and the operator $\hat{S}$ acts on H by $\hat{S}H = \{S,H\}$. The generating function is chosen to be of the form :

$$S(\boldsymbol{A},\boldsymbol{\varphi}) = (z + y\boldsymbol{\Omega}\cdot\boldsymbol{A})\sin\varphi_1, \tag{II.3}$$

with :

$$z = -f_1/\omega_1,$$
$$y = -(g_1 - f_1\Omega_1/\omega_1)/\omega_1.$$

Then the generating function S satisfies :

$$\{S,H_0\} = (Q(\boldsymbol{A}) - h_1(\boldsymbol{A}))\cos\varphi_1,$$

with $Q(\boldsymbol{A}) = y\Omega_1(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2$. The image of a Hamiltonian H given by Eq. (II.1) is :

$$\begin{aligned} H' =& H_0 + h_1\cos\varphi_1 + h_2\cos\varphi_2 + h_3\cos\varphi_3 + \{S,H_0\} + \{S,h_1\cos\varphi_1\} \\ &+\{S,h_2\cos\varphi_2\} + \{S,h_3\cos\varphi_3\} + \frac{1}{2}\{S,\{S,H_0\}\} + O(\varepsilon^3). \end{aligned}$$

We neglect the term $Q(\boldsymbol{A})\cos\varphi_1$ produced by the terms $\{S,H_0\} + h_1\cos\varphi_1$, i.e. we neglect the angle-dependence of the quadratic terms in the actions. The term $\{S,h_1\cos\varphi_1\}$ is of degree one in the actions and contains the modes $\mathbf{0}$ and $2\boldsymbol{\nu}_1$. We neglect the mode $2\boldsymbol{\nu}_1$ and we eliminate the mode $\mathbf{0}$ of the linear term in the actions by a shift : $\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{a}$ where $\boldsymbol{a}$ is of order ε^2 (in the direction of $\boldsymbol{\Omega}$). We neglect the modes with frequency vectors $\boldsymbol{\nu}_1 \pm \boldsymbol{\nu}_2$ produced by the term $\{S,h_2\cos\varphi_2\}$. The term $\{S,h_3\cos\varphi_3\}$ generates the modes

$\boldsymbol{\nu}_1 \pm \boldsymbol{\nu}_3$. We neglect the mode $\boldsymbol{\nu}_1 + \boldsymbol{\nu}_3$ and keep the next relevant Fourier mode $\boldsymbol{\nu}_1 - \boldsymbol{\nu}_3$ whose amplitude is denoted $h_3' = f_3' + g_3' \boldsymbol{\Omega} \cdot \boldsymbol{A}$, where

$$f_3' = (z g_3 \Omega_1 + y f_3 \Omega_3)/2, \tag{II.4}$$

$$g_3' = (\Omega_1 + \Omega_3) y g_3 / 2. \tag{II.5}$$

The term $\{S, Q\cos\varphi_1\}/2$ produced by $\{S, \{S, H_0\}\}/2$ gives a contribution to the quadratic part of the Hamiltonian. The new quadratic part is equal to $m(\boldsymbol{\Omega} \cdot \boldsymbol{A})^2/2$ where m is given by :

$$m = 1 + \frac{3}{2} y^2 \Omega_1^2. \tag{II.6}$$

Then after the elimination of the mode $\boldsymbol{\nu}_1$, the new Hamiltonian is equal to :

$$H' = \boldsymbol{\omega} \cdot \boldsymbol{A} + m(\boldsymbol{\Omega} \cdot \boldsymbol{A})^2/2 + h_2 \cos\varphi_2 + h_3 \cos\varphi_3 + h_3' \cos(\varphi_1 - \varphi_3), \tag{II.7}$$

where h_3' is given by Eqs. (II.4) and (II.5).

We shift the Fourier modes according to the following linear canonical transformation :

$$(\boldsymbol{A}, \boldsymbol{\varphi}) \mapsto (\boldsymbol{A}', \boldsymbol{\varphi}') = (L^{-1}\boldsymbol{A}, \tilde{L}\boldsymbol{\varphi}),$$

such that $\cos\varphi_2 = \cos\varphi_1'$, $\cos\varphi_3 = \cos\varphi_2'$ and $\cos(\varphi_1 - \varphi_3) = \cos\varphi_3'$. The vectors $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}$ are changed in the following way :

$$\boldsymbol{\Omega}' = \frac{\tilde{L}\boldsymbol{\Omega}}{\|\tilde{L}\boldsymbol{\Omega}\|} = (\Omega_2, \Omega_3, \Omega_1 - \Omega_3)/(1 + \Omega_3(\Omega_3 - 2\Omega_1)),$$
$$\boldsymbol{\omega}' = \frac{\tilde{L}\boldsymbol{\omega}}{(\tilde{L}\boldsymbol{\omega})_3} = (\omega_2/(\omega_1 - 1), 1/(\omega_1 - 1), 1).$$

We rescale time in such a way that the linear term in the actions of H_0 is equal to $\boldsymbol{\omega}'$: we multiply the Hamiltonian H' by a factor $1/(\omega_1 - 1)$. The quadratic term of the new Hamiltonian is equal to $m\|\tilde{L}\boldsymbol{\Omega}\|^2(\boldsymbol{\Omega}' \cdot \boldsymbol{A})^2/(2\omega_1 - 2)$. In order to map this quadratic part into $(\boldsymbol{\Omega}' \cdot \boldsymbol{A})^2/2$, we rescale the actions by a factor

$$\lambda_L = m\|\tilde{L}\boldsymbol{\Omega}\|^2/(\omega_1 - 1),$$

by changing the Hamiltonian H' into $\lambda_L H'(\boldsymbol{A}/\lambda_L, \boldsymbol{\varphi})$.

After these elimination and rescaling procedures, the Hamiltonian H is mapped into

$$H'' = \boldsymbol{\omega}' \cdot \boldsymbol{A} + \frac{1}{2}(\boldsymbol{\Omega}' \cdot \boldsymbol{A})^2 + h_1'' \cos\varphi_1 + h_2'' \cos\varphi_2 + h_3'' \cos\varphi_3,$$

where $h_i'' = f_i'' + g_i''\boldsymbol{\Omega}' \cdot \boldsymbol{A}$ and

$$f_1'' = f_2 c_f,$$

$$f_2'' = f_3 c_f,$$

$$f_3'' = f_3' c_f,$$

$$g_1'' = g_2 c_g,$$

$$g_2'' = g_3 c_g,$$

$$g_3'' = g_3' c_g,$$

with $c_f = m\|\tilde{L}\boldsymbol{\Omega}\|^2/(\omega_1 - 1)^2$ and $c_g = \|\tilde{L}\boldsymbol{\Omega}\|/(\omega_1 - 1)$.

The renormalization transformation $\mathcal{R}_L$ is equivalent to the 10-dimensional map

$$(\boldsymbol{\omega}, \boldsymbol{\Omega}, f_i, g_i; i = 1,2,3) \mapsto (\boldsymbol{\omega}', \boldsymbol{\Omega}', f_i'', g_i''; i = 1,2,3),$$

given by the above equations.

C Renormalization transformation $\mathcal{R}$

Combining the above two operators, the renormalization $\mathcal{R} = \mathcal{R}_L^2 \mathcal{R}_P \mathcal{R}_L$ acts in the following way : a canonical transformation eliminates the three main Fourier modes and produce the next three resonances which are the main Fourier modes at a smaller scale in phase space. A rescaling procedure shifts these resonances into the original Fourier modes, and normalizes the new Hamiltonian by rescaling the time and the actions.

The rescaling procedure acts on $\boldsymbol{\omega}_0$ in the following way : $\boldsymbol{\omega}_0$ is changed into $\boldsymbol{\omega}^{(1)} = (1/(3-\tau^2), 1/(3\tau - \tau^3), 1)$ by $\mathcal{R}_L$, then into $\boldsymbol{\omega}^{(2)} = (3 - \tau^2, \tau^{-1}, 1)$ by $\mathcal{R}_P$, then into $\boldsymbol{\omega}^{(3)} = (1/(2\tau - \tau^3), 1/(2 - \tau^2), 1)$ by $\mathcal{R}_L$, and after the final step $\mathcal{R}_L$, into $\boldsymbol{\omega}^{(4)} = \boldsymbol{\omega}_0$.

Thus the renormalization $\mathcal{R}$ is defined for a *fixed* frequency vector $\boldsymbol{\omega}_0$. In the same way, the action of $\mathcal{R}$ on $\boldsymbol{\Omega}$ reduces to the map :

$$\boldsymbol{\Omega}' = \frac{\tilde{N}\boldsymbol{\Omega}}{\|\tilde{N}\boldsymbol{\Omega}\|}. \tag{II.8}$$

The spectrum of $\tilde{N}$ is real and consists of the following eigenvalues $-\tau \approx -1.2469$, $(\tau+1)^{-1} \approx 0.4450$, $1+\tau^{-1} \approx 1.8019$. Thus on iterating the map (II.8), the vector $\boldsymbol{\Omega}$ converges to the eigenvector (denoted $\boldsymbol{\Omega}^{(3)}$, with euclidean norm one) associated with the largest eigenvalue of $\tilde{N}$. Restricting to $\boldsymbol{\Omega} = \boldsymbol{\Omega}^{(3)}$, the renormalization $\mathcal{R}$ reduces to a 6-dimensional map $(f_i, g_i; i=1,2,3) \mapsto (f_i', g_i'; i=1,2,3)$.

Before analyzing the dynamics of the renormalization transformation, we note that its domain of definition can be also extended to the larger family of Hamiltonians :

$$H(\boldsymbol{A},\boldsymbol{\varphi}) = \boldsymbol{\omega}_0 \cdot \boldsymbol{A} + \frac{1}{2}\boldsymbol{A}\cdot M\boldsymbol{A} + \boldsymbol{g}(\boldsymbol{\varphi})\cdot\boldsymbol{A} + f(\boldsymbol{\varphi}), \tag{II.9}$$

where M is a 3×3 symmetric matrix with non-zero trace, and $\boldsymbol{g}$ is a three-dimensional vector-valued function. Applying the renormalization changes the matrix M into :

$$\mathcal{R}(M) = \frac{\tilde{N}MN}{\mathrm{tr}\,(\tilde{N}MN)}, \tag{II.10}$$

where tr $(\tilde{N}MN)$ is the trace of the matrix $\tilde{N}MN$. By iterating the map (II.10), the matrix converges to $\boldsymbol{\Omega}^{(3)}\otimes\boldsymbol{\Omega}^{(3)}$. Moreover, $\boldsymbol{g}$ is renormalized into $\tilde{N}\boldsymbol{g}$ which tends to align with $\boldsymbol{\Omega}^{(3)}$ under iteration. Then the Hamiltonians (II.9) tend to the degenerate Hamiltonians (I.1) under iteration of the renormalization. Thus, we expect that the Hamiltonians (II.9) belong to the same universality class as the Hamiltonians (I.1).

III. RENORMALIZATION DYNAMICS

The renormalization transformation $\mathcal{R}$ has the following properties : $\mathcal{R}$ has an attractive integrable fixed point H_0 given by

$$H_0(\boldsymbol{A}) = \boldsymbol{\omega}_0\cdot\boldsymbol{A} + \frac{1}{2}(\boldsymbol{\Omega}^{(3)}\cdot\boldsymbol{A})^2,$$

and another non-integrable fixed point H_* which lies on the boundary of the domain of attraction of H_0. Outside the closure of the domain of attraction of H_0, the iterations of renormalization diverge to infinity. The renormalization dynamics has the same qualitative features as the renormalization for the golden mean torus for Hamiltonian systems with two degrees of freedom [2, 3]. The properties of critical tori are given by the analysis of the renormalization around the non-trivial fixed point H_*. In particular, the existence of a non-trivial fixed point with only one unstable direction for an exact renormalization would imply self-similarity of critical invariant tori. The stable manifold of H_* is of codimension one; the linearized renormalization around H_* has only one eigenvalue of modulus larger than one : $\delta \approx 4.5591$. The value of the total rescaling coefficient in the actions is $\lambda \approx 13.1890$, and the rescaling coefficient of time is $\tau + 1 \approx 2.2469$.

We also find a critical cycle with period 7 on the critical surface. This periodic cycle is obtained from the non-trivial fixed point H_* by changing the sign of the Fourier coefficients f_i and g_i. The existence of this cycle can be explained by symmetry reasons, as was done for the critical cycle with period 3 for the golden mean case [15]. In particular, it has the same critical exponents and scaling factors as the non-trivial fixed point, and belongs to the same universality class.

IV. CONCLUSION

We have defined two elementary operators $\mathcal{R}_P$ and $\mathcal{R}_L$. With these operators, we have defined an approximate renormalization in order to study invariant tori with frequency vector $\boldsymbol{\omega}_0 = (\tau^2 + \tau, \tau, 1)$. The renormalization has a hyperbolic fixed point with codimension-one stable manifold. Consequently, we expect invariant tori with this frequency vector to break in a self-similar way. In order to give a firm basis to this statement, it will be interesting to investigate numerically the break-up in one-parameter families.

We note also that the approximate renormalization we define in this paper can be generalized to an arbitrary incommensurate frequency vector $\boldsymbol{\omega}_0$ by applying the operators $\mathcal{R}_P$ and $\mathcal{R}_L$

in the order given by the Farey sequence of $\boldsymbol{\omega}_0$. For a periodic Farey sequence, we would expect to find a critical set consisting of hyperbolic fixed points or periodic cycles, or possibly normally hyperbolic invariant circles or singular analogues (as for the spiral mean [9]). For non-periodic Farey sequences, the critical set is likely to inherit chaotic dynamics from the Farey process, as for the Siegel disk [16] and other cases, but this point has not been investigated yet.

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