Isochronous Potentials

S Bolotin
Faculty of Mathematics and Mechanics,
Moscow State University, RUSSIA

RS MacKay
Mathematics Institute, University of Warwick,
Coventry CV4 7AL, U.K.

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Abstract

A potential $V : \mathbb{R} \to \mathbb{R}$ is said to be *isochronous* if all periodic solutions of $\ddot{x} = -V'(x)$ have the same period, and there is at least one. Harmonic oscillators are examples, but there are many others. In this note, we construct examples of isochronous potentials which are analytic on the whole real axis, and we construct all C^2 isochronous potentials.

1 Introduction

Systems of the form

$$\ddot{x} = -V'(x) \tag{1}$$

are fundamental in mechanics. As is well known, they conserve energy

$$H(x,p) = \frac{1}{2}p^2 + V(x), \tag{2}$$

(where $p = \dot{x}$) and every bounded connected component of a level set $H^{-1}(E) = \{(x,p) \in \mathbb{R}^2 : H(x,p) = E\}$ containing no critical points of H is a periodic orbit. The period is given by

$$T = \sqrt{2}\int_{x_-}^{x_+} \frac{dx}{\sqrt{E - V(x)}}, \tag{3}$$

where $x_- < x_+$ are the extremes of the orbit. We say the potential V is *isochronous* if (i) every periodic orbit has the same period, and (ii) there exists at least one periodic orbit (to rule out trivial cases).

We define *harmonic oscillators* to be those systems for which V is quadratic, $V(x) = V_0 + \frac{1}{2}c(x - x_0)^2, c > 0$ (though some people define them to be systems for which every solution is sinusoidal). Every orbit of a harmonic oscillator (except the equilibrium $x = x_0$) is periodic and has the same period $T = 2\pi/\sqrt{c}$. Thus harmonic oscillators are isochronous.

In the physics community, the term "anharmonic" has become a synonym for "non-isochronous". For even potentials this is correct, by the solution of the "tautochrone problem" (the only potentials such that the time to the minimum is independent of starting position are the quadratics) (e.g. §214 of [A]). There are many other isochronous potentials, however. One can take two half-parabolae of different curvatures joined at their minima, or the limiting case of one half-parabola and a hard wall at its minimum. There are smoother examples, like

$$V(x) = x^2 + x^{-2} \tag{4}$$

(communicated to us by S.Flach), which arises for non-zero angular momentum in the rotationally symmetric two-dimensional harmonic oscillator, and $V(x) = 1 + x - \sqrt{1+2x}$ [U], but both of these have a singularity, at $x = 0$ and $x = -\frac{1}{2}$ respectively.

We show below that a C^2 potential is isochronous if and only if its graph arises by horizontally shearing the graph of a parabola, by a shear which preserves monotonicity of the two sides. We construct the general C^2 isochronous potential and a large class of examples which are real analytic on the whole real line.

2 Properties of isochronous potentials

Suppose that V is C^2 and isochronous. Then firstly, periodic orbits of energy E correspond to bounded connected components of $\{x \in \mathbb{R} : V(x) < E\}$, so every periodic orbit encloses at least one local minimum of V, thus V has a local minimum. Secondly, V has no critical points other than local minima, else there would be a family of periodic orbits which come close to one of these other critical points and hence its period goes to infinity so is not constant. By continuity of V', there is precisely one point, or possibly interval, of local minimum. The case of an interval of local minima cannot be isochronous because the period would go to infinity for the family of periodic orbits shrinking onto this interval. If there is a single point of local minimum, it must be non-degenerate, else the period of oscillation around it would again go to infinity as the amplitude goes to zero. We also deduce that $V(x) \to \infty$ as $x \to \pm\infty$, else the period of the family of periodic orbits oscillating about the minimum would go to infinity as E approaches $\min(\lim_{x\to\infty} V(x), \lim_{x\to-\infty} V(x))$, so would not be constant.

We deduce that by shifts of origin and scaling time, we can put any C^2 isochronous potential into what we will call *class A*: the C^2 potentials V such that $V(0) = 0$, $V'(0) = 0$, $V''(0) = 2$, $V'(x) > 0$ for $x > 0$ and $V'(x) < 0$ for $x < 0$, and $V(x) \to \infty$ as $x \to \pm\infty$. Class A contains the subsets where V is C^r, for $r \in \{3, 4, \dots, \infty, \omega\}$ (where C^ω denotes real analytic). Then V defines two inverse functions $x_+ : [0, \infty) \to [0, \infty)$ and $x_- : [0, \infty) \to (-\infty, 0]$, such that $V(x_\pm(\phi)) = \phi$ for all $\phi \in [0, \infty)$. By the inverse function theorem, they are as smooth as V on $(0, \infty)$, because $V' \neq 0$ there. This means that if V is C^r for some $r \in \{2, 3, 4, \dots, \infty, \omega\}$, then so are $x_\pm$ on $(0, \infty)$. Near $\phi = 0$, however, they behave like square roots:

$$x_\pm(\phi) \sim \pm\sqrt{\phi} \text{ as } \phi \to 0, \tag{5}$$

since $V''(0) = 2$.

For a class A potential, every periodic orbit has energy $E \in (0, \infty)$, and every level set in this range is a single periodic orbit. The integral defining the period can be split into its contributions from the left and right-hand sides of zero. Following §214 of [A] (who attributes the idea to Puiseux), or §12 of [LL] (we are grateful to R de la Llave for pointing out this reference), we change variables from x to ϕ using the functions $x_\pm$ on the right and left respectively, to obtain

$$T = \sqrt{2} \int_0^E \frac{\Delta'(\phi)}{\sqrt{E - \phi}}\, d\phi, \tag{6}$$

where

$$\Delta(\phi) = x_+(\phi) - x_-(\phi). \tag{7}$$

Write $\phi = Eu$ to obtain

$$T = \sqrt{2E} \int_0^1 \frac{\Delta'(Eu)}{\sqrt{1 - u}}\, du. \tag{8}$$

Differentiating with respect to E shows that V is isochronous iff

$$2\phi\Delta''(\phi) + \Delta'(\phi) = 0 \tag{9}$$

for all $\phi \in (0, \infty)$. This has the general solution $\Delta(\phi) = A\sqrt{\phi} + B$, A, B arbitrary. Equation (5) imposes the restriction $\Delta(\phi) \sim 2\sqrt{\phi}$ as $\phi \to 0$. We deduce that a class A potential V is isochronous if and only if

$$\Delta(\phi) = 2\sqrt{\phi}. \tag{10}$$

Note that for an even potential, $x_- = -x_+$, so $\Delta = 2x_+$, thus it is isochronous iff $x_\pm(\phi) = \pm\sqrt{\phi}$. Hence the only even isochronous potential (up to our class A normalisation conditions) is $V(x) = x^2$.

Returning to the general class A potential, equation (10) is equivalent to

$$x_\pm(\phi) = \bar{x}(\phi) \pm \sqrt{\phi} \tag{11}$$

for some function $\bar{x}$ on $[0, \infty)$. Since $x_\pm$ come from a class A potential, $\bar{x}$ is C^2 on $(0, \infty)$ and satisfies $\bar{x}(\phi) = o(\sqrt{\phi})$ as $\phi \to 0$ and

$$|\bar{x}'(\phi)| < \frac{1}{2\sqrt{\phi}}, \tag{12}$$

for all $\phi > 0$.

3 Construction of isochronous potentials

Any C^r function $\bar{x}$ on $[0,\infty)$ ($r \in \{2,3,\ldots,\infty,\omega\}$) satisfying $\bar{x}(0) = 0$ and having $C < 1$ such that

$$|\bar{x}'(\phi)| \leq \frac{C}{2\sqrt{\phi}} \tag{13}$$

generates a C^r isochronous class A potential V. Simply join together the inverse graphs of the functions $x_\pm$ defined by Eq. (11), which are strictly monotone and go to $\pm\infty$ by Eq. (13), to construct the graph of V. It remains to check that V is C^r.

For this, it is most convenient to rearrange (11) to express the graph of V as the zero set of the function $K : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ defined by

$$K(x,\phi) = \phi - (x - \bar{x}(\phi))^2. \tag{14}$$

If $\bar{x}$ is C^r, then so is K. The partial derivative $\partial K/\partial\phi = 1 + 2(x - \bar{x}(\phi))\bar{x}'(\phi)$. On the zero-set, this takes the value $1 \pm 2\sqrt{\phi}\bar{x}'(\phi)$ which is positive for $\phi > 0$ by condition (13) and is 1 for $\phi = 0$. Thus by the implicit function theorem, the zero-set of K is the graph of a C^r function $V : \mathbb{R} \to \mathbb{R}_+$.

Different functions $\bar{x}$ generate different potentials V. There is an infinite-dimensional set of such functions $\bar{x}$, even of analytic ones. To be concrete, we give one family of examples. Take $\bar{x}(\phi) = \sqrt{1+\alpha\phi} - 1$ for any $\alpha \in [0,1)$. This yields the analytic isochronous potentials

$$V(x) = \frac{2 + (1+\alpha)(x^2+2x) - 2(x+1)\sqrt{1+\alpha x(x+2)}}{(1-\alpha)^2}, \tag{15}$$

sketched in Figure 1 for several values of α. Dorignac pointed out that as $\alpha \to 1$ this potential converges on $(0,\infty)$ to the example eq. (4), so it interpolates between this one and the harmonic one.

4 General isochronous potential

The above construction does not quite give all class A isochronous potentials. Although to obtain a C^r potential it is necessary to have $\bar{x} \in C^r$ on $(0,\infty)$ and Eq. (12) (inverse function theorem), it is not necessary to require $\bar{x}$ to be C^r at $\phi = 0$. We will construct the general isochronous class A potential.

It is given by taking

$$\bar{x}(\phi) = \sqrt{\phi}X(\sqrt{\phi}) \tag{16}$$

for any function X which is C^2 on $(0,\infty)$, satisfies

$$|(uX(u))'| < 1 \tag{17}$$

there (where $'$ denotes d/du), and such that $X(u), uX'(u), u^2X''(u) \to 0$ as $u \to 0$, and $u(\pm 1 + X(u)) \to \pm\infty$ as $u \to \infty$.

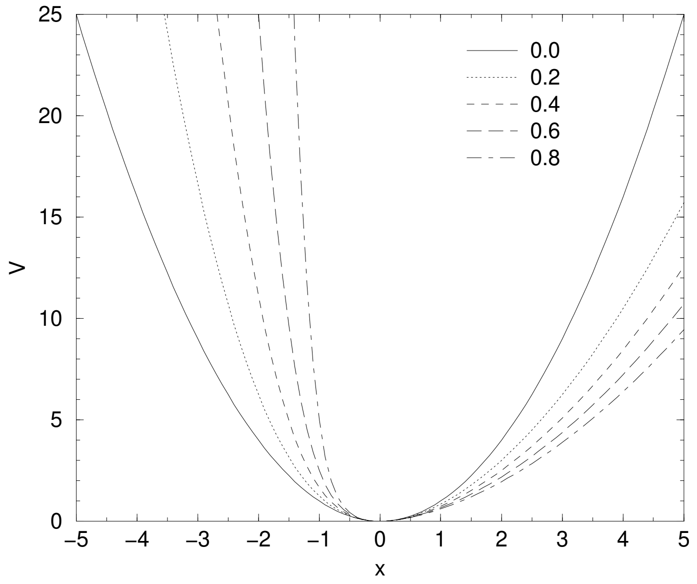

Figure 1: The potentials (15) for α from 0.0 to 0.8 in steps of 0.2.

To see this, write $\phi = u^2$. Then Eq. (11) becomes

$$x_\pm = u(\pm 1 + X(u)). \tag{18}$$

To make the potential C^2 off zero it is necessary and sufficient that X be C^2 off zero, Eq. (17) hold and $x_\pm \to \pm\infty$ as $u \to \infty$. Denote the inverse functions to Eq. (18) by $u_\pm(x)$ for $x > 0, x < 0$ respectively. They go to zero monotonically through all positive values as x goes to zero. Then

$$V(x) = u^2 = x^2/(\pm 1 + X(u_\pm(x)))^2$$

for $x > 0, x < 0$ respectively. So $V''(0)$ does not exist if $X(u)$ does not go to 0 as $u \to 0$. We deduce that $X(u) \to 0$, and the only possible value for $V''(0)$ is 2.

Differentiating Eq. (18) for $x \neq 0$, we deduce that

$$u'_\pm = (\pm 1 + X + uX')^{-1} \tag{19}$$

and

$$u''_\pm = -\frac{2X' + uX''}{(\pm 1 + X + uX')^3}. \tag{20}$$

Hence

$$V'(x) = 2uu' = \frac{2u}{\pm 1 + X + uX'} \tag{21}$$

goes to 0 as required, because of Eq. (17), and

$$V''(x) = 2u'^2 + 2uu'' = \frac{2(\pm 1 + X - uX' - u^2X'')}{(\pm 1 + X + uX')^3}. \tag{22}$$

This goes to 2 for both signs iff both $X + uX'$ and $X - uX' - u^2X''$ go to zero as $u \to 0$. We already proved that $X \to 0$, so the first condition gives $uX' \to 0$. Using this, the second condition gives $u^2X'' \to 0$. This completes the proof.

An example of a function $\bar{x}$ satisfying all the conditions but which is not C^2 at 0 is generated by $X(u) = \sqrt{u}$.

5 Potentials with the same period function

The approach generalises to answer the question of when two smooth potentials have the same period function (the period function of a family of periodic orbits is the period T as a function of the energy E). By Eq. (6), or §12 of [LL], the period function depends only on the function Δ specifying the length of the orbit as a function of energy. Hence any two potentials which are related by just a shear $x \mapsto x + \bar{x}(\phi)$ of their graphs $\phi = V(x)$ have the same period function. By the theory of the Abel transform (e.g. explained in §12 of [LL]), different functions Δ give different period functions, so this is the only way to obtain potentials with the same period function.

A simple example is the pair of potentials $V(x) = \tanh^2 x$ (the sech^2 well) and $(1 - e^{-x})^2$ (the Morse potential), for both of which $\Delta(\phi) = 2\tanh^{-1}\sqrt{\phi}$ and which have the same period function $T(E) = \pi/\sqrt{1-E}$ (e.g. differentiate formulae (A2.6 and 7) of [PR]).

The issues of how much smoothness is required of the shear to preserve a given amount of smoothness of the potential are the same as in the isochronous case, with identical results in the case of potentials with non-degenerate minima.

6 Relation with previous work

The problem is an old one, dating back at least to Huygens. An equation for Δ in terms of the period function $T(E)$ was given in §12 of [LL], assuming V to have a single minimum, which on inserting $T(E) = $ constant, solves the problem up to smoothness questions and the necessity to restrict to class A. [U] gives a necessary and sufficient condition for an isochronous potential (and [LS] a slight extension), but it is not clear from their papers that there are globally defined examples satisfying this. An equation equivalent to the combination of our equations (14) and (16) appears in [O] but without the condition (12) (or (17)) essential for constructing a solution from it. After writing the first version of our paper (April 1999), we became aware of [CMV], which contains the results of our equations (10) and (12), and gives a criterion for isochronous potentials in terms of "strict involutions", but it is not obvious from this how to make the general C^2 isochronous potential or any examples analytic on the whole of $\mathbb{R}$. There are many other papers dealing with the question of when Hamiltonian systems of more general form than (2) are isochronous, e.g. [CD], but they do not answer the question for form (2) specifically.

7 Multidimensional examples

The method allows us to construct many isochronous potentials of more than one degree of freedom: simply take a sum of independent isochronous potentials of the same period. One can also make non-separable rotationally invariant two-dimensional potentials which are isochronous for a particular angular momentum value L: horizontally shear the graph of the radial potential $U(r) = \frac{1}{2}L^2r^{-2} + r^2$ by a function that goes to zero sufficiently fast as the energy goes to infinity, to obtain $\tilde{U}(r) = \frac{1}{2}L^2r^{-2} + V(r)$, and then V will have the required property. The question remains, however, what is the most general multidimensional isochronous potential.

Acknowledgements

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