Periodic and chaotic trajectories of the second species for the n-centre problem

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Abstract

For the n-centre problem of one particle moving in the potential of attracting centres of small mass fixed in an arbitrary smooth potential and magnetic field, we prove the existence of periodic and chaotic trajectories shadowing sequences of collision orbits. In particular, we obtain large subshifts of solutions of this type for the circular restricted 3-body problem of celestial mechanics. Poincaré had conjectured existence of the periodic ones and given them the name "second species solutions".

Keywords: n-centre problem, three-body problem, second species orbits, collisions, regularisation

1 Introduction

Let Q be a smooth manifold and $P = \{p_1, \dots, p_n\}$ a finite set in Q. Consider a Lagrangian system (L_ε) with configuration space $Q \setminus P$ and Lagrangian

$$L_\varepsilon(q, \dot q) = L(q, \dot q) - \varepsilon V(q). \tag{1.1}$$

We assume that L is C^4 on TQ and quadratic in the velocity:

$$L(q, \dot q) = T(q, \dot q) + \langle \omega(q), \dot q\rangle - W(q), \tag{1.2}$$

where $T = \langle A(q)\dot q, \dot q\rangle/2$ is a Riemannian metric on Q, and that V is a C^4 function on $Q \setminus P$ having Newtonian singularities at the points $p_i \in P$. This means that in a neighbourhood U_i of any point $p_i \in P$,

$$V(q) = -\frac{f_i(q)}{\text{dist}\,(q, p_i)}, \tag{1.3}$$

where f_i is a C^4 function on U_i with $f_i(p_i) > 0$, and the distance is defined by means of the Riemannian metric T. We call the system (L_ε) with the Lagrangian (1.1) the n-centre problem. We will study this problem for small $\varepsilon > 0$ as a perturbation of the Lagrangian system (L) with the Lagrangian (1.2). Examples are given by the classical integrable 2-centre problem of Euler and by the restricted circular 3-body problem in the rotating frame, when one of the masses is small. For physical reasons (and also to simplify regularisation of singularities) we assume that Q is 2 or 3-dimensional.

Let

$$H_\varepsilon = H + \varepsilon V, \qquad H(q, \dot q) = T(q, \dot q) + W(q),$$

be the energy integral. We fix E such that the domain $D = \{q \in Q \mid W(q) < E\}$ contains the set P and study the system (L_ε) on the energy level $\{H_\varepsilon = E\} \subset T(Q \setminus P)$.

Suppose that the system (L) with the Lagrangian (1.2) has nondegenerate (definition to be recalled shortly) collision trajectories $\gamma_k : [0, \tau_k] \to D$, $k \in K$ (a finite set), of energy E such that $\gamma_k(0), \gamma_k(\tau_k) \in P$ and γ_k does not pass through other points of the set P. More precisely, $\gamma_k(0) = p_{\alpha_k}$, $\gamma_k(\tau_k) = p_{\beta_k}$, and $\gamma_k((0, \tau_k)) \subset D \setminus P$. A sequence $(k_i \in K)_{i \in \mathbf{Z}}$ will be called a *chain* if $\gamma_{k_i}(\tau_{k_i}) = \gamma_{k_{i+1}}(0)$ and $\dot\gamma_{k_i}(\tau_{k_i}) \neq \pm\dot\gamma_{k_{i+1}}(0)$ for all $i \in \mathbf{Z}$.

For each $k \in K$ let W_k be a neighbourhood of $\gamma_k([0, \tau_k])$. We say that a trajectory $\gamma : \mathbf{R} \to D \setminus P$ *shadows* the chain $(\gamma_{k_i})_{i \in \mathbf{Z}}$ if there exists an increasing sequence $(t_i)_{i \in \mathbf{Z}}$ such that $\gamma([t_i, t_{i+1}]) \subset W_{k_i}$.

Theorem 1.1 *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and any chain $(\gamma_{k_i})_{i \in \mathbf{Z}}$:*

- *there exists a trajectory $\gamma : \mathbf{R} \to D \setminus P$ of energy E for the system (L_ε) shadowing the chain $(\gamma_{k_i})_{i \in \mathbf{Z}}$, and it is unique (up to a time shift) if the W_k are chosen small enough.*

- *the orbit γ converges to the chain of collision orbits as $\varepsilon \to 0$: there exists an increasing sequence $(t_i)_{i \in \mathbf{Z}}$ such that*

$$\max_{t_i \le t \le t_{i+1}} \text{dist}\,(\gamma(t), \gamma_{k_i}([0, \tau_{k_i}])) \le C\varepsilon, \tag{1.4}$$

  where the constant $C > 0$ depends only on the set K of collision orbits.

- *the orbit γ avoids collision by a distance of order ε: there exists a constant $c \in (0, C)$, depending only on K such that*

$$c\varepsilon \le \min_{t_{i-1} \le t \le t_{i+1}} \text{dist}\,(\gamma(t), p_{\alpha_{k_i}}). \tag{1.5}$$

A more precise formulation is given below in Theorem 5.1. Theorem 1.1 implies that there is an invariant subset in $\{H_\varepsilon = E\}$ on which the system is a suspension of a subshift of finite type. The topological entropy is positive provided the graph with the set of vertices K and the set of edges

$$G = \{(k,l) \in K^2 \mid \beta_k = \alpha_l, \dot\gamma_k(\tau_k) \neq \pm\dot\gamma_l(0)\} \tag{1.6}$$

has a connected branched subgraph. Note that in the case of a periodic sequence $(k_i)_{i\in\mathbf{Z}}$, uniqueness of the trajectory γ implies that it is also periodic, with the minimal period consistent with shadowing the periodic sequence.

The condition that γ_k is *nondegenerate* can be expressed in five equivalent ways.

1. The first definition is that the points $p_{\alpha_k} = \gamma_k(0)$ and $p_{\beta_k} = \gamma_k(\tau_k)$ are not conjugate along γ_k on the energy level $\{H = E\}$. More precisely, if $\xi(t) \not\equiv 0$ is a solution of the variational equation along $\gamma_k(t)$ such that $\xi(0) = 0$ and the energy is fixed, i.e. $\langle L_{\dot q}(\gamma_k(0), \dot\gamma_k(0)), \dot\xi(0)\rangle = 0$, then $\xi(\tau_k)$ is not tangent to γ_k at p_{β_k} (linearised solutions for which $\xi(\tau_k)$ is tangent to γ_k at p_{β_k} lead to nearby full solutions which pass through p_{β_k} to first order but in a possibly first order different time). Note that $\dot\xi(0)$ is correctly defined (independent of the coordinate system), because $\xi(0) = 0$.

2. Let $S_\alpha = T_{p_\alpha}Q \cap \{H = E\}$. Then S_α is a sphere in $T_{p_\alpha}Q$. Let $\phi_t : TQ \to TQ$ be the phase flow of the system (L) and $\pi : TQ \to Q$ the projection. Trajectories connecting p_α with p_β correspond to solutions of the equation $g(v,\tau) = \pi(\phi_\tau(v)) = p_\beta$, $v \in S_\alpha$, $\tau > 0$. Then γ is a nondegenerate collision orbit if $(\dot\gamma_k(0), \tau_k)$ is a solution of this equation at which rank $dg(\dot\gamma_k(0), \tau_k)$ is maximal, i.e. equals $\dim Q$.

3. The third formulation is variational. Let $\Omega = \Omega(Q, p_\alpha, p_\beta)$ be the space of $W^{1,2}$ curves $u : [0,1] \to Q$ such that $u(0) = p_\alpha$, $u(1) = p_\beta$. Any point $(u,\tau) \in \Omega \times \mathbf{R}^+$ defines a curve $\gamma : [0,\tau] \to Q$ by $\gamma(t) = u(t/\tau)$. Let

$$F(u,\tau) = \int_0^\tau (L(\gamma(t), \dot\gamma(t)) + E)\, dt \tag{1.7}$$

be its action. Then F is a C^2 functional on $\Omega \times \mathbf{R}^+$ and its critical points correspond to trajectories of energy E connecting p_α with p_β. A collision orbit $\gamma_k : [0,\tau_k] \to D$ is nondegenerate if (u,τ_k), where $u(t) = \gamma_k(t\tau_k)$, is a nondegenerate critical point for F.

4. The next definition is similar, but F is replaced by the Maupertuis–Jacobi functional $J : \Omega \to \mathbf{R}$,

$$J(\gamma) = \int_0^1 g_E(\gamma(t), \dot\gamma(t))\, dt, \qquad g_E(q,\dot q) = 2\sqrt{(E - W(q))T(q,\dot q)} + \langle\omega(q), \dot q\rangle$$

(g_E is called the Jacobi metric). Since $W|_D < E$, the functional J is well defined on the set of curves $\gamma : [0,1] \to D$. The functional J is independent of the parametrisation of γ, so nondegeneracy means nondegeneracy on the set of curves with a fixed choice of parametrisation, for example, parametrised so that $T(\gamma(t), \dot{\gamma}(t)) = \text{const}$.

5. The final definition is the most suitable one for verification in concrete examples: represent the general solution of the system (L) as $q = q(\lambda, t)$, where λ is a parameter of dimension $2 \dim Q$. Then $H(q(\lambda, t), \dot{q}(\lambda, t)) = h(\lambda)$ for some function h. Collision orbits with energy E connecting p_α with p_β correspond to solutions of the system of equations

$$q(\lambda, 0) = p_\alpha, \quad q(\lambda, \tau) = p_\beta, \quad h(\lambda) = E, \tag{1.8}$$

in the variables λ, τ. The nondegeneracy condition is that the rank of the system (1.8) at the solution is maximal, i.e. equals $2 \dim Q + 1$.

By standard results of the calculus of variations [24], these definitions of non-degeneracy are equivalent. For example, the equivalence of 3 and 4 follows from the inequality $F(u, \tau) \geq J(u)$ which turns out to be an equality precisely when $H \equiv E$ along the curve $\gamma(t) = u(t/\tau)$.

Remarks. 1. The assertion of Theorem 1.1, apart from the uniqueness of γ, also holds if the assumption about the nondegeneracy of γ_k is replaced by the minimality of γ_k. More precisely, it is sufficient that there exist a neighbourhood W_k of $\gamma_k([0, \tau_k])$ such that

$$\int_0^\tau (L(\gamma(t), \dot{\gamma}(t)) + E)\, dt > \int_0^{\tau_k} (L(\gamma_k(t), \dot{\gamma}_k(t)) + E)\, dt$$

for any curve $\gamma : [0, \tau] \to W_k$ connecting p_{α_k} to p_{β_k} that is different from γ_k. We will prove the nondegenerate version of Theorem 1.1. In the degenerate case, instead of the implicit function theorem one uses the calculus of variations as in [5, 6], but for reasons of space we do not give the proof here. The assertion of Theorem 1.1 also holds if γ_k is an isolated minimax critical point for the action functional: one can use the multibump construction as in e.g. [8].

2. One can allow the nonsingular part L of the Lagrangian L_ε also to depend on ε. Then all the results remain true with L replaced by $L|_{\varepsilon=0}$. Also V may depend on ε. For simplicity of notation, however, we assume that L and V are independent of ε.

3. If we do not mind the shadowing orbits coming arbitrarily close to collision, or even passing through the singularities (Newtonian singularities can be regularised), then the condition $\dot{\gamma}_{k_i}(\tau_{k_i}) \neq -\dot{\gamma}_{k_{i+1}}(0)$ in the definition of the chain can be skipped. Theorem 1.1 remains true with the exception of the last statement (inequality (1.5)).

Note that the conditions of Theorem 1.1 are formulated in terms of the Lagrangian L and the set P only, not involving the potential V provided it has Newtonian singularities at P.

Corollary 1.1 *Suppose Q is a closed manifold and*

$$E > -\min_{q,\dot q} L(q,\dot q) = \min_q \left(\frac{1}{2}\langle A^{-1}(q)\omega(q), \omega(q)\rangle + W(q)\right).$$

Then for any $n \geq 2$ and almost all points $p_1, \dots, p_n \in Q$, for small $\varepsilon > 0$, there exist chaotic trajectories of energy E close to chains of collision orbits.

Proof: Indeed, the assumption implies that the Jacobi metric g_E is a positive definite Finsler metric on Q, and by Morse theory any two generic points can be connected by an infinite number of nondegenerate geodesics of the metric g_E, i.e. trajectories of energy E. Thus any two points in a generic finite set $P \subset Q$ can be connected by an infinite number of nondegenerate collision trajectories of energy E, and any other points in P do not lie on these trajectories. ∎

For example, if $n = 2$ and $Q = S^2$ with $L = T$ the standard Riemannian metric on S^2, then there exist chaotic trajectories provided p_1 and p_2 are not antipodal points of S^2.

To further illustrate the use of Theorem 1.1, we apply it to the restricted 3-body problem in sections 2 and 3, proving existence of a type of orbit conjectured 100 years ago by Poincaré but for which rigorous justification had been provided in only a few special cases. The remainder of the paper (sections 4–6) contains the proof of Theorem 1.1, which proceeds via regularisation of the Newtonian singularities, construction of a discrete action functional, and the implicit function theorem.

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2 Application to the planar circular restricted 3-body problem

Consider the planar restricted circular 3-body problem (Sun, Jupiter, and Asteroid) and suppose that the mass of Jupiter is small compared to the Sun. We suppose the masses to be normalised to $1-\varepsilon$ (Sun), ε (Jupiter), and 0 (Asteroid), with the centre of mass O stationary and the first two masses in circular orbits about O, having separation and angular frequency both normalised to 1.

To apply Theorem 1.1, consider the motion of the Asteroid in the frame rotating at angular frequency 1 about the Sun. This yields a Lagrangian system of the form (1.1), where L and V are given by

$$L = \frac{1}{2}|\dot q|^2 + x\dot y - y\dot x + \frac{1}{2}|q|^2 + \frac{1}{|q|} \tag{2.1}$$

and

$$V(q) = \frac{1}{|q|} - \frac{1}{|q-j|} + x. \tag{2.2}$$

Here, $q = (x, y) \in \mathbf{R}^2$ and $j = (1, 0)$ is the position of Jupiter (the Sun is at $(0, 0)$). Hence L has the form (1.2), V has the form (1.3), and $Q = \mathbf{R}^2 \setminus \{0\}$. For the restricted 3-body problem, the energy integral in the rotating coordinate frame,

$$H_\varepsilon = T + W + \varepsilon V = \frac{1}{2}|\dot{q}|^2 - \frac{1}{2}|q|^2 - \frac{1-\varepsilon}{|q|} - \frac{\varepsilon}{|q-j|} + \varepsilon x,$$

is called the Jacobi integral (sometimes with $\frac{\varepsilon^2}{2}$ subtracted, and multiplied by -2). The energy constant $H_\varepsilon = E$ is traditionally replaced by the Jacobi constant $C = -2E$. Denote the energy of the Asteroid in the fixed coordinate frame by $\mathcal{E}$ and its angular momentum about O by h. Then the Jacobi constant equals $C = 2(h - \mathcal{E})$.

Theorem 2.1 *In the planar circular restricted three-body problem with masses $1-\varepsilon, \varepsilon, 0$, for all values of the Jacobi constant $C \in (-\sqrt{8}, +3)$ there exists a dense subset S_C of rationals in the set A_C of allowed frequencies for Kepler ellipses crossing the unit circle, such that for all finite subsets $T \subset S_C$ there exists $\varepsilon_0 > 0$ such that for any sequence $\sigma = (\Omega_n = m_n/k_n)_{n\in\mathbf{Z}}$ in T and $0 < \varepsilon < \varepsilon_0$ there is a unique trajectory of Jacobi constant C near to a chain of collision trajectories formed by transforming ellipses of frequencies Ω_n traversed m_n times to the rotating frame, and it converges to the chain as $\varepsilon \to 0$.*

For periodic sequences σ, these solutions are a special case of the "second species solutions" whose existence Poincaré claimed to prove in chapter XXXII of [18] (published 100 years ago). "Second species solutions" are ones which in the limit as $\varepsilon \to 0$ tend to chains of collision orbits, which he called "orbites à chocs". He claimed to treat the "unrestricted case" of masses $1, \varepsilon m_1, \varepsilon m_2$ and to construct non-planar solutions too. His sketch proof of all of these, however, leaves a lot to be desired (see Levy's comments on p.629 of [19]).

The planar circular restricted case is already highly non-trivial to analyse. Many numerics were performed: for a nice survey, see [11]. Some mathematical results were obtained by Alexeyev (reviewed in part II of [1]), Perko (several references, e.g. [17]), Henrard [12], Bruno [7], Gomez and Olle [10], and Marco and Niederman [15]. Several of the above proofs, however, are incomplete from a mathematical viewpoint. The paper [15] gives a mathematical proof of the existence of two-segment second species periodic orbits, meaning ones whose orbite à chocs consists of alternation between two collision trajectories; it was not checked there that these collision trajectories do not suffer additional collisions between the intended ones, but Marco filled this gap in a private communication to us. The phenomenon of additional collisions really occurs for many other collision trajectories as the Jacobi constant is varied, so is a genuine problem. We go a long way beyond these references (and the original goal of Poincaré) by constructing solutions shadowing both periodic sequences with an arbitrary number of collisions per period and aperiodic sequences of collision orbits.

Aperiodic sequences of collision orbits were treated in the collinear case by Saari and Xia [20], but their shadowing orbits are constructed for a regularised system and collide just as often as the given sequence of collision orbits, so do not generate solutions of the three-body problem.

In the next section we construct a set of collision trajectories with the given value of C for the case $\varepsilon = 0$ (Lemma 3.1), check their non-degeneracy (Lemma 3.2) and construct a non-trivial set of chains of collision trajectories which change direction at each collision (Lemma 3.3). These three things are most easily done in the non-rotating frame, to which we shall now revert.

Remarks: 1. Given a choice of subset T in Theorem 2.1, (1.5) shows that there exists $c > 0$ such that the constructed orbits remain at least $c\varepsilon$ from collision, so the result can even be applied to an extended spherically symmetric Jupiter (its external field is equivalent to that of a point mass), provided its radius is less than $c\varepsilon$. Similarly, the Sun can be an extended spherically symmetric body provided the eccentricity of the ellipses with frequencies in T is bounded away sufficiently from 1 to prevent penetration of the Asteroid into the Sun. Spherical symmetry can even be abandonned if the Sun and Jupiter maintain the same orientations relative to each other (as for the Earth-Moon system). The potential V still has the form (1.3) outside a sphere S_J containing Jupiter in the concentric sphere of half the radius, and then V can be artificially extended to satisfy (1.3) inside S_J too. As already mentioned, Theorem 1.1 can be modified to allow V to depend on ε. The resulting solutions avoid S_J if its radius is less than $c\varepsilon$. The effect of asphericity of the Sun simply modifies the potential W, so requires no further analysis.

2. It would be interesting to see whether our Theorem 1.1 could also be used to prove existence of second species orbits not confined to the plane of rotation of the Sun and Jupiter, which Poincaré also claimed to exist. Poincaré observed that non-planar collision trajectories (when Jupiter's mass is zero) must either connect the same point of Jupiter's orbit or connect points on opposite sides of the Sun [18]. The set of ellipses in 3D with given Jacobi constant, starting and ending at a given point at given times is determined by five conditions on the six orbital elements, so the required non-degeneracy does not hold for any of this type of collision trajectory. On the other hand, the set of ellipses in 3D joining two given points of Jupiter's orbit on opposite sides of the Sun at given times with given Jacobi constant is determined by six conditions on the six orbital elements, thus most collision trajectories of this type are likely to be non-degenerate. If one could prove non-degeneracy for a subset of them, one could apply Theorem 1.1 again.

3 Construction of collision orbits

Firstly, we summarise the well known facts (e.g. [11]) about which orbits of the Kepler problem cross the unit circle, as this is a prerequisite for a collision orbit.

We restrict attention to the elliptical ones as they will suffice for our purposes.

Denote the semi-major axis of an elliptical orbit by a and its eccentricity by e. It is given in polar coordinates (r,θ) about O, with $\theta = 0$ corresponding to its perihelion, by

$$r = \frac{a(1-e^2)}{1+e\cos\theta}.$$

Its Jacobi constant has the value

$$C = a^{-1} \pm 2\sqrt{a(1-e^2)}, \tag{3.1}$$

with $+$ sign for motion in the same direction as the rotation of the Sun and Jupiter (direct or prograde) and $-$ for motion in the opposite direction (retrograde). Thus instead of (a,e) we can give (a^{-1},C), and deduce $e \in [0,1]$ from

$$e^2 = 1 - \frac{1}{4a}(C-a^{-1})^2. \tag{3.2}$$

The sense of motion is prograde if $C > a^{-1}$, retrograde if $C < a^{-1}$, and the case $C = a^{-1}$ has to be excluded because it gives degenerate ellipses of eccentricity 1 which correspond to line segments through O.

The ellipses crossing the unit circle are those for which $a(1-e^2)$ lies between $(1-e)$ and $(1+e)$, i.e. $a^{-1} \in (1-e, 1+e)$. In the variables (a^{-1},C) this translates to

$$(C-a^{-1})^2 < 8 - 4a^{-1}.$$

This region is drawn in Figure 1. Note the limiting cases of ellipses tangent to the unit circle ($(C-a^{-1})^2 = 8-4a^{-1}$), line segments ($C = a^{-1}$), and parabolae ($a^{-1} = 0$). The angular frequency (mean motion) Ω of the ellipse is given by

$$\Omega = a^{-3/2}. \tag{3.3}$$

Thus given $C \in (-\sqrt{8}, +3)$ we obtain a non-empty set A_C of frequencies of ellipses that cross the unit circle, which consists of one interval for $C \in [2,3)$ or $(-\sqrt{8}, 0]$ and two intervals if $C \in (0,2)$.

One strategy (though not the only one) to obtain a collision orbit of given C is to choose a so that Ω is rational, say m/k in lowest terms, start Jupiter and the Asteroid at either of the two intersection points, and let Jupiter make k complete revolutions and the Asteroid make m complete revolutions (in the direction corresponding to the sign of $C - a^{-1}$), after which they will collide again (at exactly the same point). The only catch is that they might collide earlier, at the other intersection point of the ellipse with the unit circle, which we call an "early collision". This would not be a great problem, as it still generates a collision orbit, just a shorter one, but it is not so easy to determine whether it is non-degenerate.

So our strategy is to determine for each C a set of rationals such that the ellipses of that frequency and in the direction of motion corresponding to the

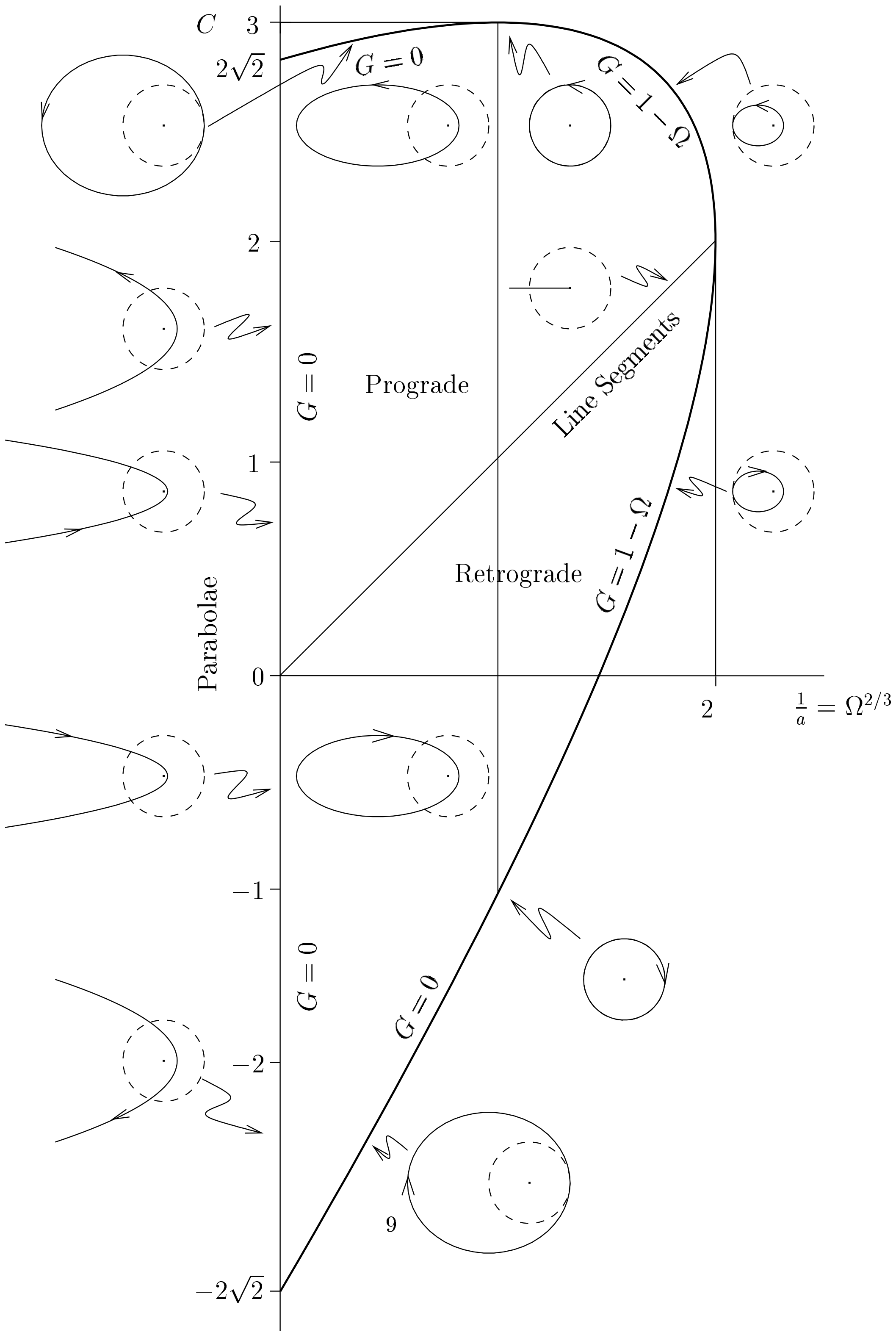

Figure 1: The region of (a^{-1}, C) for which the Kepler ellipse crosses the unit circle.

sign of $C - a^{-1}$, do not have an early collision. Then we shall prove their non-degeneracy, and finally we shall construct chains of these collision orbits for which the direction changes at each collision.

Lemma 3.1 *For all $C \in (-\sqrt{8}, +3)$ there is a dense set S_C of rational Ω in the allowed set A_C such that there is no early collision.*

Proof: Given (a^{-1}, C) in the region where the ellipse crosses the unit circle, define $f \in [0, \pi]$ to be the angle between the perihelion and either of the intersections with the unit circle, and $M \in [-\pi, \pi]$ to be 2π times the fraction of the area of the ellipse swept out by this angle f, signed $+$ for prograde and $-$ for retrograde motion (see Figure 2). Applying Kepler's law of equal areas swept

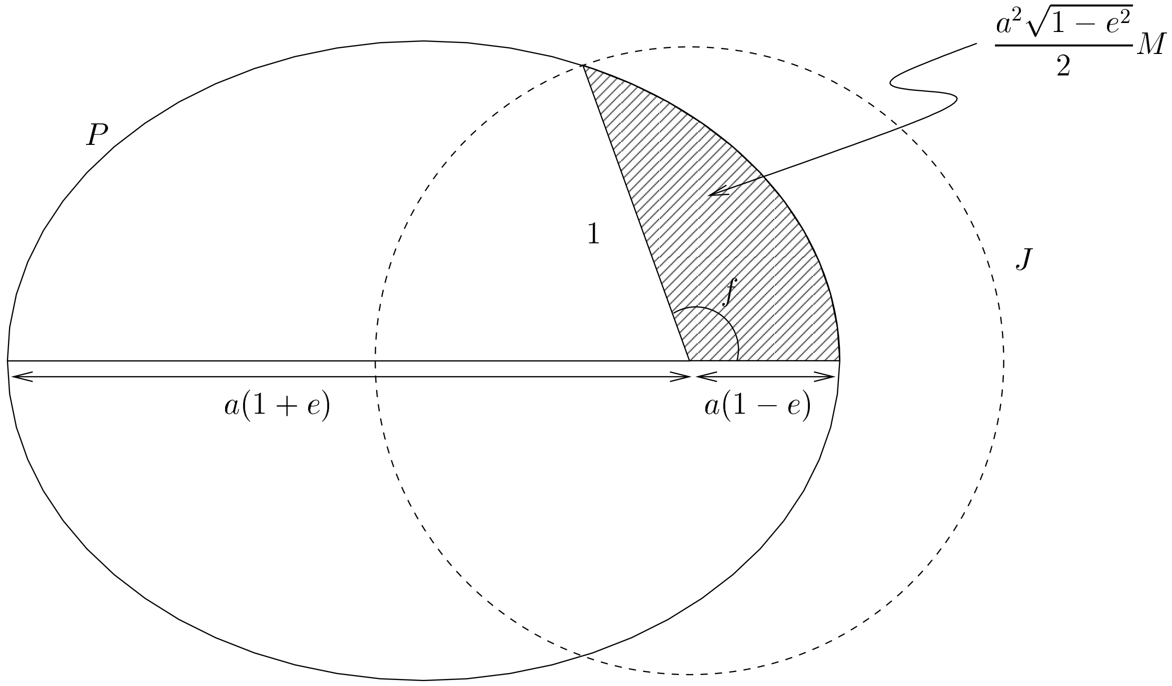

Figure 2: Illustration of the angle f and the area defining M.

in equal times to both the second and third masses, one sees that there is a collision some time at the other intersection iff

$$2\pi m' - 2M = (2\pi k' - 2f)\Omega$$

for some non-zero $m', k' \in \mathbf{Z}$. Define

$$G_C(\Omega) = \frac{1}{\pi}(M - f\Omega). \tag{3.4}$$

Then there is collision at the other intersection iff $G_C(\Omega) = m' - k'\Omega$. For $\Omega = m/k$, this implies that $G_C(m/k) = (m'k - k'm)/k \in \mathbf{Z}/k$. In particular, early collision implies that $G_C(m/k) \in \mathbf{Z}/k$. This can be seen to be false at certain rationals, e.g. $m/k = 1/1$, as will be checked below, but to obtain a dense set of rationals where it is false, we pursue a more subtle line.

The function G_C is analytic on A_C. This is because firstly the angle $f \in [0, \pi]$ is given in terms of C and Ω by

$$e \cos f = \frac{1}{4}(C - a^{-1})^2 - 1, \tag{3.5}$$

using equations (3.2) and (3.3) for e and a^{-1}. For $\Omega \in A_C$, f avoids 0 and π so is analytic in Ω (and C). Secondly, the angle M can be obtained via the "eccentric anomaly" $E \in [-\pi, \pi]$ which is defined by

$$e \cos E = 1 - a^{-1}, \tag{3.6}$$

with sign + for prograde motion, − for retrograde. For $\Omega \in A_C$, E avoids $\{0, \pm\pi\}$ so is analytic in Ω (and C). Then M is given by the Kepler equation

$$M = E - e \sin E, \tag{3.7}$$

so is analytic. $G_C(\Omega)$ is obtained via equations (3.2)–(3.7), and so G_C is analytic on A_C.

Suppose for given $C \in (-\sqrt{8}, +3)$ there is a non-empty open interval $A \subset A_C$ such that for all rationals $m/k \in A$ we have $G_C(m/k) \in \mathbf{Z}/k$. Then choose any $m_\infty/k_\infty \in A$ and sequence of distinct rationals $m_n/k_n \in A$ tending to m_∞/k_∞. For each n, there exists $\tilde{m}_n/\tilde{k}_n \in A$ such that $|\tilde{m}_n k_n - m_n \tilde{k}_n| = 1$ (a Farey neighbour). Then for all $j \geq 0$,

$$\frac{j m_n + \tilde{m}_n}{j k_n + \tilde{k}_n} \in A.$$

Thus

$$G_C\left(\frac{j m_n + \tilde{m}_n}{j k_n + \tilde{k}_n}\right) \in \mathbf{Z}/(j k_n + \tilde{k}_n).$$

Then

$$\Delta G_{C,n} := G_C\left(\frac{j m_n + \tilde{m}_n}{j k_n + \tilde{k}_n}\right) - G_C\left(\frac{m_n}{k_n}\right) \in \frac{\mathbf{Z}}{(j k_n + \tilde{k}_n) k_n}. \tag{3.8}$$

Since G_C is analytic on A_C, it is differentiable at m_n/k_n. Thus

$$\Delta G_{C,n} = \frac{G_C'(\frac{m_n}{k_n})}{(j k_n + \tilde{k}_n) k_n} + o\bigg(\frac{1}{(j k_n + \tilde{k}_n) k_n}\bigg). \tag{3.9}$$

Letting $j \to \infty$ we deduce that $G_C'(m_n/k_n) \in \mathbf{Z}$.

Since G_C is analytic on A_C, so is its derivative G'_C. We just showed that the latter is integer-valued at a sequence $m_n/k_n \in A$ with a limit point $m_\infty/k_\infty \in A$. Thus it is constant on the connected component of A_C containing A: $G'_C(\Omega) = N \in \mathbf{Z}$. This is already a very strong constraint, but it will be convenient to do even better.

By assumption, for each $m/k \in A$ there exists an integer $n(m,k)$ such that $G_C(m/k) = n(m,k)/k$. Thus for each pair of rationals in A,

$$\frac{n(m,k)}{k} - \frac{n(m',k')}{k'} = N\left(\frac{m}{k} - \frac{m'}{k'}\right). \tag{3.10}$$

This can be rearranged to show that $(n(m,k) - Nm)/k$ is the same for all rationals $m/k \in A$. Since it is rational for each, call the common value p/q in lowest terms. Then for all $m/k \in A$, $n(m,k) - Nm = pk/q$. If $q > 1$ then there are values of k (those missing some prime factor of q) for which this is impossible. Thus $q = 1$. Hence

$$G_C(\Omega) = N\Omega + p \tag{3.11}$$

on A, and by uniqueness of analytic continuation the same holds on the whole of the component of A_C containing A.

Thus to prove density of frequencies with no early collision in a component of A_C, it suffices to find a contradiction to Eq. (3.11) in that component. To achieve this, we divide the (a^{-1}, C) plane into several cases.

Case 1 : $C \in [1,3)$, prograde. Evaluate G_C at $\Omega = 1$:

$$G_C(1) = \frac{1}{2} - \frac{e}{\pi} - \frac{1}{\pi}\cos^{-1}(-e),$$

with

$$e^2 = 1 - \frac{1}{4}(C-1)^2, \tag{3.12}$$

and $\cos^{-1}$ defined to take value in $[0,\pi]$. Thus $G_C(1) \in (-\frac{1}{\pi} - \frac{1}{2}, 0)$ which is disjoint from $\mathbf{Z}$, so there is no early collision for $\Omega = 1$ for this range of C, and by Eq. (3.11) there is a dense set of rationals in the prograde component of A_C for which there is no early collision.

Case 2 : $C \in (-1,1]$, retrograde. Evaluating G_C at $\Omega = 1$ again gives

$$G_C(1) = -\frac{1}{2} + \frac{e}{\pi} - \frac{1}{\pi}\cos^{-1}(-e),$$

again with e given by (3.12) and $\cos^{-1}$ taking value in $[0,\pi]$. So $G_C(1) \in (1 - \frac{3\pi}{2}, -1)$, which is again disjoint from $\mathbf{Z}$.

Case 3 : $C \in (-\sqrt{8}, -1]$, retrograde. It is immediate for this range of C that G_C goes to zero at the two boundaries of A_C. Thus to contradict Eq. (3.11) for a given C, it suffices to show that G_C is not identically zero. We will prove this by evaluating $G_C(\Omega)$ to first order near $\Omega = 0$. Some calculation for a^{-1} small yields

$$\begin{aligned} f &\sim 2\cos^{-1}\left(\frac{|C|}{\sqrt{8}}\right), \\ E &\sim -\sqrt{2 - \frac{C^2}{4}} a^{-1/2}, \\ M &\sim -\frac{1}{3}\sqrt{2 - \frac{C^2}{4}}\left(1 + \frac{C^2}{4}\right)\Omega. \end{aligned}$$

Thus

$$G_C(\Omega) \sim -\left(\frac{1}{3}\sqrt{2 - \frac{C^2}{4}}\left(1 + \frac{C^2}{4}\right) + 2\cos^{-1}\left(\frac{|C|}{\sqrt{8}}\right)\right)\frac{\Omega}{\pi}.$$

The coefficient of Ω is negative for $C \in (-\sqrt{8}, -1]$, hence for each C in this range G_C is not identically zero.

Case 4 : $C \in (0, 1]$, prograde. Again we evaluate $G_C(\Omega)$ for Ω small.

$$\begin{aligned} f &\sim 2\cos^{-1}\left(\frac{|C|}{\sqrt{8}}\right), \\ M &\sim \frac{1}{3}\sqrt{2 - \frac{C^2}{4}}\left(1 + \frac{C^2}{4}\right)\Omega. \end{aligned}$$

So $G_C(\Omega) \sim D_C\Omega$ with

$$D_C = \frac{1}{\pi}\left(\frac{1}{3}\sqrt{2 - \frac{C^2}{4}}\left(1 + \frac{C^2}{4}\right) - 2\cos^{-1}\left(\frac{|C|}{\sqrt{8}}\right)\right).$$

We claim that D_C lies in $(-1, 0)$ for all $C \in (0, 1]$, indeed for all $C \in (0, \sqrt{8})$, because $D_0 = -1 + \frac{\sqrt{2}}{3\pi}$, $D_{\sqrt{8}} = 0$ and

$$\frac{\partial D}{\partial C} = \frac{1 + \frac{C^2}{4} - \frac{C^3}{16}}{\pi\sqrt{2 - \frac{C^2}{4}}} > 0$$

for all $C \in (0, \sqrt{8})$. Thus Eq. (3.11) is not possible for prograde motion for any C in this range.

Case 5 : $C \in (1,2)$, retrograde. For this case, we examine the behaviour of $G_C(\Omega)$ for Ω near the boundary corresponding to tangency, which is given by the solution a^{-1} of $(C-a^{-1})^2 = 8-4a^{-1}$. On this part of the boundary, $e = a^{-1}-1$. For a small change $-\tilde{\beta}$ in a^{-1} (keeping C constant), we obtain the leading order change $\tilde{e}$ to e from Eq. (3.2):

$$2e\tilde{e} \sim \frac{1}{4}(a^{-1}-C)(3a^{-1}-C)\tilde{\beta}.$$

Writing the angle f as $\pi - \tilde{\phi}$, we obtain from Eq. (3.5):

$$\frac{e}{2}\tilde{\phi}^2 - \tilde{e} \sim -\frac{1}{2}(a^{-1}-C)\tilde{\beta}.$$

So

$$\tilde{\phi}^2 \sim \frac{1}{e}(a^{-1}-C)\left(\frac{1}{4e}(3a^{-1}-C)-1\right)\tilde{\beta}.$$

Writing $E = -\pi + \tilde{\gamma}$, we obtain from Eq. (3.6):

$$\frac{e}{2}\tilde{\gamma}^2 - \tilde{e} \sim \tilde{\beta}.$$

So

$$\tilde{\gamma}^2 \sim \frac{2}{e}\left(\frac{1}{8e}(a^{-1}-C)(3a^{-1}-C)+1\right)\tilde{\beta}.$$

Next, inserting $M = -\pi + \tilde{m}$ into Eq. (3.7) yields

$$\tilde{m} \sim (1+e)\tilde{\gamma} \sim a^{-1}\tilde{\gamma}.$$

Finally the change $\tilde{\Omega}$ in Ω is

$$\tilde{\Omega} \sim -\frac{3}{2}a^{-1/2}\tilde{\beta}.$$

Thus the change $\tilde{G}$ in G_C is

$$\begin{aligned}\tilde{G} &\sim \frac{1}{\pi}(\tilde{m} + \tilde{\phi}\Omega - \pi\tilde{\Omega}) \\ &\sim \frac{1}{\pi}(a^{-1}\tilde{\gamma} + \Omega\tilde{\phi}) - \frac{3}{2}a^{-1/2}\tilde{\beta}.\end{aligned}$$

Since the first two terms are positive and of order $\sqrt{\tilde{\beta}}$ we can neglect the third one. We deduce that $\tilde{G} \sim K_C\sqrt{-\tilde{\Omega}}$ for some $K_C > 0$ (whose formula we do not bother to give), which is incompatible with Eq. (3.11).

This completes the proof of Lemma 3.1. ∎

Lemma 3.2 *For $C \in (-\sqrt{8}, +3)$, the collision orbit at $\varepsilon = 0$ in the rotating frame, corresponding to a whole number m of revolutions of an ellipse with rational frequency $m/k \in A_C$ starting and ending at collision with Jupiter, is non-degenerate.*

Proof: We use the fifth formulation of nondegeneracy listed in section 1. Energy translates here to Jacobi constant. Instead of considering all nearby solutions we can enforce the first equation of (1.6) and reduce the dimension of parameter λ to $\dim Q$. We choose to parametrise the set of planar Kepler ellipses through a given point P (with the Sun S at one focus) by the position $F \in \mathbf{R}^2$ of the second focus. Then, taking the distance $SP = 1$, we have

$$\begin{aligned} a &= \frac{1}{2}(1 + PF), \\ e &= SF/2a, \\ C &= a^{-1} \pm 2\sqrt{a - SF^2/4a}. \end{aligned}$$

The argument of the square root is positive since $e < 1$ in A_C. For $\Omega \in A_C$, the orbit crosses the unit circle transversely at non-zero speed and Jupiter moves along the unit circle at non-zero speed, so the second equation of (1.6) is satisfied nondegenerately for $\Omega = m/k$, that is the distance PF is fixed. Thus all we have to check is that the third equation of (1.6) is satisfied nondegenerately. But since $e > 0$ in A_C,

$$\left.\frac{\partial C}{\partial SF}\right|_a = \pm \frac{SF}{2a\sqrt{a - SF^2/4a}} \neq 0.$$

∎

Remark: It is fortunate that we want nondegeneracy for fixed Jacobi constant rather than fixed time (or fixed energy), because any point P on a Kepler ellipse is equal-time self-conjugate (and equal-energy self-conjugate): the family of elliptical orbits of fixed a through P return to P at exactly the same time (and have the same energy).

Lemma 3.3 *Given any sequence $(\Omega_n)_{n\in\mathbf{Z}} \in S_C$ there exists an "orbite à chocs" consisting of ellipses of frequency Ω_n connecting collisions at Jupiter, which leave each collision in neither the same nor the opposite direction as they arrive.*

Proof: For each $C \in (-\sqrt{8}, 3)$ and $\Omega \in A_C$ and position of Jupiter on the unit circle, there are two possible ellipses leaving it with frequency Ω, and they leave in different and non-opposite directions (see Figure 3). Thus at least one of them leaves in a different and non-opposite direction from that in which the Asteroid arrived along the previous collision trajectory. ∎

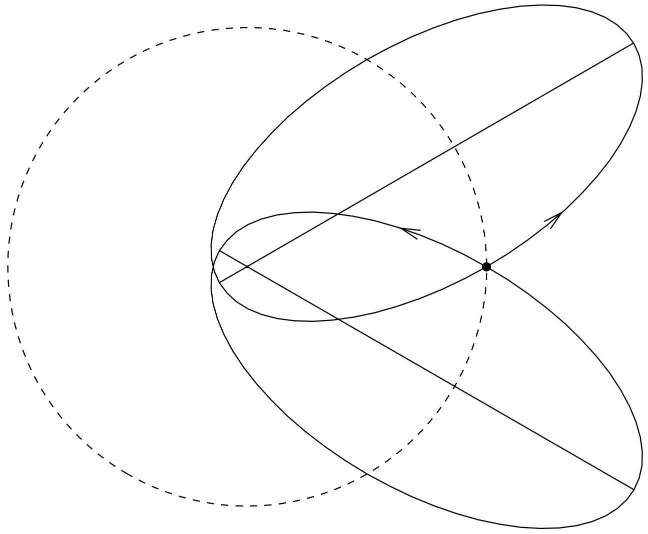

Figure 3: The two ellipses which leave the same point of the unit circle with the same values of (a^{-1}, C).

4 Boundary value problem

The rest of the paper is devoted to the proof of Theorem 1.1.

The main result of this section is the existence, for small $\varepsilon > 0$, of orbits with energy E connecting two points a, b in a small ball U_i centred at p_i. The proof is postponed to section 6. In section 5 this local result will be used to glue together the collision trajectories γ_i and prove Theorem 1.1.

For any $a \in U_i$, there is a unique trajectory $\gamma_a^+ : [0, \tau^+(a)] \to U_i$ of energy E for the system (L) with the Lagrangian (1.2) connecting a to p_i. Similarly, for any $b \in U_i$, there is a unique trajectory $\gamma_b^- : [\tau^-(b), 0] \to U_i$ of energy E connecting p_i to b. Denote

$$S^+(a) = \int_0^{\tau^+(a)} (L(\gamma_a^+(t), \dot\gamma_a^+(t)) + E)\, dt, \tag{4.1}$$

$$S^-(b) = \int_{\tau^-(b)}^0 (L(\gamma_b^-(t), \dot\gamma_b^-(t)) + E)\, dt. \tag{4.2}$$

Then $S^\pm$ are continuous functions on U_i and C^3 on $U_i \setminus \{p_i\}$. They are not differentiable at p_i, however: $S^+(a)$ behaves like $\mathrm{dist}\,(a, p_i)$ and $S^-(b)$ like $\mathrm{dist}\,(p_i, a)$, where the (nonsymmetric) distance is defined by the Jacobi metric.

Denote by

$$u^+(a) = \dot\gamma_a^+(\tau^+(a)), \quad u^-(b) = \dot\gamma_b^-(\tau^-(b)),$$

the tangent vectors to γ_a^+, γ_b^- at the point $p_i = \gamma_a^+(\tau^+(a)) = \gamma_b^-(\tau^-(b))$.

Let $\Sigma_i = \partial U_i$. Fix a small $\delta > 0$ and let

$$X_i = \{(a,b) \in \Sigma_i^2 \mid \|u_+(a) - u_-(b)\| \geq \delta\}. \tag{4.3}$$

Equivalently, a pair of points $(a,b) \in \Sigma_i$ belongs to X_i if the solution of the system (L) with energy E connecting a to b does not pass too close to the centre p_i. Let

$$Y_i = \{(a,b) \in X_i \mid \|u_+(a) + u_-(b)\| \geq \delta\}. \tag{4.4}$$

Lemma 4.1 *There exists $\varepsilon_0 > 0$ such that:*

- *for any $\varepsilon \in (0, \varepsilon_0]$ and $(a,b) \in X_i$, there exists a unique trajectory $\gamma = \gamma_{a,b}^\varepsilon : [0,\tau] \to U_i$ of energy E for the system (L_ε) connecting a to b: $\gamma_{a,b}^\varepsilon(0) = a$, $\gamma_{a,b}^\varepsilon(\tau) = b$.*
- *$\tau = \tau(a,b,\varepsilon)$ is a C^2 function on $X_i \times (0, \varepsilon_0]$ and $\tau(a,b,\varepsilon) \to \tau^+(a) + \tau^-(b)$ uniformly as $\varepsilon \to 0$.*
- *$\gamma_{a,b}^\varepsilon|_{[0,\tau_+(a)]} \to \gamma_a^+$ and $\gamma_{a,b}^\varepsilon(\cdot + \tau)|_{[\tau_-(b),0]} \to \gamma_b^-$ uniformly as $\varepsilon \to 0$. More precisely, there exists a constant $C > 0$ depending only on δ such that*

$$\begin{aligned} &\max\nolimits_{0 \leq t \leq \tau_+(a)} \operatorname{dist}(\gamma_{a,b}^\varepsilon(t), \gamma_a^+(t)) \leq C\varepsilon, \\ &\max\nolimits_{\tau + \tau_-(b) \leq t \leq \tau} \operatorname{dist}(\gamma_{a,b}^\varepsilon(t), \gamma_b^-(t - \tau)) \leq C\varepsilon. \end{aligned} \tag{4.5}$$

- *the action of the trajectory γ:*

$$S(a,b,\varepsilon) = \int_0^\tau (L_\varepsilon(\gamma(t), \dot\gamma(t)) + E)\, dt \tag{4.6}$$

  is a C^2 function on $X_i \times (0, \varepsilon_0]$ and

$$S(a,b,\varepsilon) = S^+(a) + S^-(b) + \varepsilon s(a,b,\varepsilon), \tag{4.7}$$

  where s is uniformly C^2 bounded on X_i as $\varepsilon \to 0$.

- *if, additionally, $(a,b) \in Y_i$, then the trajectory $\gamma_{a,b}^\varepsilon$ does not pass too close to p_i:*

$$\min_{0 \leq t \leq \tau} \operatorname{dist}(\gamma_{a,b}^\varepsilon(t), p_i) \geq c\varepsilon, \qquad c > 0. \tag{4.8}$$

Without the condition $(a,b) \in Y_i$ the Lemma still holds except for the last statement: the trajectory $\gamma_{a,b}^\varepsilon$ may pass through or close to p_i. The proof is given in section 6. It is particularly clear in the 2D case, when it follows from Levi-Civita regularisation [23] and the λ-lemma [16]. A similar argument was used in [12, 15] to prove the existence of symmetric periodic orbits of the second species for the circular restricted 3-body problem, but the connection property was not proved there and hence no chaotic orbits established. For the 3D case, the proof is based on KS regularisation [14]. The proof of Lemma 4.1 is simple for the Kepler problem, for whose solutions we have explicit formulae.

Remark: The proof of Lemma 4.1 is the only place in the proof of Theorem 1.1 where the assumption $\dim Q = 2$ or 3 is used. This is because in the proof we apply Levi-Civita or KS regularisation of double collisions. By using different methods, it is possible to prove Lemma 4.1 for arbitrary dimension. For example, one can replace KS-regularisation by its n-dimensional analogue using Clifford algebras. Hence Theorem 1.1 holds also in the multidimensional case, but we are not aware of any physical application of this generalisation.

5 Shadowing collision orbits

In this section we prove Theorem 1.1.

For any $k \in K$, let $x_k \in \Sigma_{\alpha_k}$ and $y_k \in \Sigma_{\beta_k}$ be the intersection points of γ_k with Σ_{α_k} and Σ_{β_k} respectively. Then $\gamma_k(t) = \gamma_{x_k}^-(t - \tau^-(x_k))$ for $0 \le t \le -\tau^-(x_k)$, $\gamma_k(t) = \gamma_{y_k}^+(t + \tau_k - \tau^+(y_k))$ for $\tau_k - \tau^+(y_k) \le t \le \tau_k$. Without loss of generality it can be assumed that the points x_k and y_k are not conjugate (on the fixed energy level) along γ_k for all k. If not, we change the radius of the balls U_i a little to make the new intersection points non-conjugate.

Let $A_k \subset \Sigma_{\alpha_k}$ be a small neighbourhood of x_k, $B_k \subset \Sigma_{\beta_k}$ a small neighbourhood of y_k, and W_k be a small neighbourhood of $\gamma_k([0, \tau_k])$. We may assume that $A_k = W_k \cap \Sigma_{\alpha_k}$ and $B_k = W_k \cap \Sigma_{\beta_k}$. If the neighbourhoods A_k, B_k, and W_k are small enough, by the non-conjugacy of x_k and y_k along γ_k and the implicit function theorem, for any $u \in A_k$, $v \in B_k$, there exists a unique solution $\sigma = \sigma_{uv}^\varepsilon : [0, \tau] \to W_k$, $\tau = \tau_{uv}^\varepsilon$, of energy E for the system (L_ε) such that $\sigma(0) = u$, $\sigma(\tau) = v$, which is close to $\gamma_k(t - \tau_-(x_k))$ for $0 \le t \le \tau$. This solution is a C^3 function of u, v. Let

$$f_\varepsilon(u, v) = \int_0^\tau (L_\varepsilon(\sigma(t), \dot\sigma(t)) + E)\, dt$$

be the action of the trajectory σ_{uv}^ε. Then f_ε is a C^3 function on $A_k \times B_k$.

Lemma 5.1 *The function $g_k(u, v) = f_0(u, v) + S_{\alpha_k}^-(u) + S_{\beta_k}^+(v)$ on $A_k \times B_k$ has a nondegenerate critical point at $z_k = (x_k, y_k)$.*

Here $S_i^\pm$ are the functions $S^\pm$ in (4.1)–(4.2) corresponding to the points p_i.

Proof: Lemma 5.1 follows from the assumption that γ_k is a non-degenerate critical point of the action functional (1.7). Indeed, $g_k(u, v)$ is the action of the piecewise smooth trajectory of the system (L) obtained by gluing together the trajectories γ_u^-, $\sigma_{u,v}^0$, and γ_v^+ (with appropriate shifts of time parametrisation). Hence g_k is the restriction of the action functional (1.7) to a finite-dimensional submanifold consisting of broken trajectories (with break points u, v) connecting p_α to p_β. ∎

Let us choose some coordinates u, v in A_k and B_k. Lemma 5.1 implies that if the neighbourhoods A_k, B_k are small enough, there exists $C > 0$ such that

$$\|(g_k''(s))^{-1}\| \leq C \quad \text{for} \quad s \in A_k \times B_k, \tag{5.1}$$

where $s = (u, v)$ and $\| \ \|$ is the max norm in $A_k \times B_k$.

Let $G \subset K^2$ be the set (1.6). Taking the neighbourhoods A_k, B_k small enough, it can be assumed that for all $(k,l) \in G$, $B_k \times A_l \subset Y_j$, where $j(k,l) = \beta_k = \alpha_l$ and Y_j is defined in (4.4).

The next result is a more precise version of Theorem 1.1. We assume that the neighbourhoods W_k are sufficiently small.

Theorem 5.1 *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and any chain $(k_i \in K)_{i \in \mathbf{Z}}$ of collision orbits there exists a unique (up to a time shift) trajectory $\gamma : \mathbf{R} \to (\cup_{k \in K} W_k) \setminus P$ of energy E for the system (L_ε) and a sequence*

$$\cdots < a_i < b_i < a_{i+1} < b_{i+1} < \cdots,$$

such that for all $i \in \mathbf{Z}$:

- $\gamma([a_i, b_i]) \subset W_{k_i}$, $\gamma(a_i) \in A_{k_i}$, $\gamma(b_i) \in B_{k_i}$*;*
- $\gamma([b_i, a_{i+1}]) \subset U_j$, $j = \beta_{k_i} = \alpha_{k_{i+1}}$.

The asymptotic behavior of this trajectory as $\varepsilon \to 0$ is as follows:

- $b_i - a_i \to \tau_{k_i} - \tau^-(x_{k_i}) - \tau^+(y_{k_i})$ *as* $\varepsilon \to 0$*;*
- $\gamma(t)$ *is* $O(\varepsilon)$*-close to* $\gamma_{k_i}([\tau^-(x_{k_i}), \tau - \tau^+(y_{k_i})])$ *for* $a_i \leq t \leq b_i$*;*
- $\gamma(t) = \gamma^\varepsilon_{q(b_i), q(a_{i+1})}(t - b_i)$ *for all* $t \in [b_i, a_{i+1}]$.

The constant $\varepsilon_0 > 0$ depends only on the set $\{\gamma_k\}_{k \in K}$ of collision orbits and is independent of the sequence $(k_i \in K)$. Thus $\gamma(t)$ is $O(\varepsilon)$-close to a chain of collision orbits. By (4.8) the trajectory $\gamma|_{[b_i, a_{i+1}]}$ avoids p_j by a distance of order ε. If we skip the second condition in the definition of the chain, then Theorem 5.1 still holds, but now a trajectory means a trajectory of the regularised system, so it may reflect from singularities. Mathematically such trajectories are perfectly well defined but they have no astronomical interest.

Proof: The strategy is continuation from the case $\varepsilon = 0$, which in the language of [4] is an "anti-integrable limit".

Given a sequence $(k_i)_{i\in\mathbf{Z}}$ with $(k_i,k_{i+1})\in G$ for all $i\in\mathbf{Z}$, let

$$Y=\prod_{i\in\mathbf{Z}}(A_{k_i}\times B_{k_i}),$$

with supremum norm in the chosen charts on A_k and B_k. Then Y is a ball in the Banach space $Z=l_\infty$. Choose $\varepsilon_0>0$ as in Lemma 4.1 and let $\varepsilon\in(0,\varepsilon_0]$. Then trajectories of system (L_ε) of energy E near the chain $(\gamma_{k_i})_{i\in\mathbf{Z}}$ correspond to critical points of the (formal) functional

$$F_\varepsilon(u,v)=\sum_{i\in\mathbf{Z}}(f_\varepsilon(u_i,v_i)+S(v_i,u_{i+1},\varepsilon)),$$

over sequences $(u,v)=(u_i,v_i)_{i\in\mathbf{Z}}\in Y$, where S is the function (4.6) corresponding to the centre p_j, $j=\beta_{k_i}$ (we omit reference to the centre, because the corresponding functions are defined on disjoint sets, so there can be no misunderstanding). Indeed, $F_\varepsilon(u,v)$ is the Maupertuis action for the concatenation of trajectories $\sigma^\varepsilon_{u_iv_i}$ connecting u_i to v_i and trajectories $\gamma^\varepsilon_{v_iu_{i+1}}$ connecting v_i to u_{i+1} within the balls U_j.

By (4.7),

$$F_\varepsilon(u,v)=\sum_{i\in\mathbf{Z}}(g_{k_i}(u_i,v_i)+\varepsilon s(v_i,u_{i+1},\varepsilon)),$$

where g_k is defined in the statement of Lemma 5.1. Thus, trajectories of (L_ε) of energy E correspond to zeroes of the mapping $\Phi_\varepsilon=\nabla F_\varepsilon:Y\to Z$, $(u,v)\mapsto(U,V)$ defined by

$$U_i=\frac{\partial g_{k_i}}{\partial u}(u_i,v_i)+\varepsilon\frac{\partial s}{\partial u}(v_{i-1},u_i,\varepsilon),$$
$$V_i=\frac{\partial g_{k_i}}{\partial v}(u_i,v_i)+\varepsilon\frac{\partial s}{\partial v}(v_i,u_{i+1},\varepsilon).$$

The sequences $(U,V)\in Z$ represent jumps in the components of momentum tangent to Σ_j at the points u_i,v_i between the trajectories between and within the balls U_j, and are also measured in supremum norm. Since all the trajectories $\sigma^\varepsilon_{u_iv_i}$ and $\gamma^\varepsilon_{v_iu_{i+1}}$ have the same energy E, zero jumps in the tangential components of the momentum implies that the concatenation of these trajectories is a smooth curve and hence a solution of (L_ε).

Since the function s is uniformly C^2 bounded as $\varepsilon\to0$, and the second derivative matrix of g is uniformly invertible, the chain of collision trajectories is a nondegenerate zero of Φ_0 and the implicit function theorem gives a locally unique continuation for a range of ε independent of the sequence (k_i). More precisely, Φ_ε has a unique zero $(u,v)\in Y$ provided that

$$\varepsilon_0^{-1}>C\max_{(k,l)\in G}\max_{B_k\times A_l}\|s''\|,$$

where C is the constant in (5.1). ■

6 Regularisation of collisions

In this section we prove Lemma 4.1.

Replacing W by $W - E$, without loss of generality let $E = 0$. Then $W(p_i) < 0$. It can be assumed that $\omega(p_i) = 0$ for all i. Indeed, if this is not true, let ϕ be a smooth function on Q such that $\nabla\phi(p_i) = \omega(p_i)$. We can subtract $\langle \nabla\phi(q), \dot q\rangle$ from L without changing the Lagrange equations.

The Hamiltonian corresponding to the Lagrangian L_ε in (1.1) has the form

$$H_\varepsilon(q,p) = \frac{1}{2}\langle A^{-1}(q)(p-\omega(q)), p-\omega(q)\rangle + W(q) + \varepsilon V(q). \tag{6.1}$$

First consider the 2D case. Then in local conformal coordinates $q \in \mathbf{R}^2 = \mathbf{C}$ near p_k, we have, up to adding a full derivative,

$$L_\varepsilon(q,\dot q) = \frac{1}{2}a(q)|\dot q|^2 + i\lambda(q)(q\dot{\bar q} - \bar q\dot q) - W(q) + \frac{\varepsilon f(q)}{|q|} \tag{6.2}$$

with $a(0) > 0$, $W(0) < 0$, and $f(0) > 0$. Without loss of generality it can be assumed that $a(0) = 1$, $f(0) = 1/4$, and $W(0) = -1/8$. If not, we scale the variables, change time and rescale ε. The Hamiltonian corresponding to the Lagrangian (6.2) equals

$$H_\varepsilon(q,p) = \frac{1}{2}|p - i\lambda(q)q|^2 + W(q) - \varepsilon\frac{f(q)}{|q|}.$$

Let $h : \mathbf{R}^2 \to \mathbf{R}^2$ be the squaring map $h(x) = x^2$, $x \in \mathbf{C}$.

Lemma 6.1 (Levi-Civita regularisation) *There exists a C^4 Hamiltonian*

$$\mathcal{H}(x,y) = \frac{1}{2}(|y|^2 - |x|^2) + O_4(x,y) \tag{6.3}$$

on $\mathbf{R}^4\{x,y\}$ such that for $x \neq 0$ the canonical transformation $g : (x,y) \to (q,p)$, $q = h(x)$, $y = h'(x)^* p$, takes trajectories of the system with Hamiltonian $\mathcal{H}$ on the energy level $\mathcal{H} = \varepsilon$ to the trajectories of the system with Hamiltonian H_ε on the energy level $H_\varepsilon = 0$.

The transformation g does not preserve the time parametrisation of the solutions, but it preserves the actions: $\int_\gamma \langle y, dx\rangle = \int_{g(\gamma)} \langle p, dq\rangle$. For trajectories of H_ε with energy 0, $\int\langle p, dq\rangle = \int L_\varepsilon\, dt$, so the action (1.7) transforms to $\int\langle y, dx\rangle$ for $\mathcal{H}$.

Lemma 6.1 is a well known fact [23, 15], so we omit the proof. The analogous but less well known 3D case, however, is treated below in full detail. Note that Levi–Civita regularisation can lead to elliptic equilibria (Kepler problem with negative energy), but in our case, the role of energy is played by $-W(0) > 0$, hence the corresponding equilibrium is always hyperbolic.

Consider a more general Hamiltonian system on $\mathbf{R}^{2m}$ with a C^{3+} (3rd derivatives satisfy the Lipschitz condition) Hamiltonian of the form (6.3). The hyperbolic equilibrium 0 has m-dimensional local stable and unstable manifolds $W^{\pm}_{\rm loc}$ [16]. Since $W^{\pm}_{\rm loc}$ are Lagrangian manifolds and project diffeomorphically to $\mathbf{R}^m\{x\}$, they are defined by C^{3+} generating functions $s^{\pm}$ on a small ball U with centre 0 in $\mathbf{R}^m$:

$$W^{\pm}_{\rm loc} = \{(x,y) \mid y = \mp\nabla s^{\pm}(x), x \in U\}. \tag{6.4}$$

The functions $s^{\pm}$ have nondegenerate minimum 0 at the point 0.

By the definition of $W^{\pm}_{\rm loc}$, for any point $a \in U$ there exists a unique trajectory $\omega_a^+ : [0,\infty) \to U$ such that $\lim_{t\to\infty} \omega_a^+(t) = 0$ and $\omega_a^+(0) = a$. Similarly, there exists a unique trajectory $\omega_a^- : (-\infty, 0] \to U$ such that $\lim_{t\to-\infty} \omega_a^-(t) = 0$ and $\omega_a^-(0) = a$. By (6.4) the actions of these trajectories equal

$$\int_{\omega_a^{\pm}} \langle y, dx\rangle = s^{\pm}(a).$$

For the trajectories $\omega_a^{\pm}(t)$, let $z_a^{\pm}(t) \in W^{\pm}_{\rm loc}$ be the corresponding orbits in the phase space.

Lemma 6.2 *Let $T > 0$ be sufficiently large. Then for any points $a, b \in U$ and $\tau \geq T$:*

- *there exists a unique trajectory*

  $$z(t) = (x(t), y(t)) = f(a,b,\tau,t), \quad (\tau,t) \in D_T = \{(\tau,t) \mid \tau \geq T,\ 0 \leq t \leq \tau\},$$

  such that $x(0) = a$ and $x(\tau) = b$.

- *the map f is C^2 on $U^2 \times D_T$ and*

  $$f(a,b,\tau,t) = z_a^+(t) + z_b^-(t-\tau) + e^{-\tau}\phi(a,b,\tau,t). \tag{6.5}$$

  where ϕ is uniformly C^2 bounded on $U^2 \times D_T$.

- *the action*

  $$S(a,b,\tau) = \int_0^{\tau} \langle y, dx\rangle$$

  of the trajectory $z(t)$ is C^2 on $U^2 \times [T,\infty)$ and

  $$S(a,b,\tau) = s^+(a) + s^-(b) + e^{-\tau}R(a,b,\tau) + \tau h(a,b,\tau), \tag{6.6}$$

  where R is uniformly C^2 bounded as $\tau \to \infty$, and $h(a,b,\tau)$ is the energy of z.

Lemma 6.2 can be deduced from Shilnikov's lemma [21], or the strong λ-lemma [9]. For the proof of (6.5) see [5]. Eq.(6.6) follows from the fact that $S(a,b,\tau) - \tau h(a,b,\tau)$ is the generating function for the canonical map $z(0) \mapsto z(\tau)$. ∎

Lemma 6.3 *The energy $h(a,b,\tau)$ of the trajectory $z(t)$ is a C^2 function on $U^2 \times [T,\infty)$ and has the form*

$$h(a,b,\tau) = e^{-\tau}(h_0(a,b) + h_1(a,b,\tau)),$$

where

$$h_0(a,b) = 2\langle v^+(a), v^-(b)\rangle, \qquad v^\pm(a) = \lim_{t\to\pm\infty}(e^{\pm t}\dot\omega_a^\pm(t)), \tag{6.7}$$

and $\|h_1\|_{C^2(U^2\times[\tau,\infty))} \to 0$ as $\tau \to \infty$.

Here $v^\pm(a)$ are tangent vectors at 0 to the asymptotic trajectories $\omega_a^\pm$, and $\langle\, , \rangle$ is the Euclidean scalar product.

Proof: There exist local symplectic coordinates $(u,v) \in \mathbf{R}^{2m}$ in a neighbourhood of the equilibrium 0 such that $W^-_{\rm loc} = \{v = 0\}$, $W^+_{\rm loc} = \{u = 0\}$, and

$$\mathcal{H}(u,v) = \langle u,v\rangle(1 + O(u,v)). \tag{6.8}$$

The symplectic transformation $(x,y) \mapsto (u,v)$ is given by the formulae

$$y = \frac{u+v}{\sqrt2} + O_3(u,v), \quad x = \frac{u-v}{\sqrt2} + O_3(u,v).$$

The Hamiltonian system on the unstable manifold $W^-_{\rm loc}$ takes the form

$$\dot u = u + O_2(u),$$

where the right hand side is of class C^{2+}. This equation can be transformed [2] to a linear equation $\dot\xi = \xi$ by a C^2 change of variables $\xi = f(u)$. Hence the phase flow on $W^-_{\rm loc}$ takes the form

$$g_{-t}(u,0) = (f^{-1}(e^{-t}f(u)),0) = e^{-t}(f(u) + G(u,t),0), \tag{6.9}$$

where $\|G\|_{C^2} \to 0$ uniformly on $W^-_{\rm loc}$ as $t \to \infty$.

A similar representation holds for the flow on the stable manifold:

$$g_t(0,v) = e^{-t}(0, g(v) + E(v,t)), \tag{6.10}$$

where $\|E\|_{C^2} \to 0$ uniformly on $W^+_{\rm loc}$ as $t \to \infty$. Note that

$$(f(u),0) = \lim_{t\to\infty} e^t g_{-t}(u,0), \quad (0,g(v)) = \lim_{t\to\infty} e^t g_t(0,v). \tag{6.11}$$

Put $t = \tau/2$ in (6.5). By (6.9) and (6.10),

$$z(\tau/2) = (e^{-\tau/2} f(u), e^{-\tau/2} g(v)) + e^{-\tau/2} F(u, v, \tau), \tag{6.12}$$

where $u = u(\tau)$ and $v = v(0)$, and $\|F\|_{C^2} \to 0$ as $\tau \to \infty$. Since $\mathcal{H}$ is a first integral, substituting (6.12) into (6.8) gives an estimate for the energy:

$$h(a, b, \tau) = \mathcal{H}(z(\tau/2)) = e^{-\tau}(\langle f(u), g(v)\rangle + h_1(u, v, \tau)),$$

where $\|h_1\|_{C^2} \to 0$ as $\tau \to \infty$. Now passing to the variables x, y and using (6.11), we obtain Lemma 6.3. ∎

A result like Lemma 6.3 (for the case of simple eigenvalues with minimal absolute value) was the basis for the theorem of Turayev and Shilnikov [22], although it was not formulated in [22]. See also [5, 6].

Take small $\nu > 0$ and let $B = \{(a, b) \in U^2 \mid h_0(a, b) \geq \nu\}$. Then for $(a, b) \in B$ the function $h_0(a, b)$ is bounded away from zero. Thus $h(a, b, \tau)$ is monotone in τ for sufficiently large τ. For small $\varepsilon > 0$, solving the equation $h(a, b, \tau) = \varepsilon$ for τ yields a C^2 function $\tau = \tau_\varepsilon(a, b)$. This gives:

Proposition 6.1 *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$:*

- *for any $(a, b) \in B$, there exists a unique trajectory $z^\varepsilon_{a,b} = (x^\varepsilon_{a,b}, y^\varepsilon_{a,b}) : [0, \tau] \to U \times \mathbf{R}^m$ of energy ε connecting the points a and b.*
- *the time $\tau = \tau_\varepsilon(a, b)$ is a C^2 function on B and*

  $$\tau_\varepsilon(a, b) = -\log \varepsilon + \mu(a, b, \varepsilon),$$

  where the function μ is uniformly bounded on B as $\varepsilon \to 0$.
- *we have*

  $$z^\varepsilon_{a,b}(t) = z^+_a(t) + z^-_b(t) + \varepsilon\zeta(a, b, \varepsilon), \tag{6.13}$$

  where the function ζ is uniformly C^1 bounded as $\varepsilon \to 0$.
- *the action $f_\varepsilon(a, b) = S(a, b, \tau_\varepsilon(a, b))$ of the trajectory $z^\varepsilon_{a,b}$ is a C^2 function on B and*

  $$f_\varepsilon(a, b) = s^+(a) + s^-(b) + \varepsilon r(a, b, \varepsilon) - \varepsilon \log \varepsilon, \tag{6.14}$$

  where r is uniformly C^2 bounded on B as $\varepsilon \to 0$.

Note that (6.7) and (6.13) imply that

$$\min_{0 \leq t \leq \tau} |x^\varepsilon_{a,b}(t)|^2 = 2\varepsilon(|v^+(a)||v^-(b)| - \langle v^+(a), v^-(b)\rangle) + o(\varepsilon). \tag{6.15}$$

Hence $x^\varepsilon_{a,b}(t)$ avoids 0 by a distance of order $\sqrt{\varepsilon}$ provided that $v^+(a) \neq v^-(b)$.

Proof of Lemma 4.1 in the 2D case: This is based on Proposition 6.1 and Lemma 6.1. Collision solutions $\gamma^+_{h(a)} : [0, \tau^+(h(a))] \to U_i$ with p_i for the system (L) correspond (up to time reparametrisation) to asymptotic orbits $\omega^+_a : [0, +\infty) \to U = h^{-1}(U_i)$ to the equilibrium 0 for the system with Hamiltonian $\mathcal{H}$ on the level set $\mathcal{H} = 0$. In the conformal coordinates with centre p_i, the map h takes the trajectory $\omega^+_a : [0, +\infty) \to U$ to the trajectory $\gamma^+_{h(a)} : [0, \tau^+(h(a))] \to U_i$ (with changed time parametrization), i.e. $\gamma^+_{h(a)}(t) = \omega^+_a(t_+(t))$, where $t_+ : [0, +\infty] \to [0, \tau^+(h(a))]$. Similarly, $\gamma^-_{h(a)}(t) = \omega^-_a(t_-(t))$, where $t_- : [-\infty, 0] \to [\tau^-(h(a)), 0]$.

Using the definition of the squaring map h, we obtain that for any $a \in U$,

$$h(v^\pm(a)) = \mp \lambda u^\pm(h(a)) \tag{6.16}$$

with some $\lambda > 0$. A calculation shows that $\lambda = |v^\pm(a)|^2/2$. By conservation of the action, $S^\pm(h(a)) = s^\pm(a)$.

Note that $h(y) = \lambda h(x)$, $\lambda \le 0$, is equivalent to $y \perp x$. Hence $u^+(h(a)) \ne u^-(h(b))$ iff $v^+(a)$ and $v^-(b)$ are not orthogonal.

Take two points $\tilde{a}, \tilde{b} \in \Sigma_i$ such that $|u_+(\tilde{a}) - u_-(\tilde{b})| \ge \delta$. Then there exist $a, b \in U$ such that $\tilde{a} = h(a)$, $\tilde{b} = h(b)$, and

$$h_0(a, b) = 2\langle v^+(a), v^-(b)\rangle \ge \nu,$$

provided $\nu > 0$ is sufficiently small. By Proposition 6.1 we can connect a to b by a trajectory $z^\varepsilon_{a,b} = (x^\varepsilon_{a,b}, y^\varepsilon_{a,b})$ of energy ε for the system with Hamiltonian (6.3). Under the map h and an appropriate time reparametrization, $h(x^\varepsilon_{a,b})$ gives a trajectory of energy 0 for the system (L_ε) connecting $\tilde{a} = h(a)$ to $\tilde{b} = h(b)$. Since the action is invariant under the transformation h, (6.14) implies (4.7) with $s(h(a), h(b), \varepsilon) = r(a, b, \varepsilon)$. Condition (4.8) follows from (6.15) and $|h(x)| = |x|^2$. Lemma 4.1 is proved for the 2D case. ∎

A similar argument works in the 3D case. The difference is that instead of Levi-Civita regularisation, one uses KS-regularisation [14].

In geodesic coordinates $q \in \mathbf{R}^3$ in a neighbourhood U_i of the point p_i,

$$T = \frac{1}{2}\langle A(q)\dot{q}, \dot{q}\rangle, \qquad A(q) = I + O_2(q).$$

Then

$$\operatorname{dist}^2(q, p_i) = |q|^2 + O_4(q).$$

Since $\omega(p_i) = 0$,

$$L_\varepsilon(q, \dot{q}) = \frac{1}{2}\langle (I + O_2(q))\dot{q}, \dot{q}\rangle + \langle O(q), \dot{q}\rangle - W(q) + \frac{\varepsilon f(q)}{|q|},$$

where $f(q) = (1 + O_4(q)/|q|^2)^{-1/2}$. Hence the Hamiltonian (6.1) has the form

$$\begin{aligned} H_\varepsilon(q,p) &= \frac{1}{2}|p - O(q)|^2 + \langle O_2(q)(p - O(q)), p - O(q)\rangle + W(q) - \frac{\varepsilon f(q)}{|q|} \\ &= \frac{1}{2}|p|^2 + \langle O_2(q)p, p\rangle + \langle p, O(q)\rangle + O_4(q) + W(q) - \frac{\varepsilon f(q)}{|q|}. \quad (6.17) \end{aligned}$$

By scaling the variables it can be assumed that $f(0) = 1/4$ and $W(0) = -1/8$.

The KS-regularisation [14] is defined by a quadratic Hopf map $h : \mathbf{R}^4 \to \mathbf{R}^3$ such that:

- $|h(x)| = |x|^2$;
- $h(e^{Jt}x) = h(x)$ for any $t \in \mathbf{R}$, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$;
- if $\langle Jx, y\rangle = 0$, then $|h'(x)y| = 2|x||y|$ and $|(h'(x)^*)^{-1}y| = |y|/(2|x|)$.

The map $h : \mathbf{R}^4 = \mathbf{C}^2 \to \mathbf{R}^3 = \mathbf{C} \times \mathbf{R}$ can be defined by the formula $h(z,w) = (2z\bar{w}, |z|^2 - |w|^2)$.

Lemma 6.4 (KS regularisation) *There exists a C^{3+} Hamiltonian $\mathcal{H}$ of the form (6.3) on $\mathbf{R}^8\{x,y\}$, invariant under the symmetry group $\Phi_s : \mathbf{R}^8 \to \mathbf{R}^8$, $\Phi_s(x,y) = (e^{Js}x, e^{-Js}y)$, such that the exact canonical transformation $g : (x,y) \to (q,p)$, $q = h(x)$, $y = h'(x)^*p$, defined on*

$$N = \{(x,y) \mid x \neq 0, F(x,y) = \langle Jx, y\rangle = 0\},$$

takes trajectories of the system with Hamiltonian $\mathcal{H}$ on the level

$$M_\varepsilon = \{\mathcal{H} = \varepsilon\} \cap N$$

of the first integrals $\mathcal{H}, F$ to the trajectories of the system with Hamiltonian H_ε on the zero energy level.

Here the the partly defined transformation $g : N \to U_i \times \mathbf{R}^3$ is said to be exact canonical, because $\int_\gamma \langle y, dx\rangle = \int_{g(\gamma)} \langle p, dq\rangle$ for any curve γ in N. The Hamiltonian $\mathcal{H}$ is C^4 everywhere except for $x = 0$. In other words, the system with Hamiltonian H_ε on $\{H_\varepsilon = 0\}$ is a time reparametrisation of the Hamiltonian reduction with respect to the symmetry group Φ_s of the system with Hamiltonian $\mathcal{H}$ on $\{\mathcal{H} = \varepsilon\}$.

Proof: Let $x \in \mathbf{R}^4 \setminus \{0\}$ and $p \in \mathbf{R}^3$. Set $q = h(x)$ and $y = h'(x)^*p$. Then $\langle Jx, y\rangle = 0$, and so $(x,y) \in N$. Hence $|q| = |x|^2$ and $|y| = 2|x||p|$. By (6.17), for the function $\tilde{H} = H_\varepsilon \circ g$ on N we have

$$\tilde{H}(x,y,\varepsilon) = H_\varepsilon(q,p) = \frac{|y|^2}{8|x|^2} + \langle O_2(x)y, y\rangle + \langle y, O(x)\rangle + O_8(x) + W(h(x)) - \varepsilon\frac{f(h(x))}{|x|^2}.$$

Extend $\tilde{H}$ to a function on $(\mathbf{R}^4 \setminus \{0\}) \times \mathbf{R}^4$ by using the same formula and multiply by $|x|^2/f(h(x))$. Since $f(h(x))$ is a C^4 function,

$$\mathcal{H}(x,y) = \frac{|x|^2}{f(h(x))}\tilde{H}(x,y,\varepsilon) + \varepsilon = \frac{1}{8f(0)}|y|^2 + \frac{W(0)}{f(0)}|x|^2 + O_4(x,y)$$

is a C^{3+} function on $\mathbf{R}^8$. By the scaling of the variables, the coefficents equal $\pm 1/2$ respectively. Then we get a Hamiltonian $\mathcal{H}$ of the form (6.3).

Since $\{\tilde{H} = 0\} = \{\mathcal{H} = \varepsilon\}$, the trajectories of the corresponding Hamiltonian systems on the energy levels coincide. Since h is invariant under e^{Js}, $\mathcal{H}$ is invariant under the symmetry group Φ_s, and F is the corresponding momentum integral. Solutions $(x(t), y(t))$ of Hamilton's equations with the Noether integral $F = \langle y, Jx\rangle = 0$ and energy $\mathcal{H} = \varepsilon$ correspond to trajectories of the system with Hamiltonian H_ε on the energy level $H_\varepsilon = 0$ via the map $q = h(x)$, $p = (h'(x)^*)^{-1}y$. Lemma 6.4 is proved. ■

Proof of Lemma 4.1 in the 3D case: We proceed as in the 2D case. According to Lemma 6.4, to connect the points $\tilde{a} = h(a)$ and $\tilde{b} = h(b)$ by a trajectory of energy 0, the points a and b in Proposition 6.1 should be chosen in such a way that the connecting trajectory z_{ab}^ε has the value of the first integral $F = 0$. Only then $h(x_{ab}^\varepsilon)$ will be a trajectory of the system with Hamiltonian H_ε. A calculation shows that the value of F on z_{ab}^ε equals

$$F|_{z_{ab}^\varepsilon} = f(a,b,\varepsilon) = \varepsilon\langle Jv_+(a), v_-(b)\rangle + o(\varepsilon).$$

We need the following property of the Hopf map $h : \mathbf{R}^4 \to \mathbf{R}^3$. The condition $h(x) = \lambda h(y)$ with $\lambda < 0$ is equivalent to $\langle x, y\rangle = 0$ and $\langle Jx, y\rangle = 0$, or, equivalently, $\langle e^{Jt}x, y\rangle \equiv 0$.

Equation (6.16) holds also in the 3D case. Suppose that $|u^+(\tilde{a}) - u^-(\tilde{b})| \ge \delta$. Then $h(v^+(a)) \ne \lambda h(v^-(b))$ for all $\lambda < 0$. In this case we can replace b by $e^{Js}b$ with some $s \in \mathbf{R}$ so that for the new pair of points, $f(a, e^{Js}b, \varepsilon) = 0$ and

$$h_0(a, e^{Js}b) = 2\langle v_+(a), e^{Js}v_-(b)\rangle \ge \nu,$$

provided $\nu > 0$ is sufficiently small. Indeed,

$$\langle Jv_+(a), e^{Jt}v_-(b)\rangle = -\frac{d}{dt}\langle v_+(a), e^{Jt}v_-(b)\rangle,$$

and the function $t \to \langle v_+(a), e^{Jt}v_-(b)\rangle$ has a nondegenerate maximum point with positive value. Hence for small $\varepsilon > 0$ the function $t \to f(a, e^{Jt}b, \varepsilon)$ has a nondegenerate zero $s = g(a,b,\varepsilon)$ with $h_0(a, e^{Js}b) \ge \nu$.

Then by Proposition 6.1 the points a and $e^{Js}b$ can be connected by a trajectory with zero value of the integral F and energy $\mathcal{H} = \varepsilon$. Its projection under the Hopf map h gives the desired trajectory of energy 0 for the system (L_ε) connecting $h(a)$ to $h(b)$. Now Lemma 4.1 is proved also in the 3D case. ■

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